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Finite-Time Adaptive Consensus Tracking Control Based on Barrier Function and Cascaded High-Gain Observer

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Abstract: This paper studies the consensus tracking control for a class of uncertain high-order nonlinear multi-agent systems under an undirected leader-following architecture. A novel distributed finite-time adaptive control framework is proposed based on the barrier function. The distributed cascaded high-gain observers are introduced to solve the problem of robust consensus tracking with unmeasured intermediate states in multi-agent systems based on the proposed control framework. The proposed control schemes guarantee the finite-time consensus of multi-agent systems, which is proven by the finite-time Lyapunov stability and singular perturbation theory. In conclusion, numerical simulations verify the proposed control protocols' effectiveness, and their performance advantages are shown by comparing them with another existing method.

Keywords: robust consensus tracking; disturbance observer; state estimation; barrier function; singular perturbation

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1. Introduction

Cooperative control for multi-agent systems (MASs) has attracted attention from researchers worldwide owing to its broad application in various fields, such as robots [1,2], unmanned aerial vehicles [3,4], unmanned underwater vehicles [5,6], and smart grids [7,8], just to name a few. Many efforts have been devoted to the fundamental consensus problem in MAS cooperation control [9–11]. The consensus control problem for Markov jumping MASs under undirected graphs is also widely studied [12,13]. It should be noted that synchronous control and formation control, which are widely used in practical systems to greatly improve the capacity and efficiency of the system, such as synchronous control of multiple electrohydraulic actuators [14] and formation control of multiple UAVs [15], can be translated into the consensus tracking control problem for MASs. This problem generally falls into two categories: leader–follower consensus (also called consensus tracking) and leaderless consensus. The work [16] proposes a distributed containment control strategy for MASs to cope with composite attacks, which is based on the leader–follower architecture and combined with the Luenberger observer. Distributed leaderless model-independent consensus control of multiple Euler–Lagrange systems was studied in [17]. The work [18] developed a leaderless consensus tracking control strategy for multiple quadrotor systems with bounded disturbances in a directed topology.

Finite-time control strategies typically have higher control accuracy, better robustness, and faster convergence to the equilibrium than asymptotic control strategies [19]. Moreover, in many practical applications, the finite-time stabilization makes more sense than asymptotic stabilization, such as in the orbiting state of satellite systems and other applications that focus on the state behavior over a finite period [20], and the trajectory control of spacecraft and other such applications that need to maintain the system's state within a specific time frame without exceeding a predetermined bound [21]. Therefore, the finite-time consensus tracking problem for MASs has attracted much attention [22–24]. For instance, a nonlinear finite-time consensus control protocol based on continuous state feedback was developed for single-integrator MASs with directional link failure in [25]. For second-order MASs, the finite-time consensus tracking control protocols based on the terminal sliding mode method were presented in [1,26,27]. In the work [28], the finite-time nonlinear consensus protocols were constructed based on a novel sliding surface design for each double-integrator MAS. A consensus tracking control based on a time-varying function approach for the second-order MASs under signed directed topology structures was proposed in [29]. The output consensus control of high-order MASs with external disturbances under directed networks was investigated in [30]. Combined with the recursive method, a finite-time consensus control strategy was designed for a class of high-order MASs in [31].

To cope with the unknown disturbances and relax the assumption of disturbances, two types of barrier functions (a positive definite barrier function and a semi-positive definite barrier function) were given for the first time. An adaptive control strategy was designed for the first-order system to avoid excessive control gains in [32]. Based on this, a series of sliding mode controllers based on barrier functions were developed [33–35]. In the work [36], a distributed control scheme based on the barrier function was devised for second-order systems. Nevertheless, the estimation and compensation of disturbances were not considered.

Although effective, the above works require the input of the full states of MASs. In case of limited states, where full states are unmeasurable or unavailable, several methods have been developed to solve the consensus tracking problem for MASs with limited state measurements. The work [37] proposed a finite-time consensus tracking control strategy on account of output feedback for second-order MASs. However, the reference signals need to meet specific dynamics. An output feedback-based consensus tracking control protocol for a class of second-order MASs with an undirected graph was designed without consideration of the uncertainties and disturbances in [38]. In the work [39], a distributed adaptive control scheme employing the extended state observer was devised to achieve consensus tracking for second-order underactuated MASs under an undirected graph. Unfortunately, the uncertainties and disturbances of MASs were not considered in either study. The uncertain second-order MASs without velocity measurements were considered in [40], where the corresponding distributed robust controller based on the uncertainty and disturbance estimator was designed to realize the consensus tracking (synchronization) of MASs. However, asymptotic convergence can only be guaranteed when the derivatives of the disturbances satisfy specific conditions. In the work [41], the proposed finite-time consensus control protocols also required additional assumptions about the derivatives of disturbances. It should be noted that the state estimation techniques involved in the above methods all suffer from the contradiction between steady-state accuracy and transient performance. To alleviate this problem, Khalil designed a cascaded high-gain observer that uses saturation constraints, making it possible to avoid transient peak effects while ensuring estimation accuracy [42].

This paper is motivated by the need to address the challenges regarding the uncertainty and disturbance of MASs. The finite-time consensus control design problem is studied for uncertain high-order MASs with either full-state measurements or unmeasurable intermediate states under an undirected communication graph. Compared with the existing literature, the original contributions of this paper can be summarized as follows:

(1) A distributed finite-time adaptive disturbance observer is designed based on the barrier function. Then, the neighbor-based finite-time adaptive consensus tracking control protocols are developed. The proposed approach does not require additional assumptions on the existence and boundness of the derivatives of the lumped disturbances (which include model uncertainties and external disturbances).

(2) The cascade high-gain observer is first introduced into the proposed finite-time consensus tracking control framework based on the barrier function while retaining its original performance advantages. For the case of a MAS without intermediate state measurements, the finite-time bounded consensus between the leader and followers is guaranteed. Moreover, the ultimate error bounds can be monotonously adjusted by the designed perturbation parameter.

(3) The proposed strategy is suitable for heterogeneous MASs to account for the different dynamics of each agent. Moreover, the designed control scheme can be applied to solve the robust cooperative control problem of multiple UAVs.

The remainder of this paper is organized as follows. Section 2 gives the problem statement. Finite-time adaptive consensus tracking control protocols are shown in Section 3. Section 4 analyzes stability of the closed-loop system. Two simulation cases are provided in Section 5 to verify the theoretical findings. Finally, Section 6 concludes the paper.

2. Problem Statement

2.1. Notations and Preliminaries

The information interaction topology among n agents is described by a graph, which can be denoted as $\mathcal{G}(\mathcal{V},\mathcal{E})$. Here, $\mathcal{V} = \{v_1, v_2, \ldots, v_{N-1}, v_N\}$ represents the set of N nodes, the node v_i ($i = 1, 2, \ldots, N-1, N$) denotes the i-th agent, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the sets of edges. In this paper, the edge $(v_i, v_j) \in \mathcal{E}$ means the node v_i is a neighbor of the node v_j, and they can receive information from each other. Hence, the graph $\mathcal{G}$ is undirected; that is, if $(v_i, v_j) \in \mathcal{E}$, $(v_j, v_i) \in \mathcal{E}$ is also established. The set of all the neighbors of the node v_i is a subset of $\mathcal{V}$ and can be denoted as $N_i = \{j | (v_i, v_j) \in \mathcal{E}\}$. If there is a sequence of ordered edges $(v_i, v_s), (v_s, v_t), \cdots (v_p, v_q), (v_q, v_j)$, the set composed of the above edges can be regarded as a path from the node v_i to node v_j. The graph $\mathcal{G}$ has a spanning tree if all other nodes can be reached from a certain node called the root node. Moreover, if there is a path between any two different nodes, the graph $\mathcal{G}$ is connected. The adjacency matrix of the graph $\mathcal{G}$ is $A = [a_{ij}] \in \mathcal{R}^{N\times N}$, where $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The in-degree matrix $H = \mathrm{diag}\{h_i\} \in \mathcal{R}^{N\times N}$, where $h_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix of a graph $\mathcal{G}$ is $L = [l_{ij}] \in \mathcal{R}^{N\times N}$, which satisfies $L = H - A$. Suppose there is a virtual leader indexed as a node 0. The adjacency matrix of the leader is denoted as $B = \mathrm{diag}\{b_i\} \in \mathcal{R}^{N\times N}$, where $b_i > 0$ if the i-th agent can directly receive the leader's information; otherwise, $b_i = 0$. $\mathrm{sgn}(\cdot)$ denotes the standard symbolic function, $\mathrm{sat}(\cdot)$ denotes the standard unity saturation function, and $\mathrm{sat}(x) = \min\{|x|, 1\} \cdot \mathrm{sgn}(x)$ holds. The symbol $\otimes$ denotes the Kronecker product, $x^{(n)}$ the n-th derivative of x, $\|\cdot\|$ the Euclidean norm, $\|\cdot\|_\infty$ the infinite norm, and $|\cdot|$ the absolute value.

2.2. Problem Formulation

The dynamics of the i-th ($i = 1, 2, \cdots, N$) agent is described as follows:

$$\begin{cases} \dot{x}_i = \bar{A}x_i + \bar{B}[f_i(x_i) + g_i(x_i)u_i + d_i(x_i,t)], \\ y_i = \bar{C}x_i, \end{cases} \tag{1}$$

where $x_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]^T \in \mathcal{R}^{n\times 1}$ is the system state, $y_i \in \mathcal{R}$ is the measurable system output, $f_i(x_i) \in \mathcal{R}$ and $g_i(x_i) \in \mathcal{R}\backslash\{0\}$ are the continuously differentiable bounded nonlinear functions of known structure, $u_i \in \mathcal{R}$ is the control input, $d_i(x_i,t) \in \mathcal{R}$ is the uncertainty and disturbance of the system, and matrices ($\bar{A}$, $\bar{B}$ and $\bar{C}$) are given as follows:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{R}^{n\times n}, \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathcal{R}^{n\times 1}, \bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{R}^{1\times n}. \tag{2}$$

Remark 1. *By selecting different dimensions of the state/input matrix in Equation (1), real physical objects, such as a quadrotor, robot, or manipulator, can be described. For example, the position model of a quadrotor can be linearized [43] in a second-order system, where* $\bar{A} \in \mathcal{R}^{2\times 2}$ *and* $\bar{B} \in \mathcal{R}^{2\times 1}$.

In this paper, some assumptions are made as follows.

