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Concerning Two Classes of Non-Diophantine Arithmetics †

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Abstract: We present two classes of abstract prearithmetics, $\{\mathbf{A}_M\}_{M\geq 1}$ and $\{\mathbf{B}_M\}_{M>0}$. The first one is weakly projective with respect to the nonnegative real Diophantine arithmetic $\mathbf{R}_+ = (\mathbb{R}_+, +, \times, \leq_{\mathbb{R}_+})$, and the second one is projective with respect to the extended real Diophantine arithmetic $\overline{\mathbf{R}} = (\overline{\mathbb{R}}, +, \times, \leq_{\overline{\mathbb{R}}})$. In addition, we have that every $\mathbf{A}_M$ and every $\mathbf{B}_M$ is a complete totally ordered semiring. We show that the projection of any series of elements of $\mathbb{R}_+$ converges in $\mathbf{A}_M$, for any $M \geq 1$, and that the projection of any non-indeterminate series of elements of $\mathbb{R}$ converges in $\mathbf{B}_M$, for all $M > 0$. We also prove that working in $\mathbf{A}_M$, for any $M \geq 1$, and in $\mathbf{B}_M$, for all $M > 0$, allows to overcome a version of the paradox of the heap.

Keywords: non-Diophantine arithmetics; convergence of series; paradox of the heap

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1. Introduction

Although the conventional arithmetic—which we call Diophantine from Diophantus, the Greek mathematician who first approached this branch of mathematics—is almost as old as mathematics itself, it sometimes fails to correctly describe natural phenomena. For example, in [1], Helmoltz points out that adding one raindrop to another one leaves us with one raindrop, while in [2], Kline notices that Diophantine arithmetic fails to correctly describe the result of combining gases or liquids by volume. Indeed, one quarter of alcohol and one quarter of water only yield about 1.8 quarters of vodka. To overcome this issue, scholars started developing inconsistent arithmetics, that is, arithmetics for which one or more Peano axioms were at the same time true and false. The most striking one was ultraintuitionism, developed by Yesenin-Volpin in [3], that asserted that only a finite quantity of natural numbers exists. Other authors suggested that numbers are finite (see, e.g., [4,5]), while different scholars adopted a more moderate approach. The inconsistency of these alternative arithmetics lies in the fact that they are all grounded in the ordinary Diophantine arithmetic. The first consistent alternative to Diophantine arithmetic was proposed by Burgin in [6], and the name non-Diophantine seemed perfectly suited for this arithmetic. Non-Diophantine arithmetics for natural and whole numbers have been studied by Burgin in [6–9], while those for real and complex numbers have been studied by Czachor in [10,11]. A complete account on non-Diophantine arithmetics can be found in the recent book by Burgin and Czachor [12].

There are two types of non-Diophantine arithmetics: dual and projective. In this paper, we work with the latter. We start by defining an abstract prearithmetic $\mathbf{A} := (A, +_A, \times_A, \leq_A)$, where $A \subset \mathbb{R}$ is the *carrier* of $\mathbf{A}$ (that is, the set of the elements of $\mathbf{A}$), $\leq_A$ is a partial order on A, and $+_A$ and $\times_A$ are two binary operations defined on the elements of A. We conventionally call them addition and multiplication, but that can be any generic operation. Naturally, the conventional Diophantine arithmetic $\mathbf{R} = (\mathbb{R}, +, \times, \leq_{\mathbb{R}})$ of real numbers is an abstract prearithmetic; we denote by $+$ the usual addition, $\times$ the usual multiplication, and $\leq_{\mathbb{R}}$ the usual partial order on $\mathbb{R}$.

