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Relation "Greater than or Equal to" between Ordered Fuzzy Numbers

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Abstract: The ordered fuzzy number (OFN) is determined as an ordered pair of fuzzy number (FN) and its orientation. FN is widely interpreted as imprecise number approximating real number. We interpret any OFN as an imprecise number equipped with additional information about the location of the approximated number. This additional information is given as orientation of OFN. The main goal of this paper is to determine the relation "greater than or equal to" on the space of all OFNs. This relation is unambiguously defined as an extension of analogous relations on the space of all FN. All properties of the introduced relation are investigated on the basis of the revised OFNs' theory. It is shown here that this relation is a fuzzy one. The relations "greater than" and "equal to" also are considered. It is proven that the introduced relations are independent on the orientation of the compared OFNs. This result makes it easier to solve optimization tasks using OFNs.

Keywords: ordered fuzzy number; fuzzy relation; preorder; strict order; equivalence relation

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1. Introduction

The concept of ordered fuzzy number (OFN) was intuitively introduced by Kosiński [1–4] as an extension of the notion of fuzzy number (FN) which is widely interpreted as imprecise approximation of real number. OFNs' usefulness follows from the fact that it is interpreted as FN with additional information about the location of the approximated number. Kosiński [1–4] has determined arithmetic for OFNs as an extension of results obtained by Goetschel and Voxman [5] for FNs. For formal reasons, the Kosiński' theory was revised [6] in such a way that revised OFN definition fully corresponds to the intuitive Kosiński's definition of OFN. OFNs are always defined without use of any ordering relation between FNs. Knowing this fact makes it easier to read the section on ordering relationship between OFNs. This paper is linked to the revised OFNs' theory.

In decision analysis, economics and finance, OFNs are frequently employed to evaluate the alternatives in modelling a real-world problem [7–19]. On the other hand, the OFN theory has an important disadvantage. This disadvantage is due to the lack of formal mathematical models associated with OFNs. Therefore, an important goal of further formal research should be to fill these theoretical gaps.

If any alternatives are evaluated by OFNs then their ranking leads to OFNs' arrangement which is pre-given as an ordering relation "greater than or equal to" between OFNs.

Since the notion of OFN is interpreted as an extension of the notion of FN, any formal model of order between OFNs should be consistent with the fixed ordering relation between FNs. Unlike in the case of real numbers, FNs have no natural order. A straightforward approach to the ordering of FNs is to convert each compared FN into a real number. Any procedure of this conversion is called a "defuzzification method" [20]. Representative examples of FNs' arrangement using different defuzzification methods are presented in [20–56]. Each individual defuzzification method, however, pays attention to a special aspect of an FN. As a consequence, each approach suffers from some defects that only one real number is associated with each FN. Freeling [57] pointed out that "by reducing the whole of our analysis to a single number, we are losing much of the information. We have purposely been keeping throughout our calculations".

Kosiński and Sztyma [58] introduced defuzzification methods for OFNs. Some applications of OFN arrangement using defuzzification methods are presented in [8,16,18,19]. On the other side, in [17], it is shown that the use of defuzzification methods has a significant impact on the ordering of OFNs. In an extreme case, the use of defuzzification procedures can totally blur the true picture of arrangement of OFNs. It can lead to results deviating from real ranking of decision alternatives, which will increase the hazard of making a wrong decision. For this reasons, OFNs arrangement should be described by a fuzzy relation which compares OFNs pairwise. In this way, we can compare OFNs without losing information about the imprecision and orientation of evaluated OFNs. This approach is more realistic.

For FNs, fuzzy order relations can be defined in two ways. First of all, fuzzy order of FNs can be determined using α-cuts. Representative examples of FNs' arrangement using α-cuts are presented in [59–61]. At present, the α-cuts theory dedicated to OFNs is unknown. Therefore, in the current moment, any formal models of ordering with use of α-cuts cannot be extended to the case of OFNs. Moreover, Orlovsky [62] defined fuzzy order of FN applying the Zadeh's Extension Principle [63–65]. This method does not raise any objections.

Therefore, the main goal of presented work is to define such fuzzy order relation between OFN's which is consistent with fuzzy order introduced by Orlovsky. Setting such a relationship is needed to build each quantitative model based on comparison of OFNs. In general, the relation $\widetilde{GE}$ can be applied in any such quantitative model of the real world that a comparison of imprecise numbers is used. The tentative approach to this subject was presented in [66]. Obtained in this way fuzzy order of OFNs is applied in [12,17]. The results presented here are the final generalized version of such fuzzy ordering OFNs that it fulfils assumed condition.

The paper is organised in the following way. Section 2 presents considered models of imprecise quantity. Section 2.1 describes the basic concepts of FNs and arithmetic operations on FNs. The revised notion of OFN and arithmetic operations on OFNs are presented in Section 2.2. It is pointed out here that OFNs are always defined without use of any ordering relation between FNs. In Section 2.3, the disorientation map is introduced. Moreover, some differences between FNs and OFNs are explained here. In Section 3 the author proves that some simple properties are fulfilled by Orlovsky's fuzzy order of FN. In Section 4 the author introduced such relation "greater than or equal to" between OFNs which is consistent with Orlovsky's fuzzy order. Section 5 contains some basic problems linked with ordering of OFNs. In Section 6, all theoretical considerations are illustrated by case study devoted to the subject of investment decisions. Finally, Section 7 contains the final remarks.