Assumption 1. *The uncertainty and disturbance* $d_i(x_i, t)$ *is bounded, which satisfies*

$$\|d_i(x_i,t)\|_\infty \leq d_{\max}, \tag{3}$$

where $d_{\max}$ *is an unknown non-negative constant.*

Assumption 2. *The virtual leader is a unique system that acts as a reference command generator giving commands to partial followers of the networked group.*

Assumption 3. *The static undirected digraph* $\mathcal{G}$ *is connected and contains a spanning tree with the virtual leader as the root node.*

Assumption 4. *The reference signal* $y_0(t) \in \mathcal{R}$ *is continuously differentiable up to its i-th derivative, and there exists* $\psi_i > 0$ *for all* $t > 0$ *such that*

$$\left|\frac{d^i y_0(t)}{dt^i}\right| = \left|y_0^{(i)}(t)\right| \leq \psi_i, \text{ for } i = 0,1,2,\ldots,n. \tag{4}$$

Remark 2. *Assumption 4 is proposed to ensure that the desired trajectories are smooth enough to avoid actuator saturation due to sudden jumps caused by discontinuous control inputs.*

The following definitions and lemmas are also used to solve the consensus control problem.

Definition 1 ([32])**.** *Define the barrier function as an even continuous function* $K_b(x)$*:* $x \in (-\omega, \omega) \to K_p(x) \in [p, \infty)$ *strictly increasing on* $[0, \omega)$*, where* $\omega > 0$ *is assumed given and fixed,*

- $\lim_{|x|\to\omega} K_p(x) = +\infty$;
- *the minimum value of* $K_p(x)$ *is* $K_p(0) = p \geq 0$.

Define the positive definite barrier function as follows

$$K_p(x) = \frac{\omega p}{\omega - |x|}, \; x \in (-\omega, \omega), \tag{5}$$

where p *is a positive constant.*

Remark 3. *By definition, the value of the barrier function tends toward infinity as the function variable approaches the bound, which means that when the barrier function remains bounded, the variable* x *will always remain within the bound* $(-\omega, \omega)$*, and the value of the barrier function will adjust dynamically with the change of the variable* x *(i.e., parameter adaptive adjustment), and its positive definiteness also provides support for the development of Lyapunov-based control methods.*

Lemma 1 ([44])**.** *Assuming that at least one follower in the connected undirected graph* $\mathcal{G}$ *can receive reference commands from the virtual leader, then* $F = L + B$ *is a positive definite matrix.*

Lemma 2 ([32])**.** *Consider the following system*

$$\dot{\sigma}(t) = \mu(t) + \delta(t), \tag{6}$$

where $\delta(t)$ *is bounded and satisfies* $|\delta(t)| \leq \delta_{\max}$*, the bound* $\delta_{\max} > 0$ *exists and can be unknown.*

If the control input of system (6) is

$$\mu(t) = -K(t, \sigma(t))\mathrm{sgn}(\sigma(t)), \tag{7}$$

and

$$K(t, \sigma(t)) = \begin{cases} K_a(t), \dot{K}_a(t) = \bar{K}|\sigma(t)|, \text{ for } 0 < t \leq \bar{t}, \\ K_b(\sigma(t)), \text{ for } t > \bar{t}, \end{cases} \tag{8}$$

where $K_a(0) > 0$, and $\bar{K}$ is a positive constant. Then, the variable $\sigma(t)$ of the system (6) will converge to the domain $|\sigma(t)| \leq \sigma_1$ in a finite time T_σ, where σ_1 satisfies

$$\sigma_1 = \begin{cases} \omega\left(1 - \frac{p}{\sigma_{\max}}\right), \text{ if } p < \delta_{\max}, \\ 0, \text{ if } p \geq \delta_{\max}. \end{cases} \tag{9}$$

Lemma 3 ([45])**.** *Consider the following high-order system*

$$\begin{cases} \dot{z}_1 = z_2, \\ \vdots \\ \dot{z}_{n-1} = z_n, \\ \dot{z}_n = u_z, \end{cases} \tag{10}$$

Suppose that α_i is chosen such that the polynomial $s_n + \alpha_n s_{n-1} + \cdots + \alpha_2 s_1 + \alpha_1$ is Hurwitz, and β_i is given by

$$\begin{cases} \beta_{i-1} = \frac{\beta_i \beta_{i+1}}{2\beta_{i+1} - \beta_i}, i = 1, 2, \ldots, n \\ \beta_n = \beta, \\ \beta_{n+1} = 1. \end{cases} \tag{11}$$

For $\forall \beta \in (1-\varepsilon, 1)$, there exists $\varepsilon \in (0, 1)$ such that the system (10) states converge to zero in finite time T_z under the action of the following control law

$$u_z = -\alpha_1 \mathrm{sgn}(z_1)|z_1|^{\beta_1} - \alpha_2 \mathrm{sgn}(z_2)|z_2|^{\beta_2} - \cdots - \alpha_n \mathrm{sgn}(z_n)|z_n|^{\beta_n}. \tag{12}$$

Definition 2 ([40])**.** *The MASs are considered to approach consensus tracking if for any $y_0(t)$,*

$$\lim_{t \to T_1} |y_i(t) - y_0(t)| = 0, \tag{13}$$

or

$$\lim_{t \to T_2} |y_i(t) - y_0(t)| \leq \varphi(\varepsilon), \tag{14}$$

where T_1 and T_2 are the corresponding settling time, $\varepsilon > 0$ is a sufficiently small design parameter to be determined, and $\varphi(\varepsilon)$ is a class $\mathcal{K}$ function with respect to ε.

Remark 4. *By the definition of finite-time stability in [46], consider a system: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t)), \boldsymbol{x}(0) = \boldsymbol{x_0}$, where $\boldsymbol{x} = [x_1, x_2, \ldots x_n]^T \in R^n$, $\boldsymbol{f}(\boldsymbol{x}(t)) : R^n \to R^n$, and $f(0) = 0$. Suppose that there exists a continuous positive definite function $V(\boldsymbol{x}) : R^n \to R$, such that $\dot{V}(\boldsymbol{x}) + cV^\beta(\boldsymbol{x}) \leq 0$ $(\boldsymbol{x} \in R^n \backslash \{\boldsymbol{0}\})$, for some real numbers $c > 0$ and $\beta \in (0, 1)$. Then, the origin is a globally finite-time stable equilibrium and the settling time is $T(\boldsymbol{x_0}) \leq \frac{1}{c(1-\beta)} V^{1-\beta}(\boldsymbol{x_0})$.*

This paper aims to design a distributed controller so that the outputs described by the system (1) approach consensus with that of the virtual leader in a finite time under the conditions of Assumptions 1–4.

3. Finite-Time Adaptive Consensus Tracking Control

In this section, the distributed robust adaptive controllers are designed to achieve the consensus tracking of MASs for the following two cases. One is that all states of the agent *i* in the system (1) are available, while the other is that intermediate states of the system (1) are partially or entirely unavailable (except the states $x_{i,1}$ and $x_{i,n}$).

The consensus errors of the agent i in MASs are defined as

$$e_i = \sum_{j \in N_i} a_{ij}(y_i - y_j) + b_i(y_i - y_0) \in \mathcal{R}. \tag{15}$$

Define

$$e = [e_1, e_2, \ldots, e_N] \in \mathcal{R}^N, \tag{16}$$

$$\tilde{y}_i = y_i - y_0, \tag{17}$$

$$\tilde{y} = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_N] \in \mathcal{R}^N. \tag{18}$$

3.1. Finite-Time Adaptive Consensus Tracking Control of All-State Measurable High-Order Multi-Agent Systems

The distributed control law u_i of the i-th agent in MASs with measurable states is designed as:

$$\begin{aligned} u_i = & \frac{\left[b_i y_0^{(n)} + \sum_{j \in N_i} a_{ij}\left(f_j(x_j) + g_j(x_j)u_j + \hat{d}_j\right)\right]}{\left(\sum_{j \in N_i} a_{ij} + b_i\right) g_i(x_i)} - \\ & \frac{\left(\sum_{j \in N_i} a_{ij} + b_i\right)\left(\hat{d}_i + f_i(x_i)\right) + \chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right) - u_{s_i}}{\left(\sum_{j \in N_i} a_{ij} + b_i\right) g_i(x_i)}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \chi\left(e_i, e_i^{(1)}, \ldots e_i^{(n-1)}\right) = & \alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} + \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} + \\ & \alpha_{i,3}\mathrm{sgn}\left(e_i^{(2)}\right)\left|e_i^{(2)}\right|^{\beta_{i,3}} + \cdots + \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}, \end{aligned} \tag{20}$$

$i = 1, 2, \ldots, N-1, N$, $j \in N_i$, $\hat{d}_i$, and $\hat{d}_j$ are the uncertainty and disturbance estimates of the corresponding agent systems. The parameter selection of $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}$ and $\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,n}$ needs to meet the corresponding constraints in Lemma 3; the dynamics of u_{s_i} is given by:

$$\dot{u}_{s_i} = -K_{s_i}(t, s_i)\mathrm{sgn}(s_i), u_{s_i}(0) = 0, \tag{21}$$

and

$$K_{s_i}(t, s_i) = \begin{cases} K_{a_{i,1}}(t),\ \dot{K}_{a_{i,1}}(t) = K_{i,1}|s_i|, \text{ if } 0 < t \le t_{i,1}, \\ K_{b_{i,1}}(s_i) = \frac{\omega_{i,1} p_{i,1}}{\omega_{i,1} - |s_i|}, \text{ if } t > t_{i,1}, \end{cases} \tag{22}$$

where

$$\begin{aligned} s_i = & \alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} + \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} + \cdots + \\ & \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}} + e_i^{(n)} + \dot{s}_{d_i}, \end{aligned} \tag{23}$$

the parameters $K_{i,1}$, $\omega_{i,1}$ and $p_{i,1}$ are positive constants, and $\dot{s}_{d_i}$ will be designed in the following.

Remark 5. *It should be noted that u_j needs to be used in designing the control protocol u_i in (19). To avoid the problem of the implementation loop, we adopt a method similar to that in the work [40]; that is, use $u_j(t-\tau)$ instead of $u_j(t)$ to construct the current control input $u_i(t)$, where τ can be* *selected as the fundamental sample time. The theoretical analysis in the studies [40,47] show that this handling method will not significantly affect the stability and convergence of the closed-loop system.*

Remark 6. *Since the variable $e_i^{(n)}$ cannot be utilized directly in designing s_i, similar to that used in [48,49], it can be obtained by integrating Equation (23)*

$$h(s_i) = \int_0^t s_i d\tau = e_i^{(n-1)} + \int_0^t \chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right) d\tau + s_{d_i}, \tag{24}$$

which yields $s_i = \lim_{\tau \to 0}[h(t) - h(t-\tau)]/\tau$. Therefore, sgn$(s_i)$ is described as

$$\mathrm{sgn}(s_i) = \mathrm{sgn}[h(t) - h(t-\tau)], \tag{25}$$

and τ can be chosen as the fundamental sample time.