Abstract prearithmetic $\mathbf{A}$ is called *weakly projective* with respect to a second abstract prearithmetic $\mathbf{B} = (B, +_B, \times_B, \leq_B)$ if there exist two functions $g : A \to B$ and $h : B \to A$ such that, for all, $a_1, a_2 \in A$, $a_1 +_A a_2 = h(g(a_1) +_B g(a_2))$ and $a_1 \times_A a_2 = h(g(a_1) \times_B g(a_2))$. Function g is called the *projector* and function h is called the *coprojector* for the pair $(\mathbf{A}, \mathbf{B})$.

The *weak projection* of the sum $b_1 +_B b_2$ of two elements of B onto A is defined as $h(b_1 +_B b_2)$, while the weak projection of the product $b_1 \times_B b_2$ of two elements of B onto A is defined as $h(b_1 \times_B b_2)$.

Abstract prearithmetic $\mathbf{A}$ is called *projective* with respect to abstract prearithmetic $\mathbf{B}$ if it is weakly projective with respect to $\mathbf{B}$, with projector f^{-1} and coprojector f. We call f, that has to be bijective, the *generator* of projector and coprojector.

Weakly projective prearithmetics depend on two functional parameters, g and h—one, f, if they are projective—and recover the conventional Diophantine arithmetic when these functions are the identity. To this extent, we can consider non-Diophantine arithmetics as a generalization of the Diophantine one.

In this work, we consider two classes of abstract prearithmetics, $\{\mathbf{A}_M\}_{M \geq 1}$ and $\{\mathbf{B}_M\}_{M>0}$. They are useful to describe some natural and tech phenomena for which the conventional Diophantine arithmetic fails, and their elements allow us to overcome the version of the paradox of the heap (or sorites paradox) stated in Section 2 in [9]. The setting of this variant of the sorites paradox is adding one grain of sand to a heap of sand, and the question is, once a grain is added, whether the heap is still a heap. The heart of sorites paradox is the issue of vagueness, in this case, vagueness of the word "heap".

We show that every element $\mathbf{A}_M$ of the first class is a complete totally ordered semiring, and it is weakly projective with respect to $\mathbf{R}_+$. Furthermore, we prove that the weak projection of any series $\sum_n a_n$ of elements of $\mathbb{R}_+ := [0, \infty)$ is convergent in each $\mathbf{A}_M$.

The elements of $\{\mathbf{B}_M\}_{M>0}$ can be used to solve the paradox of the heap too. They are complete totally ordered semirings and are projective with respect to the extended real Diophantine arithmetic $\overline{\mathbf{R}} = (\overline{\mathbb{R}}, +, \times, \leq_{\overline{\mathbb{R}}})$. The projection of any non-indeterminate series $\sum_n a_n$ of terms in $\mathbb{R}$ is convergent in $\mathbf{B}_{M'}$, for all $M' > 0$.

The proofs of our claims can be found in [13], where a third interesting class of abstract prearithmetics is also presented.

The paper is divided as follows. Section 2 presents the two classes, and Section 3 is a summary of our results. Section 4 is a discussion.

2. Material and Methods

In this section, we present classes $\{\mathbf{A}_M\}_{M \geq 1}$ and $\{\mathbf{B}_M\}_{M>0}$ and their properties. We summarize them in Section 3.

2.1. Class $\{\mathbf{A}_M\}_{M \geq 1}$

For any real $M \geq 1$, we define the corresponding non-Diophantine prearithmetic as $\mathbf{A}_M = (A_M, \oplus, \otimes, \leq_{A_M})$ having the following properties: (1) the order relation $\leq_{A_M}$ is the restriction to A_M of the usual order on the reals; (2) $A_M \subset \mathbb{R}_+$ has maximal element M and minimal element 0 with respect to $\leq_{A_M}$, and is such that $0 \in A_M$, which ensures having a multiplicative absorbing and additive neutral element in our set; $1 \in A_M$, which ensures having a multiplicative neutral element in our set; there is at least an element $x \in (0, 1)$ such that $x \in A_M$; and (3) it is closed under the binary operations $a \oplus b := \min(M, a + b)$ and $a \otimes b := \min(M, a \times b)$, where $+$ and $\times$ denote the usual sum and product in $\mathbb{R}$, respectively.

