



Continuous and Pulsed Quantum Control †

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Abstract: We consider two alternative procedures which can be used to control the evolution of a generic finite-dimensional quantum system, one hinging upon a strong continuous coupling with a control potential and the other based on the application of frequently repeated pulses onto the system. Despite the practical and conceptual difference between them, they lead to the same dynamics, characterised by a partitioning of the Hilbert space into sectors among which transitions are inhibited by dynamical superselection rules.

Keywords: quantum control; dynamical decoupling; quantum zeno effect; adiabatic evolution

1. Introduction

Consider a quantum system with a finite dimensional Hilbert space \mathcal{H} , whose evolution $U(t) = e^{-itH}$ is generated by the Hamiltonian H . We are interested in some protocols which dynamically induce a partition of \mathcal{H} into superselection sectors $\mathcal{H}_\mu = P_\mu \mathcal{H}$, in the sense that if the system is initially in some state belonging to one of the superselection sectors, i.e. $|\psi\rangle \in \mathcal{H}_\mu$, it will remain in that sector during its evolution $|\psi(t)\rangle \in \mathcal{H}_\mu$, as shown pictorially in Figure 1. More precisely, given a complete set of orthogonal projections $\{P_\mu\}$ satisfying

$$\sum_{\mu=1}^m P_\mu = I, \quad P_\mu P_\nu = \delta_{\mu\nu} P_\mu = \delta_{\mu\nu} P_\mu^\dagger, \quad (1)$$

we want to engineer an effective dynamics generated by the block-diagonal Hamiltonian

$$H_Z = \sum_{\mu=1}^m P_\mu H P_\mu. \quad (2)$$

This evolution is a manifestation of a Quantum Zeno Dynamics (QZD), a generalisation of the quantum Zeno effect [1], consisting in the freezing of the state of a quantum system when it is subject to frequent measurements aimed at ascertaining if it is still in its initial state. In the case of non-selective measurements onto multi-dimensional subspaces $\mathcal{H}_\mu = P_\mu \mathcal{H}$ a non-trivial evolution can take place inside each subspace, generated by the Hamiltonian (2), with P_μ being the measurement projections. In this context the superselection sectors \mathcal{H}_μ are called quantum Zeno subspaces (QZSs) [2].

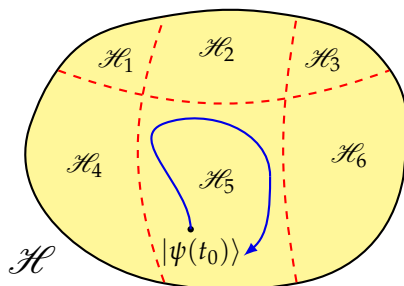


Figure 1. A pictorial representation of the partitioning of the Hilbert space \mathcal{H} into QZSs $\mathcal{H}_\mu = P_\mu \mathcal{H}$. If the system is in a given QZS at the initial time t_0 , it will evolve coherently in this subspace and will never make a transition to the other QZSs.

2. Strong Continuous Coupling

The first protocol consists in adding to the Hamiltonian H a strong coupling to a control potential V , so that the dynamics is generated by a total Hamiltonian $H_K = H + KV$, where $K > 0$ is the coupling strength. As K grows to infinity, the evolution generated by H_K is equivalent to a QZD, with the QZSs determined by the eigenprojections of the control potential V . Such result is expressed formally in Theorem 1, where we also bound the error between the actual evolution of the system and the controlled evolution when K is large but finite.

Theorem 1. Let H and V be Hermitian operators acting on a finite dimensional space \mathcal{H} , with V having the spectral decomposition

$$V = \sum_{\mu=1}^m \lambda_\mu P_\mu. \tag{3}$$

Then, defining H_Z as in Equation (2), we have

$$e^{-it(H+KV)} = e^{-itKV} e^{-itH_Z} + \mathcal{O}\left(\frac{1}{K}\right), \tag{4}$$

as $K \rightarrow \infty$. (Here and in the following the notation $\mathcal{O}(x)$ will stand for an operator $A(x)$ depending on the real parameter x such that $\|A(x)\| \leq C|x|$ for x sufficiently small and nonvanishing, and for some positive constant C).

The proof of the theorem makes use of an adiabatic theorem [3–5].

3. Pulsed Decoupling

The second protocol consists in the application of periodic pulses to the system, implemented by an instantaneous unitary transformation U_{kick} applied to the evolving state at time intervals t/n , as shown in Figure 2a. The idea at the basis of this procedure—and of the proof of Theorem 2—can be understood by looking at each step as an effective “rotation” of the Hamiltonian (see Figure 2b), so that the global effect over the whole time interval $(0, t)$ is to average out of the Hamiltonian the off-diagonal part with respect to the eigenprojections of the unitary kick [3,6]. Such result is expressed formally in Theorem 2.

Theorem 2. Let H be a Hermitian operator on a finite dimensional Hilbert space \mathcal{H} , and let U_{kick} be a unitary operator with the spectral decomposition

$$U_{\text{kick}} = \sum_{\mu=1}^m e^{-i\lambda_\mu} P_\mu. \tag{5}$$

Then, by defining H_Z as in Equation (2), we have

$$\left(U_{\text{kick}} e^{-i\frac{t}{n}H} \right)^n = U_{\text{kick}}^n e^{-itH_Z} + \mathcal{O}\left(\frac{1}{n}\right), \tag{6}$$

as $n \rightarrow \infty$.