For the sake of obtaining the uncertainty and disturbance estimate $\hat{d}_i$ of the i-agent system required in (19), a barrier function-based adaptive disturbance observer is developed in the following. For the i-th agent system, auxiliary systems are constructed as follows:

$$s_{d_i} = \phi_i - x_{i,n}, \tag{26}$$

where the dynamic equations of ϕ_i are

$$\dot{\phi}_i = f_i(x_i) + g_i(x_i)u_i + \hat{d}_i, \tag{27}$$

and $\hat{d}_i$ is designed as:

$$\hat{d}_i = K_{d_i}(t, s_{d_i})\mathrm{sgn}(s_{d_i}), \tag{28}$$

and

$$K_{d_i}(t, s_{d_i}) = \begin{cases} K_{a_{i,2}}(t),\ \dot{K}_{a_{i,2}}(t) = K_{i,2}|s_{d_i}|, \text{ if } 0 < t \le t_{i,2}, \\ K_{b_{i,2}}(s_{d_i}) = \frac{\omega_{i,2} p_{i,2}}{\omega_{i,2} - |s_{d_i}|}, \text{ if } t > t_{i,2}, \end{cases} \tag{29}$$

where $K_{i,2} > 0$, $\omega_{i,2} > 0$, and $p_{i,2} > 0$. Define the disturbance estimation errors as:

$$\tilde{d}_i = \hat{d}_i - d_i, \tag{30}$$

then $\dot{s}_{d_i} = \tilde{d}_i$ holds.

Lemma 4 ([49])**.** *Assume that the system (1) states are measurable and available. The distributed adaptive disturbance observers based on the barrier function provided in Definition 1 are designed as (26)–(29). If the parameter condition $p_{i,2} > d_{\max}$ is satisfied, the disturbance estimation errors $\tilde{d}_i$ can converge to the origin in finite time T_d, and*

$$T_d = \left\| t_{i,2} + t_{i,d} \right\|_\infty, \tag{31}$$

where

$$t_{i,d} = \frac{2V^{\frac{1}{2}}\left(s_{d_i}(t_{i,2}), K_{b_{i,2}}(s_{d_i})\right)}{\theta_i}, \tag{32}$$

$$V\left(s_{d_i}(t_{i,2}), K_{b_{i,2}}(s_{d_i})\right) = \frac{1}{2}s_{d_i}^2 + \frac{1}{2}[K_{b_{i,2}}(s_{d_i}) - K_{b_{i,2}}(0)]^2, \tag{33}$$

and $\theta_i = \sqrt{2}\left[K_{b_{i,2}}(s_{d_i}) - d_{\max}\right] \min\left\{1, \frac{\omega_{i,2} p_{i,2}}{\left(\omega_{i,2} - |s_{d_i}|^2\right)}\right\}$.

Remark 7. *In this paper, the distributed disturbance observer designed for the finite-time consensus control of MASs only requires the assumption that disturbances are bounded, as described in Assumption 1. Unlike some existing technologies, such as the extended state observer [50], high-gain observer [51], and uncertainty and disturbance estimator [52], for the proposed scheme, no additional assumptions on the derivatives of disturbances are required, no prior knowledge of the upper bounds of disturbances is needed, and finite-time convergence of the disturbance estimation errors can be guaranteed by the proposed observer.*

The control objective (13) can be achieved by the proposed control scheme (19). The closed-loop system performance and the stability analysis under the action of this strategy will be given in Section 4.

3.2. Finite-Time Adaptive Consensus Tracking Control of High-Order Multi-Agent Systems without Intermediate State Measurements

It is not easy to achieve that all system states can be measured or utilized in practical applications. On the one hand, it is hard to ensure that sensors are installed at each part of the system to measure state information. On the other hand, the intermediate state of the system may be affected by measurement noise, resulting in a severely distorted or unusable signal. Therefore, this subsection will consider the unmeasurable or unusable intermediate state of high-order MASs. We will develop a finite-time adaptive consensus tracking control scheme independent of the intermediate state information of the system (1).

Since the intermediate states $(x_{i,2}, x_{i,3}, \ldots, x_{i,n-1})$ of the system (1) are partially or entirely unavailable, a cascade high-gain observer is adopted to obtain intermediate state estimates, which are used to replace the unavailable states required in the controller design (19). Define the state estimates of the i-th agent system as $\hat{x}_i = [\hat{x}_{i,1}, \hat{x}_{i,2}, \ldots, \hat{x}_{i,n}]^T \in \mathcal{R}^{n\times 1}$. The corresponding state estimation errors $\tilde{x}_i$ can be defined as

$$\tilde{x}_i = x_i - \hat{x}_i. \tag{34}$$

The distributed cascade high-gain observers are designed as follows:

$$\dot{\varsigma}_{i,1} = \frac{\varsigma_{i,2} + \gamma_{i,1}(y_i - \varsigma_{i,1})}{\varepsilon_i}, \tag{35}$$

$$\dot{\varsigma}_{i,2} = \frac{\gamma_{i,2}(y_i - \varsigma_{i,1})}{\varepsilon_i}, \tag{36}$$

$$\hat{x}_{i,1} = \varsigma_{i,1}, \tag{37}$$

$$\hat{x}_{i,2} = M_2 \mathrm{sat}\left(\frac{\varsigma_{i,2}}{\varepsilon_i M_2}\right), \tag{38}$$

$$\dot{\varsigma}_{i,\rho} = \frac{-\gamma_{i,\rho}(\varsigma_{i,\rho} + \hat{x}_{i,\rho-1})}{\varepsilon_i}, \tag{39}$$

$$\hat{x}_{i,\rho} = M_\rho \mathrm{sat}\left(\frac{\gamma_{i,\rho}\varsigma_{i,\rho} + \gamma_{i,\rho}\hat{x}_{i,\rho-1}}{\varepsilon_i M_\rho}\right), \tag{40}$$

where $i \in N$, and the Equations (39) and (40) hold for $3 \leq \rho \leq n$. Let $M_\rho > \max_{x\in X}|x_i|$ for $2 \leq \rho \leq n$, if $x(t)$ belongs to a compact set X.

Lemma 5 ([42])**.** *If the distributed cascade high-gain observers for the system (1) are designed as (35)–(40), there exist $\varepsilon_i^* > 0$ such that for $0 < \varepsilon_i \leq \varepsilon_i^*$ ($i = 1, 2, \ldots, N$), the state estimation errors are bounded and the ultimate bounds can be reduced by decreasing ε_i. There is time $T(\varepsilon_i)$, with $\lim_{\varepsilon_i \to 0} T(\varepsilon_i) = 0$, such that:*

$$\|\tilde{x}_{i,1}\|_\infty \leq a_1 \varepsilon_i^2, \tag{41}$$

$$\left\|\tilde{x}_{i,\rho}\right\|_\infty \leq a_i \varepsilon_i, \text{ for } 2 \leq \rho \leq n, \tag{42}$$

for all $t \geq T(\varepsilon_i)$, a_1 and a_i are positive constants.

The distributed control law $\hat{u}_i$ of the i-th agent in MASs without intermediate state measurements can be designed as:

$$\hat{u}_i = \frac{\left[b_i y_0^{(n)} + \sum_{j\in N_i} a_{ij}\left(f_j(\hat{x}_j) + g_j(\hat{x}_j)\hat{u}_j + \hat{d}_j^*\right)\right]}{\left(\sum_{j\in N_i} a_{ij} + b_i\right) g_i(\hat{x}_i)} - \frac{\left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(\hat{d}_i^* + f_i(\hat{x}_i)\right) + \hat{\chi}\left(e_i, \hat{e}_i^{(1)}, \ldots, \hat{e}_i^{(n-2)}, e_i^{(n-1)}\right) - \hat{u}_{s_i}}{\left(\sum_{j\in N_i} a_{ij} + b_i\right) g_i(\hat{x}_i)}, \tag{43}$$

where

$$\begin{aligned}\hat{\chi}\left(e_i, \hat{e}_i^{(1)}, \ldots \hat{e}_i^{(n-2)}, e_i^{(n-1)}\right) =& \alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} + \alpha_{i,2}\mathrm{sgn}\left(\hat{e}_i^{(1)}\right)\left|\hat{e}_i^{(1)}\right|^{\beta_{i,2}} + \cdots + \\ & \alpha_{i,n-1}\mathrm{sgn}\left(\hat{e}_i^{(n-2)}\right)\left|\hat{e}_i^{(n-2)}\right|^{\beta_{i,n-1}} + \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}},\end{aligned} \tag{44}$$

$i = 1, 2, \ldots, N-1, N$, $j \in N_i$, and

$$\begin{cases} \hat{e}_i^{(1)} = \sum_{j\in N_i} a_{ij}\left(\hat{x}_{i,2} - \hat{x}_{j,2}\right) + b_i\left(\hat{x}_{i,2} - y_0^{(1)}\right), \\ \hat{e}_i^{(2)} = \sum_{j\in N_i} a_{ij}\left(\hat{x}_{i,3} - \hat{x}_{j,3}\right) + b_i\left(\hat{x}_{i,3} - y_0^{(2)}\right), \\ \vdots \\ \hat{e}_i^{(n-2)} = \sum_{j\in N_i} a_{ij}\left(\hat{x}_{i,n-1} - \hat{x}_{j,n-1}\right) + b_i\left(\hat{x}_{i,n-1} - y_0^{(n-2)}\right), \\ e_i^{(n-1)} = \sum_{j\in N_i} a_{ij}\left(x_{i,n} - x_{j,n}\right) + b_i\left(x_{i,n} - y_0^{(n-1)}\right), \end{cases} \tag{45}$$

the dynamic of the control components $\hat{u}_{s_i}$ are designed as follows:

$$\dot{\hat{u}}_{s_i} = -\hat{K}_{s_i}(t, \hat{s}_i)\mathrm{sgn}(\hat{s}_i), \hat{u}_{\hat{s}_i}(0) = 0, \tag{46}$$

and

$$\hat{K}_{s_i}(t, \hat{s}_i) = \begin{cases} \hat{K}_{a_{i,1}}(t),\ \dot{\hat{K}}_{a_{i,1}}(t) = \hat{K}_{i,1}|\hat{s}_i|, \text{ if } 0 < t \leq \bar{t}_{i,1}, \\ \hat{K}_{b_{i,1}}(\hat{s}_i) = \frac{\hat{\omega}_{i,1}\hat{p}_{i,1}}{\hat{\omega}_{i,1} - |\hat{s}_i|}, \text{ if } t > \bar{t}_{i,1}, \end{cases} \tag{47}$$

where

$$\begin{aligned}\hat{s}_i =& \alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} + \alpha_{i,2}\mathrm{sgn}\left(\hat{e}_i^{(1)}\right)\left|\hat{e}_i^{(1)}\right|^{\beta_{i,2}} + \cdots + \\ & \alpha_{i,n-1}\mathrm{sgn}\left(\hat{e}_i^{(n-2)}\right)\left|\hat{e}_i^{(n-2)}\right|^{\beta_{i,n-1}} + \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}} + e_i^{(n)} + \dot{\hat{s}}_{d_i},\end{aligned} \tag{48}$$