Proposition 1. *Addition $\oplus$ is associative.*

Since addition $\oplus$ is associative, we have that, for every $k \in \mathbb{N}$, $\bigoplus_{n=1}^{k} x_n$ is equal to the minimum between M and $\sum_{n=1}^{n} x_n$. By imposing on M the relative topology derived from $\mathbb{R}$, we can define the limit as $k \to \infty$ of $\bigoplus_{n=1}^{k} x_n$ as $\min(M, \sum_{n=1}^{\infty} x_n)$.

Proposition 2. $A_M = [0, M]$.

This result implies that $\mathbf{A}_M$ is a complete totally ordered semiring.

Remark 1. *Notice that $\mathbf{A}_M$ cannot be a ring because for any $a \in A_M \backslash \{0\}$, it lacks the additive inverse $-a$; this is because we defined A_M to be a subset of $\mathbb{R}_+$. Notice also that in this abstract, prearithmetic M is an idempotent element, that is, $M \oplus M \oplus \cdots \oplus M = M$.*

2.1.1. Overcoming the Paradox of the Heap

The paradox of the heap is a paradox that arises from vague predicates. A formulation of such paradox (also called the sorites paradox, from the Greek word $\sigma\omega\rho o\varsigma$, "heap"), given in Section 2 in [9], is the following: (1) one million grains of sand make a heap; (2) if one grain of sand is added to this heap, the heap stays the same; (3) however, when we add 1 to any natural number, we always obtain a new number.

This formulation of the paradox of the heap is proposed by Burgin to inspect whether adding $1 to the assets of a millionaire makes them "more of a millionaire" or leaves their fortune unchanged. We use the class $\{\mathbf{A}_M\}_{M \geq 1}$ to address paradox of the heap. Indeed, it is enough to take the element of the class for which $M = 1{,}000{,}000$, so that when we perform the addition $M \oplus 1$, we obtain M. This conveys the idea that adding a grain of sand to the heap leaves us with a heap.

The class we introduced can also be used to describe phenomena such as the one noted by Helmholtz in [1], adding one raindrop to another one gives one raindrop, or the one pointed out by Lebesgue (cf. [2]), putting a lion and a rabbit in a cage, one will not find two animals in the cage later on. In both these cases, it suffices to consider the element $\mathbf{A}_1$ of the class for which $M = 1$, so that $1 \oplus 1 = 1$.

Class $\{\mathbf{A}_M\}_{M \geq 1}$ allows us also to avoid introducing inconsistent Diophantine arithmetics, that is, arithmetics for which one or more Peano axioms were at the same time true and false. For example, in [4], Rosinger points out that electronic digital computers, when operating on the integers, act according to the usual Peano axioms for $\mathbb{N}$ plus an extra ad hoc axiom, called the machine infinity axiom. The machine infinity axiom states that there exists $\breve{M} \in \mathbb{N}$ far greater than 1 such that $\breve{M} + 1 = \breve{M}$ (for example, $\breve{M} = 2^{31} - 1$ is the maximum positive value for a 32-bit signed binary integer in computing). Clearly, Peano axioms and the machine infinity axiom together give rise to an inconsistency, which can be easily avoided by working with $\mathbf{A}_{\breve{M}}$.

In [5], Van Bendegem developed an inconsistent axiomatic arithmetic similar to the "machine" one described in [4]. He changed the Peano axioms so that a number that is the successor of itself exists. In particular, the fifth Peano axiom states that if $x = y$, then x and y are the same number. In the system of Van Bendegem, starting from some number n, all its successors will be equal to n. Then, the statement $n = n + 1$ is considered as both true and false at the same time, giving rise to an inconsistency. It is immediate to see how this inconsistency can be overcome by working with any abstract prearithmetic $\mathbf{A}_M$ in our class.