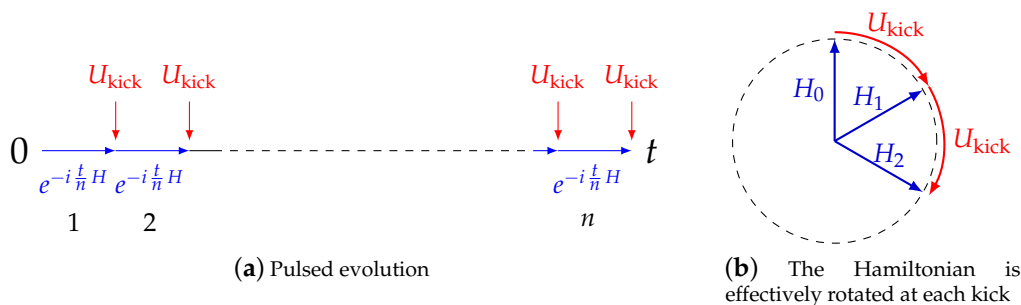


Figure 2. (a) Alternating free evolutions of duration t/n with instantaneous unitary kicks U_{kick} (b) is equivalent to a sequence of infinitesimal evolutions of duration t/n generated by Hamiltonians $H_\ell = U_{\text{kick}}^\dagger H U_{\text{kick}}$ rotated at each step by the unitary kick.

4. Example: Four-Level System

As a particular example, consider a 4-level system, where $\mathcal{H} = \mathbb{C}^4$, and a Hamiltonian H inducing Rabi transitions between adjacent levels (this scheme is very similar to that implemented in [7]):

$$H = \sum_{k=1}^4 \Omega_k (|k\rangle\langle k+1| + |k+1\rangle\langle k|). \tag{7}$$

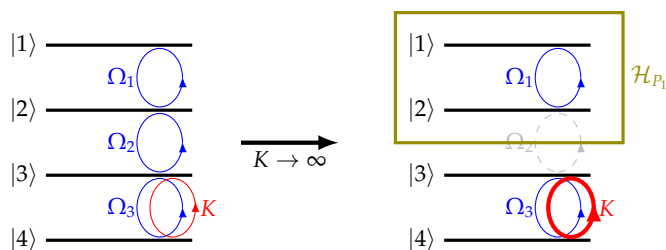


Figure 3. Effect of the strong coupling between states $|3\rangle$ and $|4\rangle$. The other two QZSs have not been highlighted in the figure since they are made of linear combinations of states $|3\rangle$ and $|4\rangle$.

Such Hamiltonian will determine a time oscillation of the populations (see Figure 4a) $P_k(t) = |\langle k|e^{-itH}|1\rangle|^2$. Using Theorem 1, we can show now that it is possible to decouple levels $|1\rangle$ and $|2\rangle$ from $|3\rangle$ and $|4\rangle$ with a strong coupling between $|3\rangle$ and $|4\rangle$:

$$KV = K(|3\rangle\langle 4| + |4\rangle\langle 3|). \tag{8}$$

The eigenprojections of this potential are $P_0 = |1\rangle\langle 1| + |2\rangle\langle 2|$ and $P_\pm = \frac{1}{2}(|3\rangle \pm |4\rangle)(\langle 3| \pm \langle 4|)$, so that the Zeno Hamiltonian, $H_Z = P_0 H P_0 + P_+ H P_+ + P_- H P_-$, is block-diagonal with respect to the QZSs

$$\mathcal{H}_1 = \text{span}\{|1\rangle, |2\rangle\}, \quad \mathcal{H}_+ = \text{span}\{|3\rangle + |4\rangle\}, \quad \mathcal{H}_- = \text{span}\{|3\rangle - |4\rangle\}. \tag{9}$$

The situation is pictorially represented in Figure 3. Figure 4b shows the behaviour of occupation probabilities $P_k(t)$ in the strong coupling regime: we can see oscillations between states $|1\rangle$ and $|2\rangle$

which belong to the same QZS, but the probability of a transition towards the states $|3\rangle$ and $|4\rangle$ vanishes since they do not belong to the initial QZS. The same result can be obtained by using instead the protocol considered in Theorem 2, with e.g. the unitary kick

$$U_{\text{kick}} = e^{-i\lambda(|3\rangle\langle 4| + |4\rangle\langle 3|)}. \tag{10}$$

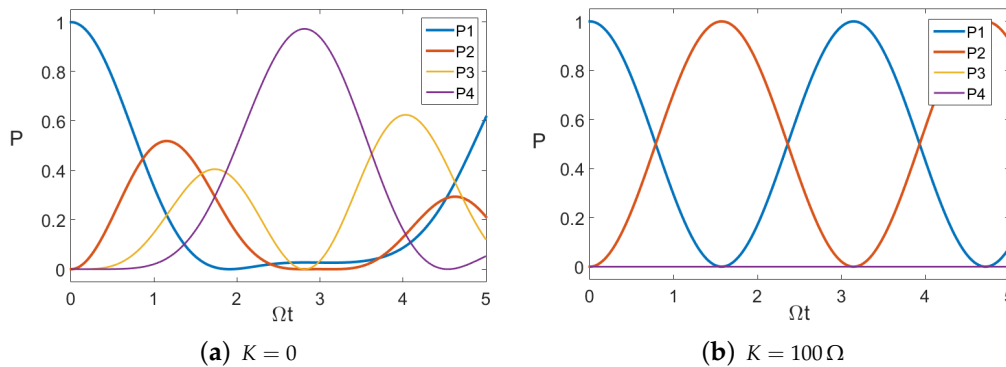


Figure 4. Populations P_k with $\Omega_1 = \Omega_2 = \Omega_3 \equiv \Omega$ without control potential (a) and with the control potential turned on with $K = 100\Omega$ (b).

In this example we have considered a particular Hamiltonian H generating the evolution of the system to be controlled. Note however that there are no assumptions on the structure of the Hamiltonian in our theorems, which are therefore valid in completely general situations, as long as we consider finite dimensional quantum systems.

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