the parameters $\hat{K}_{i,1}$, $\hat{\omega}_{i,1}$, and $\hat{p}_{i,1}$ are positive constants, $\hat{d}_i^*$, $\hat{d}_j^*$, and $\dot{\hat{s}}_{d_i}$ are uncertainty and disturbance estimates of the corresponding MASs without intermediate state measurements; for the i-th agent system, $\dot{\hat{s}}_{d_i}$ are the dynamics of the auxiliary systems constructed as follows:

$$\hat{s}_{d_i} = \hat{\phi}_i - x_{i,n}, \tag{49}$$

where the dynamic equations of $\hat{\phi}_i$ are:

$$\dot{\hat{\phi}}_i = f_i(\hat{x}_i) + g_i(\hat{x}_i)\hat{u}_i + \hat{d}_i^*, \tag{50}$$

and $\hat{d}_i^*$ is designed as:

$$\hat{d}_i^* = \hat{K}_{d_i}(t, \hat{s}_{d_i})\text{sgn}(\hat{s}_{d_i}), \tag{51}$$

and

$$\hat{K}_{d_i}(t, \hat{s}_{d_i}) = \begin{cases} \hat{K}_{a_{i,2}}(t),\ \dot{\hat{K}}_{a_{i,2}}(t) = \hat{K}_{i,2}|\hat{s}_{d_i}|, \text{ if } 0 < t \le \bar{t}_{i,2}, \\ \hat{K}_{b_{i,2}}(\hat{s}_{d_i}) = \frac{\hat{\omega}_{i,2}\hat{p}_{i,2}}{\hat{\omega}_{i,2} - |\hat{s}_{d_i}|}, \text{ if } t > \bar{t}_{i,2}, \end{cases} \tag{52}$$

where $\hat{K}_{i,2} > 0$, $\hat{\omega}_{i,2} > 0$ and $\hat{p}_{i,2} > 0$.

Remark 8. *The proposed distributed control protocol $\hat{u}_i$ (43) combines the designed distributed finite-time disturbance observer (51) with the distributed cascaded high-gain state observer (35)–(40), which not only relaxes the restriction on the disturbance assumption, i.e., has wider applicability, but also avoids the peaking phenomenon of the conventional high-gain observer and improves the transient performance. The barrier function-based controller is designed in Definition 1 with adaptive parameter adjustment. Moreover, according to its structure, it can be observed that the control gain decreases with the consensus tracking error, avoiding excessive control gain and still ensuring the finite-time convergence performance of the closed-loop system when the intermediate states of the higher-order nonlinear MASs are unavailable.*

Theorem 3. *Assume that the intermediate states of the MASs (1) under a connected undirected graph $\mathcal{G}$ and Assumptions 1–4 are unavailable. The distributed barrier function-based adaptive disturbance observers are designed as (49)–(52), and the distributed cascade high-gain observers are given by (35)–(40). Then, finite-time estimation of the lumped disturbance can be realized by selecting sufficiently large parameters $\hat{p}_{i,2}$, and the disturbance estimation errors can converge to zero in a finite time $\bar{T}_d$.*

Proof. Consider the system (1) and take the derivatives on both sides of the Equation (49); one obtains

$$\dot{\hat{s}}_{d_i} = \tilde{f}(x_i, \hat{x}_i) + \tilde{g}(x_i, \hat{x}_i)\hat{u}_i + \hat{d}_i^* - d_i, \tag{53}$$

where $\tilde{f}(x_i, \hat{x}_i) = f_i(\hat{x}_i) - f_i(x_i)$, $\tilde{g}(x_i, \hat{x}_i) = g_i(\hat{x}_i) - g_i(x_i)$. Since $f_i(x_i)$ and $g_i(x_i)$ are continuously differentiable, $f_i(x_i)$ and $g_i(x_i)$ are locally Lipschitz, then we can get

$$\|\tilde{f}(x_i, \hat{x}_i)\|_\infty = \|f_i(\hat{x}_i) - f_i(x_i)\|_\infty \le L_{f_i}\|x_i - \hat{x}_i\|_\infty = L_{f_i}\|\tilde{x}_i\|_\infty, \tag{54}$$

$$\|\tilde{g}(x_i, \hat{x}_i)\|_\infty = \|g_i(\hat{x}_i) - g_i(x_i)\|_\infty \le L_{g_i}\|x_i - \hat{x}_i\|_\infty = L_{g_i}\|\tilde{x}_i\|_\infty, \tag{55}$$

where $k_{f_i} > 0$ and $k_{g_i} > 0$ are the Lipschitz constants of $\tilde{f}(x_i, \hat{x}_i)$ and $\tilde{g}(x_i, \hat{x}_i)$, respectively. Define the lumped disturbances as

$$D_i = \tilde{f}(x_i, \hat{x}_i) + \tilde{g}(x_i, \hat{x}_i)\hat{u}_i - d_i. \tag{56}$$

Then, the above Equation (53) can be rewritten as:

$$\dot{\hat{s}}_{d_i} = \hat{d}_i^* + D_i. \tag{57}$$

Note that here we consider $\hat{u}_i$ to be bounded, and a subsequent proof will illustrate its boundedness. Owing to Assumption 1, Lemma 5 and (54)–(55), it can be deduced that D_i is bounded. As shown in (51) and (52), all conditions required by Lemma 2 have been fulfilled. Assume that $\|D_i\|_\infty \leq D_{\max}$. According to Lemma 4, it can be derived that $\bar{T}_d = \|\bar{t}_{i,2} + \bar{t}_{i,d}\|_\infty$,

$$\bar{t}_{i,d} = \frac{2V^{\frac{1}{2}}\left(\hat{s}_{d_i}(\bar{t}_{i,2}), \hat{K}_{b_{i,2}}(\hat{s}_{d_i})\right)}{\hat{\theta}_i}, \tag{58}$$

and $\hat{\theta}_i = \sqrt{2}\left[\hat{K}_{b_{i,2}}(\hat{s}_{d_i}) - D_{\max}\right] \min\left\{1, \frac{\hat{\omega}_{i,2}\hat{\rho}_{i,2}}{\left(\hat{\omega}_{i,2} - \left|\hat{s}_{d_i}\right|^2\right)}\right\}$. The proof is completed. □

In this section, we design the distributed finite-time adaptive consensus tracking controller for the case where the system states are all measurable and for the case where intermediate states are not measurable. In the next section, we will analyze the stability of the system (1) under the action of the proposed controllers.

4. Stability Analysis

First, we consider the situation where all system states can be measured. The stability and performance of the closed-loop system are investigated as follows.

Theorem 4. *Consider the MASs (1) under a connected undirected graph $\mathcal{G}$ and Assumptions 1–4, and suppose that all system states are measurable and available. Suppose the distributed controllers are designed as (19)–(23), which are employed as the control inputs. Moreover, the disturbance observers are designed as (26)–(29) to estimate the uncertainties and disturbances d_i of the MASs (1). Then, the stability of the closed-loop systems is guaranteed, and the objective (13) can be achieved. The MASs can achieve consensus tracking in a finite time T_1.*

Proof. Take the derivative of e_i (15) with respect to time t up to its n-th derivative, yields

$$e_i^{(n)} = \sum_{j\in N_i} a_{ij}(\dot{x}_{i,n} - \dot{x}_{j,n}) + b_i\left(\dot{x}_{i,n} - y_0^{(n)}\right). \tag{59}$$

Considering system (1) and substituting (19) into (59), one can obtain

$$\begin{aligned} e_i^{(n)} &= \left(\sum_{j\in N_i} a_{ij} + b_i\right)(f_i(x_i) + g_i(x_i)u_i + d_i) - \sum_{j\in N_i} a_{ij}(f_j(x_j) + g_j(x_j)u_j + d_j) - b_i y_0^{(n)} \\ &= \left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(d_i - \hat{d}_i\right) - \sum_{j\in N_i} a_{ij}\left(d_j - \hat{d}_j\right) - \chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right) + u_{s_i}. \end{aligned} \tag{60}$$

From (20)–(23), (30), and (60), it can be derived that

$$s_i = u_{s_i} + \left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j. \tag{61}$$

According to Lemma 4, we have $\tilde{d}_i = \tilde{d}_j = 0$ for all $t \geq T_d$. Then, the above Equation (61) can be simplified to

$$s_i = u_{s_i}. \tag{62}$$

Taking the first-order time derivative of Equation (62) and combining (21) yields

$$\dot{s}_i = -K_{s_i}(t, s_i)\mathrm{sgn}(s_i). \tag{63}$$

For $t > T_d + t_{i,1}$, according to (22), the above Equation (63) can be rewritten as:

$$\dot{s}_i = -K_{b_{i,1}}(s_i)\mathrm{sgn}(s_i). \tag{64}$$

From (22), Lemmas 2 and 4, we can conclude that the sliding surface s_i is asymptotically converging in a finite time T_c,

$$T_c = \|t_{i,1} + t_{i,u}\|_\infty, \tag{65}$$

where

$$t_{i,u} = \frac{2V^{\frac{1}{2}}\left(s_i(t_{i,1}), K_{b_{i,1}}(s_i)\right)}{\lambda_i}, \tag{66}$$

$$V\left(s_i(t_{i,1}), K_{b_{i,1}}(s_i)\right) = \frac{1}{2}s_i^2 + \frac{1}{2}\left[K_{b_{i,1}}(s_i) - K_{b_{i,1}}(0)\right]^2, \tag{67}$$

and $\lambda_i = \sqrt{2}K_{b_{i,1}}(s_i)\min\left\{1, \frac{\omega_{i,1}p_{i,1}}{(\omega_{i,1}-|s_i|^2)}\right\}$.

In the light of (23) and Lemma 3, it can be concluded that

$$\begin{aligned} s_i =& \alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} + \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} + \cdots + \\ & \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}} + e_i^{(n)} = 0, \end{aligned} \tag{68}$$

for $t > T_d + T_c$, we can then derive that $e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}$ converge to zero in a finite time T_1, which is defined as $T_1 = T_d + T_c + T_z$. Hence, the consensus errors e defined in (16) converge to zero in a finite time T_1. From (15)–(18) and Lemma 1, one can get

$$e = [(L + B) \otimes I_N]\tilde{y}. \tag{69}$$

and it can be derived that

$$\|\tilde{y}\|_\infty \leq \left\|[(L + B) \otimes I_N]^{-1}\right\|_\infty \|e\|_\infty. \tag{70}$$

It can be obtained that $\tilde{y}$ can converge to zero in finite time T_1. Hence, the objective (13) can be achieved.

For $t < T_1$, the boundedness of the consensus errors e and tracking errors $\tilde{y}$ will be proved. According to Lemma 4, we can find that disturbance estimation errors $\tilde{d}_i$ are bounded. From (21), we have $\|\tilde{d}_i\| \geq u_{s_i}(0) = 0$. When the control inputs u_{s_i} satisfy

$$u_{s_i} > \left\|\left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j\right\|_\infty, \tag{71}$$

then $s_i > 0$. In the light of (21) and (22), it can be concluded that $\dot{u}_{s_i} < 0$ and $u_{s_i}\dot{u}_{s_i} < 0$ hold.