2.1.2. $\mathbf{A}_M$ Is Weakly Projective with Respect to $\mathbf{R}_+$

Pick any $\mathbf{A}_{M'} \in \{\mathbf{A}_M\}$, and consider $\mathbf{R}_+ = (\mathbb{R}_+, +, \times, \leq_{\mathbb{R}_+})$. Consider then the functions $g : A_M \to \mathbb{R}_+$, $a \mapsto g(a) \equiv a$, so that g is the identity function $\mathrm{Id}|_{A_M}$, and $h : \mathbb{R}_+ \to A_M$, $a \mapsto h(a) := \min(M, a)$. Now, if we compute $h(g(a) + g(b))$, for all $a, b \in A_M$, we have that $h(g(a) + g(b)) = h(a + b) = \min(M, a + b) = a \oplus b$. Similarly, we show that $h(g(a) \times g(b)) = a \otimes b$. Hence, addition and multiplication in $\mathbb{R}_+$ are weakly projected onto addition and multiplication in $A_{M'}$, respectively. So, we can conclude that $\mathbf{A}_{M'}$ is weakly projective with respect to $\mathbf{R}_+$, for all $M' \geq 1$.

2.1.3. Weak Projection of Series in $\mathbf{R}_+$ onto $\mathbf{A}_M$

We first present the following result.

Proposition 3. *Any series $\bigoplus_n a_n$ of elements of A_M is always convergent.*

Then, we claim that the weak projection of any series of elements of $\mathbb{R}_+$ converges in A_M, for all $M \geq 1$. This is an exciting result because it allows the scholar that needs a particular series to converge in their analysis to reach that result by performing a weak projection of the series onto $\mathbf{A}_M$ and then continue the analysis in $\mathbf{A}_M$.

Consider any series $\sum_n a_n$ of elements of $\mathbb{R}_+$. It can be convergent or divergent to $+\infty$. It cannot be divergent to $-\infty$ because we are summing positive elements only, and it cannot be neither convergent nor divergent (i.e., it cannot be indeterminate), because the elements of the series cannot alternate their sign.

Proposition 4. *The weak projection* $h(\sum_{n=1}^{\infty} a_n)$ *of* $\sum_{n=1}^{\infty} a_n := \lim_{k\to\infty} \sum_{n=1}^{k} a_n$ *is convergent.*

The following lemma comes immediately from Proposition 1.

Lemma 1. *For any series* $\sum_{n=1}^{\infty} a_n$ *of elements of* $\mathbb{R}_+$*, for any* $k \in \mathbb{N}$, $h(\sum_{n=1}^{\infty} a_n) = h\left(\sum_{n=1}^{k} a_n\right) \oplus h\left(\sum_{n=k+1}^{\infty} a_n\right)$.

2.2. Class $\{\mathbf{B}_M\}_{M>0}$

In this section, we present a class of abstract prearithmetics $\{\mathbf{B}_M\}_{M>0}$ where every element is a complete totally ordered semiring, and such that the projection of a convergent or divergent series (to $+\infty$ or $-\infty$) of elements of $\mathbb{R}$ converges. Its elements can be used to solve the paradox of the heap. For every real $M > 0$, we define the corresponding non-Diophantine prearithmetic as $\mathbf{B}_M = (B_M, \dot{+}, \dot{\times}, \leq_{B_M})$ having the following properties: (1) the order relation $\leq_{B_M}$ is the restriction to B_M of the usual order on the reals; (2) $B_M = [0, M]$; (3) let $\overline{\mathbb{R}} := [-\infty, \infty]$ and consider the function

$$f : \overline{\mathbb{R}} \to B_M, \quad x \mapsto f(x) := \begin{cases} M\left(\frac{\arctan(x)}{\pi} + \frac{1}{2}\right) & \text{if } x \in \mathbb{R} \\ M & \text{if } x = \infty \\ 0 & \text{if } x = -\infty \end{cases}$$

and its inverse

$$f^{-1} : B_M \to \overline{\mathbb{R}}, \quad x \mapsto f^{-1}(x) := \begin{cases} \tan\left(\frac{\pi}{M}\left(x - \frac{M}{2}\right)\right) & \text{if } x \in (0, M) \\ \infty & \text{if } x = M \\ -\infty & \text{if } x = 0 \end{cases}.$$