Similarly, in the case that

$$u_{s_i} < \left\|\left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j\right\|_\infty, \tag{72}$$

we have $s_i < 0$. Then, $\dot{u}_{s_i} > 0$ and $u_{s_i}\dot{u}_{s_i} < 0$ hold. Therefore, we can conclude that

$$u_{s_i} \in \left[-\left\|\left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j\right\|_\infty, \left\|\left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j\right\|_\infty\right]. \tag{73}$$

Moreover, we can get

$$\begin{aligned}\|s_i\|_\infty &\le \|u_{s_i}\|_\infty + \left\| \left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j \right\|_\infty \\ &\le 2\left\| \left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j \right\|_\infty .\end{aligned} \tag{74}$$

From (15) and (23), we have

$$\begin{cases} \dot{e}_i = e_i^{(1)}, \\ \dot{e}_i^{(1)} = e_i^{(2)}, \\ \vdots \\ \dot{e}_i^{(n-2)} = e_i^{(n-1)}, \\ \dot{e}_i^{(n-1)} = -\alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}} - \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \cdots - \\ \qquad \alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}} + s_i - \dot{s}_{d_i}. \end{cases} \tag{75}$$

Since s_i and $\dot{s}_{d_i}$ are bounded, and considering Lemma 3 and the input-to-state stability[53], it can be concluded that the consensus errors e are bounded. Furthermore, according to (70), it can be derived that the tracking errors $\tilde{y}$ are also bounded. The proof is completed. □

Corollary 1. *Consider the MASs (1) under a connected undirected graph $\mathcal{G}$ and Assumptions 1–4 and suppose that all system states are measurable and available. Then, under the distributed control law (19), the consensus tracking and the closed-loop system stability are still guaranteed by introducing a sufficiently slight time delay τ in the neighbor's control input $u_j(t)$.*

Proof. Consider introducing a time delay τ in the neighbor's control input $u_j(t)$, then the distributed control law $u_i(t)$ can be rewritten as

$$\begin{aligned} u_i(t) = &\frac{\left[b_i y_0^{(n)} + \sum_{j\in N_i} a_{ij}\left(f_j(x_j) + g_j(x_j)u_j(t-\tau) + \hat{d}_j\right)\right]}{\left(\sum_{j\in N_i} a_{ij} + b_i\right) g_i(x_i)} - \\ &\frac{\left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(\hat{d}_i + f_i(x_i)\right) + \chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right) - u_{s_i}}{\left(\sum_{j\in N_i} a_{ij} + b_i\right) g_i(x_i)}. \end{aligned} \tag{76}$$

Substituting (76) into (1) and combining (60) yields

$$\begin{aligned} e_i^{(n)} = &\left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(d_i - \hat{d}_i\right) - \sum_{j\in N_i} a_{ij}\left(d_j - \hat{d}_j\right) + \sum_{j\in N_i} a_{ij}\left[u_j(t-\tau) - u_j(t)\right] - \\ &\chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right) + u_{s_i}. \end{aligned} \tag{77}$$

For $t < \tau$, it should be noted that $u_j(t-\tau) = 0$. For $t = \tau$, we can find that $u_j(t-\tau) = u_j(0) = u_i(0)$ $(i \in N_j)$. Moreover, by Theorem 4, we can obtain that for all t, the consensus error e and $\chi\left(e_i, e_i^{(1)}, \ldots, e_i^{(n-1)}\right)$ are bounded. Therefore, $u_j(t-\tau)$ is bounded for $t \le \tau$ given the bounded initial variables and the related definitions of the functions (21), (28) and (76). Further, it can be derived by iterative substitution that $u_j(t-\tau)$ and $u_j(t)$ are bounded. Hence, it can be observed that $u_i(t)$ is bounded.

From (20)–(23), (30) and (77), one obtains

$$s_i = u_{s_i} + \left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j + \sum_{j\in N_i} a_{ij}\left[u_j(t-\tau) - u_j(t)\right]. \tag{78}$$

After a derivation similar to the process of proving Theorem 4, we can get

$$\|s_i\|_\infty \le 2\left\|\left(1 - \sum_{j\in N_i} a_{ij} - b_i\right)\tilde{d}_i + \sum_{j\in N_i} a_{ij}\tilde{d}_j + \sum_{j\in N_i} a_{ij}\left[u_j(t-\tau) - u_j(t)\right]\right\|_\infty. \tag{79}$$

For $t \ge T_1$, we have

$$\|s_i\|_\infty \le 2\left\|\sum_{j\in N_i} a_{ij}\left[u_j(t-\tau) - u_j(t)\right]\right\|_\infty \triangleq \delta_i(\tau), \tag{80}$$

where $\delta_i(\tau)$ satisfies that

$$\lim_{\tau\to 0} \delta_i(\tau) = 0. \tag{81}$$

Hence, the consensus tracking and the closed-loop system stability are guaranteed by introducing a sufficiently slight time delay τ. □

Next, the closed-loop stability and performance of high-order multi-agent systems without intermediate state measurements are analyzed as follows.

Theorem 5. *Consider the MASs (1) under a connected undirected graph $\mathcal{G}$, Assumptions 1–4, and the assumption that the intermediate states of the system are unmeasurable. If the distributed controllers are designed as (43)–(48), the distributed cascade high-gain observers are designed as (35)–(40) to estimate the unmeasurable intermediate states, and the disturbance observers are designed as (49)–(52) to estimate the uncertainties and disturbances d_i of the MASs (1). Then, the stability of the closed-loop systems is guaranteed, and the consensus errors e can converge to the domain S within a finite time T_2. Moreover, the objective (14) can be achieved.*

Proof. Considering the system (1) and (59), under the action of the control input $\hat{u}_i$, yields

$$e_i^{(n)} = \left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(f_i(x_i) + g_i(x_i)\hat{u}_i + d_i\right) - \sum_{j\in N_i} a_{ij}\left(f_j(x_j) + g_j(x_j)\hat{u}_j + d_j\right) - b_i y_0^{(n)}. \tag{82}$$

Substituting (43) into (82) and combining (54) and (55), one can get

$$\begin{aligned} e_i^{(n)} =& \left(\sum_{j\in N_i} a_{ij} + b_i\right)\left[f_i(x_i) + (g_i(\hat{x}_i) - \tilde{g}(x_i,\hat{x}_i))\hat{u}_i + d_i\right] - \\ & \sum_{j\in N_i} a_{ij}\left(f_j(x_j) + g_j(x_j)\hat{u}_j + d_j\right) - b_i y_0^{(n)}, \\ =& -\left(\sum_{j\in N_i} a_{ij} + b_i\right)\left(\tilde{f}(x_i,\hat{x}_i) + \tilde{g}(x_i,\hat{x}_i)\hat{u}_i + \tilde{d}_i^*\right) + \\ & \sum_{j\in N_i} a_{ij}\left(\tilde{f}(x_j,\hat{x}_j) + \tilde{g}(x_j,\hat{x}_j)\hat{u}_j + \tilde{d}_j^*\right) - \hat{\chi}\left(e_i, \hat{e}_i^{(1)}, \ldots, \hat{e}_i^{(n-2)}, e_i^{(n-1)}\right) + \hat{u}_{s_i}, \end{aligned} \tag{83}$$

where $\tilde{d}_i^* = \hat{d}_i^* - d_i$ and $\tilde{d}_j^* = \hat{d}_j^* - d_j$.

Combining (44), (48), (56), (57) and (83), it can be derived that

$$\begin{aligned}\dot{\hat{s}}_i =&\left(1-\sum_{j\in N_i} a_{ij}-b_i\right)(\tilde{f}(x_i,\hat{x}_i)+\tilde{g}(x_i,\hat{x}_i)\hat{u}_i+\tilde{d}_i^*)+\\&\sum_{j\in N_i} a_{ij}\left(\tilde{f}(x_j,\hat{x}_j)+\tilde{g}(x_j,\hat{x}_j)\hat{u}_j+\tilde{d}_j^*\right)+\hat{u}_{s_i}.\end{aligned} \tag{84}$$

According to Theorem 3, for all $t \geq \bar{T}_d$, it can be concluded that

$$\begin{aligned}&\left(1-\sum_{j\in N_i} a_{ij}-b_i\right)(\tilde{f}(x_i,\hat{x}_i)+\tilde{g}(x_i,\hat{x}_i)\hat{u}_i+\tilde{d}_i^*)+\\&\sum_{j\in N_i} a_{ij}\left(\tilde{f}(x_j,\hat{x}_j)+\tilde{g}(x_j,\hat{x}_j)\hat{u}_j+\tilde{d}_j^*\right)=0.\end{aligned} \tag{85}$$

Then, we have

$$\dot{\hat{s}}_i = \hat{u}_{s_i}, \tag{86}$$

for all $t \geq \bar{T}_d$.

Taking the derivative of (86) with respect to time t and combining (46) and (47), one can obtain

$$\dot{\hat{s}}_i = -\hat{K}_{b,1}(\hat{s}_i)\mathrm{sgn}(\hat{s}_i), \tag{87}$$

for $t > \bar{T}_d + \bar{t}_{i,1}$.

In the light of Lemmas 2 and 4, and Theorem 4, we can deduce that the sliding surface $\hat{s}_i$ is asymptotically converging in a finite time $\bar{T}_c$,

$$\bar{T}_c = \|\bar{t}_{i,1} + \bar{t}_{i,u}\|_\infty, \tag{88}$$

where

$$\bar{t}_{i,u} = \frac{2V^{\frac{1}{2}}\left(\hat{s}_i(\bar{t}_{i,1}),\hat{K}_{b_{i,1}}(\hat{s}_i)\right)}{\hat{\lambda}_i}, \tag{89}$$

$$V\left(\hat{s}_i(\bar{t}_{i,1}),\hat{K}_{b_{i,1}}(\hat{s}_i)\right) = \frac{1}{2}\hat{s}_i^2 + \frac{1}{2}\left[\hat{K}_{b_{i,1}}(\hat{s}_i) - \hat{K}_{b_{i,1}}(0)\right]^2, \tag{90}$$

and $\hat{\lambda}_i = \sqrt{2}\hat{K}_{b_{i,1}}(\hat{s}_i)\min\left\{1, \frac{\hat{\omega}_{i,1}\hat{\rho}_{i,1}}{(\hat{\omega}_{i,1}-|\hat{s}_i|^2)}\right\}$.

According to (48) and Theorem 3, we can get

$$\begin{aligned}\hat{s}_i =&\alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}}+\alpha_{i,2}\mathrm{sgn}\left(\hat{e}_i^{(1)}\right)\left|\hat{e}_i^{(1)}\right|^{\beta_{i,2}}+\cdots+\alpha_{i,n-1}\mathrm{sgn}\left(\hat{e}_i^{(n-2)}\right)\left|\hat{e}_i^{(n-2)}\right|^{\beta_{i,n-1}}\\&+\alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}+e_i^{(n)}=0,\end{aligned} \tag{91}$$

Further, we can derive that

$$\begin{aligned}e_i^{(n)} =&-\alpha_{i,1}\mathrm{sgn}(e_i)|e_i|^{\beta_{i,1}}-\alpha_{i,2}\mathrm{sgn}\left(\hat{e}_i^{(1)}\right)\left|\hat{e}_i^{(1)}\right|^{\beta_{i,2}}-\cdots-\alpha_{i,n-1}\mathrm{sgn}\left(\hat{e}_i^{(n-2)}\right)\left|\hat{e}_i^{(n-2)}\right|^{\beta_{i,n-1}}\\&-\alpha_{i,n}\mathrm{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}.\end{aligned} \tag{92}$$

Given the definition e_i in (15) and (45), the above Equation (92) can be rewritten as:

$$e_i^{(n)} = -\alpha_{i,1}\text{sgn}(e_i)|e_i|^{\beta_{i,1}} - \alpha_{i,2}\text{sgn}\left(e_i^{(1)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,2}\right)\cdot \left|e_i^{(1)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,2}\right|^{\beta_{i,2}} - \cdots - \alpha_{i,n-1}\text{sgn}\left(e_i^{(n-2)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,n-1}\right)\cdot \left|e_i^{(n-2)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,n-1}\right|^{\beta_{i,n-1}} - \alpha_{i,n}\text{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}. \tag{93}$$

Considering the singular perturbation theory and Lemma 5, the state estimation error system can be regarded as a fast time-varying system and the consensus error system as a slowly time-varying system. Hence, when the singular parameter $\varepsilon_i \to 0$, $\|\tilde{x}_i\|_\infty \to 0$. In addition, owing to Lemma 3, we can obtain that the reduced system

$$\begin{cases} \dot{e}_i = e_i^{(1)}, \\ \dot{e}_i^{(1)} = e_i^{(2)}, \\ \vdots \\ \dot{e}_i^{(n-2)} = e_i^{(n-1)}, \\ \dot{e}_i^{(n-1)} = -\alpha_{i,1}\text{sgn}(e_i)|e_i|^{\beta_{i,1}} - \alpha_{i,2}\text{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \cdots - \\ \qquad \alpha_{i,n}\text{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}, \end{cases} \tag{94}$$

is exponentially stable at the origin.

If $\varepsilon_i \neq 0$, the consensus error system can be expressed as:

$$\begin{cases} \dot{e}_i = e_i^{(1)}, \\ \dot{e}_i^{(1)} = e_i^{(2)}, \\ \vdots \\ \dot{e}_i^{(n-2)} = e_i^{(n-1)}, \\ \dot{e}_i^{(n-1)} = -\alpha_{i,1}\text{sgn}(e_i)|e_i|^{\beta_{i,1}} - \alpha_{i,2}\text{sgn}\left(\hat{e}_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \cdots - \\ \qquad \alpha_{i,n-1}\text{sgn}\left(\hat{e}_i^{(n-2)}\right)\left|\hat{e}_i^{(n-2)}\right|^{\beta_{i,n-1}} - \alpha_{i,n}\text{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}}. \end{cases} \tag{95}$$

The system (95) can be rewritten as:

$$\begin{cases} \dot{e}_i = e_i^{(1)}, \\ \dot{e}_i^{(1)} = e_i^{(2)}, \\ \vdots \\ \dot{e}_i^{(n-2)} = e_i^{(n-1)}, \\ \dot{e}_i^{(n-1)} = -\alpha_{i,1}\text{sgn}(e_i)|e_i|^{\beta_{i,1}} - \alpha_{i,2}\text{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \cdots - \\ \qquad \alpha_{i,n}\text{sgn}\left(e_i^{(n-1)}\right)\left|e_i^{(n-1)}\right|^{\beta_{i,n}} + \Gamma, \end{cases} \tag{96}$$

where

$$\Gamma = \left(\begin{array}{l} \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \alpha_{i,2}\mathrm{sgn}\left[e_i^{(1)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,2}\right]\cdot \\ \left|e_i^{(1)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,2}\right|^{\beta_{i,2}} \end{array} \right) + \cdots + \left(\begin{array}{l} \alpha_{i,2}\mathrm{sgn}\left(e_i^{(1)}\right)\left|e_i^{(1)}\right|^{\beta_{i,2}} - \alpha_{i,n-1}\mathrm{sgn}\left[e_i^{(n-2)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,n-1}\right]\cdot \\ \left|e_i^{(n-2)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,n-1}\right|^{\beta_{i,n-1}} \end{array} \right). \tag{97}$$

Define $F_q = \mathrm{sgn}\left(e_i^{(q)}\right) - \mathrm{sgn}\left(e_i^{(q)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,q+1}\right), q = 1, 2, \ldots, n-2$, it can be found that

$$\|F_q\|_\infty \le 2. \tag{98}$$

Furthermore, we define $E_q = \left|e_i^{(q)}\right|^{\beta_{i,q+1}} - \left|e_i^{(q)} - \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,q+1}\right|^{\beta_{i,q+1}}$, $q = 1, 2, \ldots, n-2$. According to Lemma 3, we have $\beta_{i,q+1} \in (0,1)$, and

$$\|E_q\|_\infty \le \left\| \left| \left(\sum_{j\in N_i} a_{ij} + b_i\right)\tilde{x}_{i,q+1}\right|^{\beta_{i,q+1}} \right\|_\infty. \tag{99}$$

Given Lemma 5 and (97)–(99), it can be concluded that

$$\|\Gamma\|_\infty \le 2\|\alpha_{q+1}\|_\infty \left\| \left| a_i \varepsilon_i \left(\sum_{j\in N_i} a_{ij} + b_i\right)\right|^{\beta_{i,q+1}} \right\|_\infty, \tag{100}$$

where $q = 1, 2, \ldots, n-2$.

Then, there exists $\varepsilon^* > 0$ so that for all $\varepsilon_i < \varepsilon^*$, $\|\Gamma\|_\infty$ is uniformly ultimately bounded and the ultimate bound is proportional to ε_i. From the input-to-state stability and (96), it can be derived that

$$\lim_{t\to T_2} \|e\|_\infty \le \gamma(\|\Gamma\|_\infty) = S, \tag{101}$$

where γ is a $\mathcal{K}$-class function, and the finite time T_2 is defined as $T_2 = \bar{T}_d + \bar{T}_c + T_z$. According to (70), we can get

$$\|\tilde{y}\|_\infty \le \left\|[(L+B)\otimes I_N]^{-1}\right\|_\infty \cdot \gamma\left(2\|\alpha_{q+1}\|_\infty \left\| \left| a_i \varepsilon_i \left(\sum_{j\in N_i} a_{ij} + b_i\right)\right|^{\beta_{i,q+1}} \right\|_\infty\right), \tag{102}$$

and $q = 1, 2, \ldots, n-2$. Then, the objective (14) is achieved.

For $t < T_2$, the proof of the boundedness of a closed-loop system is similar to that in Theorem 4. This ends the proof. □

Remark 9. *The analysis of introducing a sufficiently short time delay τ in the term $\hat{u}_j$ of (43) is similar to Corollary 1. Therefore, it will not be repeated. Similarly, the bounded consensus tracking and the closed-loop system stability are still guaranteed.*

Remark 10. *To reduce the chattering effect induced by the discontinuous* $\mathrm{sgn}(x)$*, it can be approximated and replaced by a continuous function* $\tanh(ax)$ *in this paper, where a is a large positive constant.*

Remark 11. *It should be noted that when the number of followers is large enough, as mentioned in [54], it will greatly increase the computational burden of the system, and the reasonable cluster processing is expected to alleviate this problem.*

Remark 12. *The undirected graph's adjacency matrix is symmetrical compared with the directed graph. Considering what is mentioned in [55], additional adaptation laws, parameterized estimation, and nonlinearities are applied to cope with adaptive consensus-controlled directed communication topologies, which complicates their application due to the need for additional hardware and software resources. Therefore, this paper considers the control design for high-order MASs under undirected graphs. However, directed graphs have an advantage over undirected graphs in terms of security. Inspired by this, it is meaningful to improve the proposed scheme to make it applicable to the case of directed graphs in future work as much as possible without increasing the complexity.*

5. Numerical Examples

In this section, two numerical examples demonstrate the effectiveness of the proposed finite-time adaptive consensus tracking control strategy in Section 3. Moreover, numerical simulation comparison results with the control method proposed in [56] are presented. A multi-agent system with one virtual leader (indexed as a node 0) and four followers is considered in the simulation. Figure 1 shows the communication topology among agents.

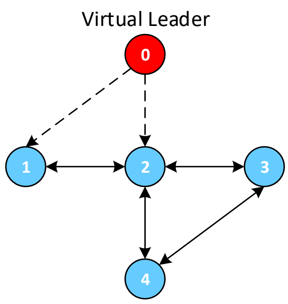

Figure 1. Communication topology graph $\mathcal{G}$ for the multi-agent system.

Then, the Laplacian matrix of the graph $\mathcal{G}$ and the adjacency matrix of the leader are given as follows:

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.1. Case 1: All States of the Multi-Agent System Are Measurable

In this case, we aim to verify the effectiveness of the finite-time adaptive consensus tracking control scheme based on the barrier function. Let the dynamics of the virtual leader be $y_0 = 2\sin(2t)$. The followers are described by the uncertain third-order nonlinear systems ($n = 3$) in the form of (1) with $f_1(x_1) = (x_{1,1} + 2)(x_{1,1} + x_{1,2})$, $f_2(x_2) = \sin(x_{2,2})(2x_{2,1} + x_{2,2})$, $f_3(x_3) = 0.5x_{3,1}^2 + 0.5x_{3,2} + x_{3,3}$, $f_4(x_4) = (x_{4,1} + x_{4,2} + x_{4,3})(0.5x_{4,1} + x_{4,2} + 2x_{4,3})$, $g_1(x_1) = 1$, $g_2(x_2) = 3 - \sin(x_{2,1}) + \cos(x_{2,2})$, $g_3(x_3) = [2 - \sin(x_{3,2})]\left(x_{3,1}^2 + 1\right)$, $g_4(x_4) = x_{4,1}^2 + x_{4,2}^2 + x_{4,3}^2 + 1$. The lumped disturbances of followers are $d_1 = 2\sin(0.1\pi t)$, $d_2 = 1 - 0.5/[\cos(0.2\pi t) + 3]$, $d_3 = 1/\left(1 + x_{3,1}^2\right) - \sin(x_{3,2}) + 0.1$, and $d_4 = 1/[\mathrm{sgn}(x_{4,1}) + 2]$. The parameter settings involved in the proposed control scheme (19) are presented in Table 1.