Then, $\mathbf{B}_M$ is closed under the binary operations $a \dot{+} b := f\left(f^{-1}(a) + f^{-1}(b)\right)$, where $+$ denotes the sum in $\overline{\mathbb{R}}$, and $a \dot{\times} b := f\left(f^{-1}(a) \times f^{-1}(b)\right)$, where $\times$ denotes the product in $\overline{\mathbb{R}}$.

Notice that we do not “force” 0 and M to be the boundary elements of B_M; they come naturally from the way addition $\dot{+}$ and multiplication $\dot{\times}$ are defined. In addition, we have that by construction $\mathbf{B}_M$ is projective with respect to $\overline{\mathbf{R}} = (\overline{\mathbb{R}}, +, \times, \leq_{\overline{\mathbb{R}}})$, and that its generator induces an homeomorphism between $\overline{\mathbb{R}}$ and $[0, M]$. This tells us immediately that $(B_M, \dot{+}, \dot{\times}, \leq_{B_M})$ is a complete totally ordered semiring, so addition $\dot{+}$ and multiplication $\dot{\times}$ are associative. The fact that $+\infty$ and $-\infty$ in $\overline{\mathbb{R}}$ correspond to M and 0, respectively, tells us that the projection $f(\sum_n a_n)$ of any series $\sum_n a_n$ of elements of $\mathbb{R}$ converges in $\mathbf{B}_M$, as long as $\sum_n a_n$ is not indeterminate. The elements of $\mathbf{B}_M$ can be used to solve the paradox of the heap; to see this, notice that $M \dot{+} a = M$, for all $a \in B_M$ and all $M > 0$. Moreover, M is an idempotent element of B_M, for all $M > 0$: $M \dot{+} M \dot{+} \cdots \dot{+} M = M$.

3. Results

The results we find in this work are summarized in Table 1.

Table 1. A summary of the properties of the classes we introduce in this paper. All of them can be used to describe those (natural and tech) phenomena for which Diophantine arithmetics fail.

$\mathbf{A}_M$	$\mathbf{B}_M$
$A_M = [0, M]$	$B_M = [0, M]$
Solves paradox of the heap	Solves paradox of the heap
Weakly projective with regard to $\mathbf{R}_+$	Projective with regard to $\overline{\mathbf{R}}$
Complete totally ordered semiring	Complete totally ordered semiring
Weak projection of a series of elements of $\mathbb{R}_+$ is convergent	Projection of a series of elements of $\mathbb{R}$ is absolutely convergent
–	Projection of a non-indeterminate series of elements of $\mathbb{R}$ is convergent

4. Discussion

In this work, we presented two classes $\{\mathbf{A}_M\}_{M\geq 1}$ and $\{\mathbf{B}_M\}_{M>0}$ of abstract prearithmetics that allow us to overcome the paradox of the heap without resorting to inconsistent Diophantine arithmetics. An element of such classes can also be used to obtain results such as $1 \oplus 1 = 1$, which reflect what occurs in many real-world applications.

We showed that, for all $M \geq 1$, $\mathbf{A}_M$ is weakly projective with respect to $\mathbf{R}_+$ and, for all $M > 0$, $\mathbf{B}_M$ is projective with respect to $\overline{\mathbf{R}}$. In addition, the weak projection of any series of elements of $\mathbb{R}_+$ is convergent in $\mathbf{A}_M$, while the projection of any non-indeterminate series of elements of $\mathbb{R}$ is convergent in $\mathbf{B}_M$.

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