Table 1. Parameter settings ($i = 1, 2, 3, 4$).

Symbol	Value	Symbol	Value	Symbol	Value
$\alpha_{i,1}$	40	$\beta_{i,3}$	$1/2$	$K_{i,1}$	100
$\alpha_{i,2}$	30	$\omega_{i,1}$	0.01	$K_{i,2}$	100
$\alpha_{i,3}$	15	$\omega_{i,2}$	0.01	τ	0.001
$\beta_{i,1}$	$1/4$	$p_{i,1}$	5	a	100
$\beta_{i,2}$	$1/3$	$p_{i,2}$	5		

The initial conditions are selected as follows: $s_{d_1}(0) = 0.1$, $s_{d_2}(0) = 0.2$, $s_{d_3}(0) = 0.2$, $s_{d_4}(0) = 0.1$, $x_{1,1}(0) = 1$, $x_{2,1}(0) = 0.5$, $x_{3,1}(0) = -0.5$, $x_{4,1}(0) = -1$, $x_{i,2}(0) = 0.2$, $x_{i,3}(0) = 0$, $K_{a_{i,1}}(0) = K_{a_{i,2}}(0) = 10$, and $u_{s_i}(0) = 0$ ($i = 1, 2, 3, 4$). The fundamental sample time is set as 0.001s.

The simulation results under the action of the control protocol proposed in (19) are presented in Figure 2. It can be seen from Figure 2a–c that the finite-time consensus tracking of the uncertain high-order nonlinear MASs with the communication topology depicted in Figure 1 can be achieved, which is consistent with the control objective (13) and the results presented in Theorem 4. From Figure 2d, the lumped disturbance of each agent can be effectively estimated by the proposed distributed disturbance observer based on the barrier function, and the disturbance estimation errors can approach zero. Moreover, the control inputs of followers are indicated in Figure 2e.

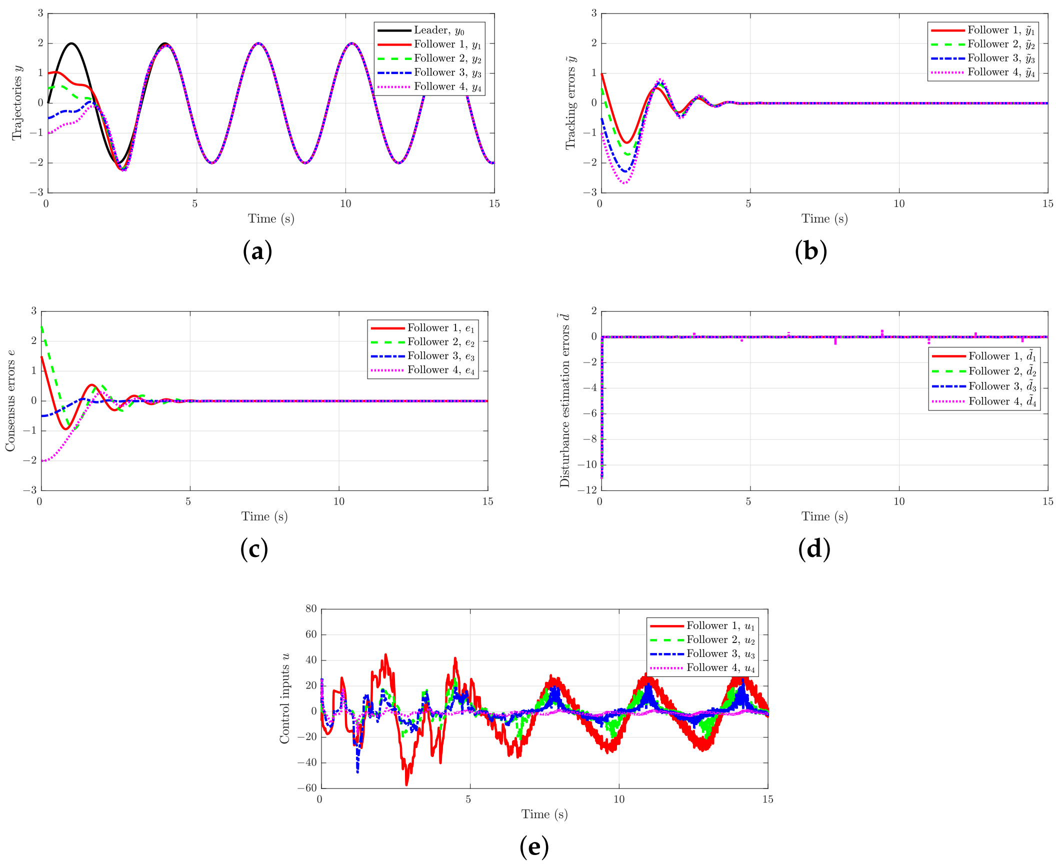

Figure 2. Simulation results under the action of the control protocol proposed in (19): (**a**) trajectories, (**b**) tracking errors, (**c**) consensus errors, (**d**) disturbance estimation errors, (**e**) control inputs.

Furthermore, to evaluate the performance advantage of the proposed control method for the consensus tracking of multi-intelligent systems in this case, a simulation comparison with the state feedback-based method developed in [56] is considered. Here, the root mean square (RMS) values of steady-state tracking errors and consensus errors are considered as key performance indicators for evaluating the control performance. The RMS value is defined as follows:

$$E_{h_i} = \sqrt{\frac{1}{M}\sum_{l=l_0}^{M}(h_i^2)}, h = \{\tilde{y}, e\}, i = \{1,2,3,4\}, \tag{103}$$

where M is the simulation step number and l_0 is the step number at the start of sampling. Figure 3 and Table 2 illustrate the simulation comparison results of tracking and consensus errors. The simulation results of the method proposed in this paper are shown as solid lines, and the results of the method in [56] are shown as dashed lines, with different colors representing different followers. It can be observed from Figure 3 that the steady-state tracking errors and consensus errors under the action of the proposed method are much more minor than the results of the state feedback-based control method in [56]. Furthermore, it is validated by the results in Table 2, and the RMS values of the errors are at least three orders of magnitude better than those of the state feedback-based control method in [56].

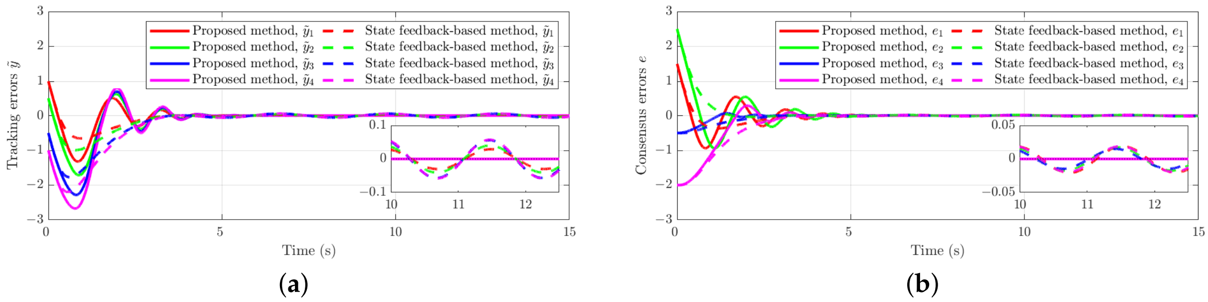

Figure 3. Comparison results of tracking errors and consensus errors between the proposed method and the state feedback-based method proposed in [56]: (**a**) tracking errors, (**b**) consensus errors.

Table 2. RMS values of tracking errors and consensus errors under different control methods (10 s $< t \leq 15$ s, i.e., M = 15,000, l_0 = 10,001).

Method	$E_{\tilde{y}_1}$	$E_{\tilde{y}_2}$	$E_{\tilde{y}_3}$	$E_{\tilde{y}_4}$	$E_{\tilde{e}_1}$	$E_{\tilde{e}_2}$	$E_{\tilde{e}_3}$	$E_{\tilde{e}_4}$
Proposed method	4.57×10^{-7}	8.87×10^{-7}	1.36×10^{-6}	1.33×10^{-6}	8.18×10^{-8}	5.72×10^{-7}	5.96×10^{-7}	5.27×10^{-7}
Control method [56] (state feedback-based)	0.0215	0.0288	0.0401	0.0413	0.0143	0.0123	0.0103	0.0138

5.2. Case 2: Intermediate States of the Multi-Agent System Are Unavailable

This example is to verify the effectiveness of the proposed finite-time adaptive consensus tracking controller based on the cascaded high-gain observer under the condition of unmeasurable intermediate states of MASs. The selection of control parameters and initial conditions are the same as that of Case 1. The parameters of the distributed cascade state observer are selected as: $\gamma_{i,1} = 2$, $\gamma_{i,2} = 1$, $\gamma_{i,3} = 1$, $M_2 = 10$, $M_3 = 5$, $\varepsilon_i = 0.005/0.001$ $(i = 1,2,3,4)$. Different parameters ε_i are selected to verify the control objective (14); that is, the smaller the parameter ε_i is, the smaller the ultimate bounds of state estimation errors, consensus errors, and tracking errors. The response results of the closed-loop system are depicted in Figures 4 and 5. We can see from Figures 4a and 5a that all followers are able to fast track the leader's trajectory. By comparing Figures 4b–d and 5b–d, it can be found that the smaller the parameter ε_i, the smaller the ultimate bounds of state estimation, tracking, and consensus errors. This conclusion is aligned with the control objective (14) and theoretical findings. The lumped disturbance estimation errors and control inputs of each follower are indicated in Figures 4e,f and 5e,f, respectively.

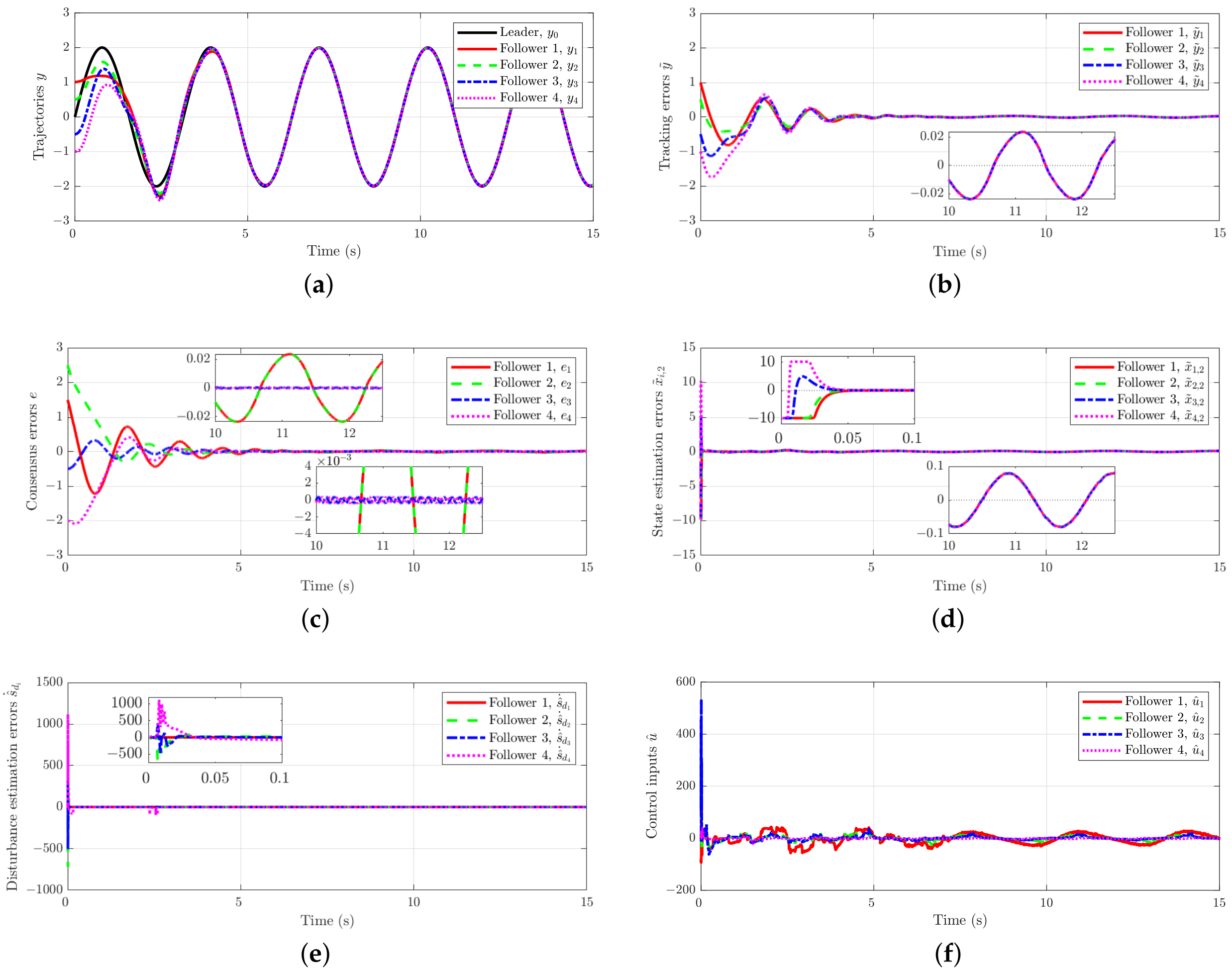

Figure 4. Simulation results under the action of the control protocol proposed in (43) with $\varepsilon_i = 0.005$: (**a**) trajectories, (**b**) tracking errors, (**c**) consensus errors, (**d**) state estimation errors, (**e**) disturbance estimation errors, (**f**) control inputs.

In this case, we also compare the simulation results of the proposed method with those of the high-gain observer-based control method in [56]. The comparison results are depicted in Figure 6 and Table 3. We can see from Figure 6 and Table 3 that the simulation results under the action of the proposed method in this paper have an advantage of at least one order of magnitude over the results of the high gain observer-based method in [56] in a steady-state performance. It should be noted that the state estimation errors under the action of the method in [56] are smaller than those under the action of the proposed method, but the transient peaking phenomenon is quite obvious. Moreover, this paper achieves a reduction in the steady-state estimation error by reducing ε_i.

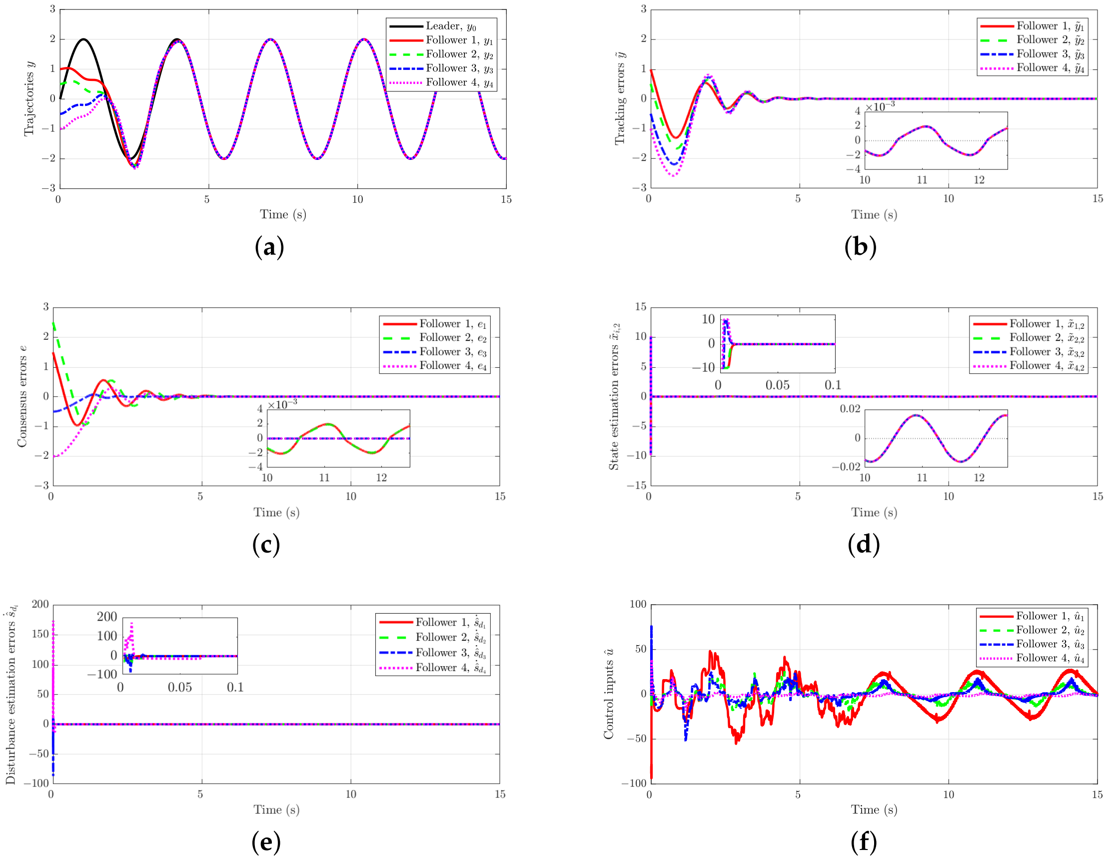

Figure 5. Simulation results under the action of the control protocol proposed in (43) with $\varepsilon_i = 0.001$: (**a**) trajectories, (**b**) tracking errors, (**c**) consensus errors, (**d**) state estimation errors, (**e**) disturbance estimation errors, (**f**) control inputs.

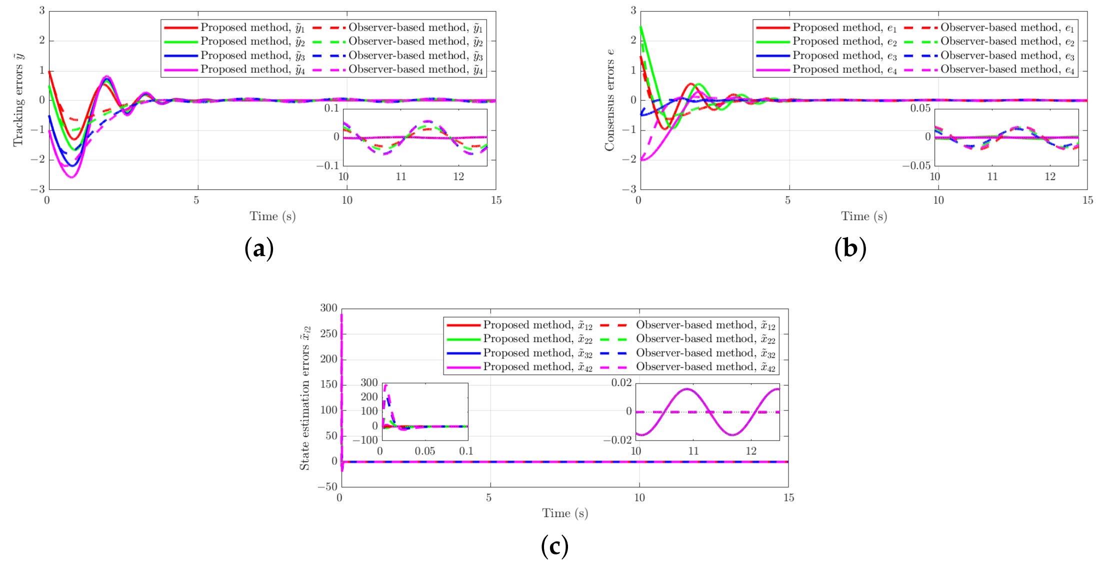

Figure 6. Comparison results of tracking errors, consensus errors and state estimation errors between the proposed method and the high-gain observer-based method proposed in [56]: (**a**) tracking errors, (**b**) consensus errors, (**c**) state estimation errors.

Table 3. RMS values of tracking errors and consensus errors under different control methods (10 s $< t \leq 15$ s, i.e., M = 15,000, l_0 = 10,001).

Method	$E_{\tilde{y}_1}$	$E_{\tilde{y}_2}$	$E_{\tilde{y}_3}$	$E_{\tilde{y}_4}$	$E_{\tilde{e}_1}$	$E_{\tilde{e}_2}$	$E_{\tilde{e}_3}$	$E_{\tilde{e}_4}$
Proposed method	0.0014	0.0014	0.0014	0.0014	0.0014	0.0014	5.74×10^{-7}	5.48×10^{-7}
Control method [56] (high-gain observer-based)	0.0214	0.0287	0.0400	0.0412	0.0144	0.0123	0.0103	0.0137

6. Conclusions

In this paper, a barrier function-based distributed finite-time adaptive control framework is designed for uncertain high-order nonlinear MASs with measurable states under a connected undirected graph. The consensus and tracking errors of the closed-loop system are asymptotically converging in a finite time. It should be noted that this control scheme does not require additional assumptions about the derivative of the lumped disturbance. By considering the unmeasurable intermediate states of MASs, a distributed cascaded high-gain observer is introduced based on the proposed control framework. Moreover, a finite-time adaptive consensus tracking control scheme is developed based on the barrier function and cascaded high-gain observer. The errors of state estimation, consensus, and tracking of the closed-loop system converge to a small bounded domain in finite time, and the size of the bound is directly proportional to the designed perturbation parameter. Two simulation cases are given to verify the effectiveness of the proposed control protocols, and the simulation results are consistent with the corresponding theoretical findings. Moreover, numerical simulation compares the proposed method with some existing methods. The results indicate that the proposed method dramatically improves the tracking control accuracy.

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