2. Imprecise Quantities—Considered Models

Objects of any considerations may be given as elements of a predefined space $\mathbb{X}$. The basic tool for imprecise classification of these elements is the notion of fuzzy set introduced by Zadeh [67]. Any fuzzy set $\mathcal{A}$ is unambiguously determined by means of its membership function $\mu_A \in [0,1]^{\mathbb{X}}$, as follows

$$\mathcal{A} = \{(x, \mu_A(x)); x \in \mathbb{X}\}. \tag{1}$$

In all our considerations we use the multivalued logic determined by Łukasiewicz [68]. The truth value of the sentence $\mathcal{P}$ will be denoted by the symbol $tv(\mathcal{P})$. From the point-view of multi-valued logic, the value $\mu_A(x)$ is interpreted as the truth value $tv($"$x \in \mathcal{A}$"$)$. By the symbol $\mathcal{F}(\mathbb{X})$ we denote the family of all fuzzy sets in the space $\mathbb{X}$. Any fuzzy set $\mathcal{A} \in \mathcal{F}(\mathbb{X})$ may be described using the following notions:

For each $\alpha \in]0,1]$, the $\alpha-$ cuts $[\mathcal{A}]_\alpha$ determined as follows

$$[\mathcal{A}]_\alpha = \{x \in \mathbb{X} : \mu_A(x) \geq \alpha\}; \tag{2}$$

The support closure $[\mathcal{A}]_{0^+}$ given in the following way

$$[\mathcal{A}]_{0^+} = \lim_{\alpha \to 0^+} [\mathcal{A}]_\alpha. \tag{3}$$

An imprecise quantity is a family of real numbers belongs to it in a different degree. In this section, the fuzzy set notion is applied for describing imprecise quantities.

2.1. Fuzzy Numbers—Some Basic Notions

A commonly used model of an imprecise number is FN, defined as a fuzzy set in real line $\mathbb{R}$. The most general definition of FN is given as follows:

The most general definition of fuzzy number is given as follows:

Definition 1 [69]. *The fuzzy number (FN) is such a fuzzy subset $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ with bounded support closure $[\mathcal{L}]_{0^+}$ that it is represented by its upper semi-continuous membership function $\mu_L \in [0;1]^{\mathbb{R}}$ satisfying the conditions:*

$$\exists_{x \in \mathbb{R}} \ \mu_L(x) = 1 \tag{4}$$

$$\forall_{(x,y,z) \in \mathbb{R}^3} \quad x \leq y \leq z \implies \mu_L(y) \geq \min\{\mu_L(x); \mu_L(z)\}. \tag{5}$$

The set of all FN we denote by the symbol $\mathbb{F}$. Let us consider any arithmetic operation $*$ defined on $\mathbb{R}$. The symbol $\circledast$ denotes an extension of arithmetic operation $*$ to $\mathbb{F}$. In [70], arithmetic operations on FN are introduced in such way that they are coherent with the Zadeh's Extension Principle. In line with it, for any pair $(\mathcal{K}, \mathcal{L}) \in \mathbb{F}^2$ represented by their membership functions $\mu_K, \mu_L \in [0,1]^{\mathbb{R}}$, the FN

$$\mathcal{M} = \mathcal{K} \circledast \mathcal{L} \tag{6}$$

is described by its membership function $\mu_M \in [0,1]^{\mathbb{R}}$ determined by means of the identity:

$$\mu_M(z) = \sup\{\min\{\mu_K(x), \mu_L(y)\} : z = x * y, (x,y) \in \mathbb{R}\}. \tag{7}$$

Thanks to the results obtained in [5], we have that any FN can be equivalently defined as follows:

Theorem 1 [71]. *For any FN $\mathcal{L}$ there exists such a non-decreasing sequence $(a,b,c,d) \subset \mathbb{R}$ that $\mathcal{L}(a,b,c,d,L_L,R_L) = \mathcal{L} \in \mathcal{F}(\mathbb{R})$ is determined by its membership function $\mu_L(\cdot|a,b,c,d,L_L,R_L) \in [0,1]^{\mathbb{R}}$ described by the identity*

$$\mu_L(x|a,b,c,d,L_L,R_L) = \begin{cases} 0, & x \notin [a,d], \\ L_L(x), & x \in [a,b], \\ 1, & x \in [b,c], \\ R_L(x), & x \in [c,d], \end{cases} \tag{8}$$

where the left reference function $L_L \in [0,1]^{[a,b]}$ and the right reference function $R_L \in [0,1]^{[c,d]}$ are upper semi-continuous monotonic ones meeting the conditions:

$$L_L(b) = R_L(c) = 1, \tag{9}$$

$$[\mathcal{L}]_{0^+} = [a,d]. \tag{10}$$

The FN $\mathcal{L}(a,a,a,a,L_L,R_L) = [\![a]\!]$ represents the real number $a \in \mathbb{R}$. Therefore, we can say $\mathbb{R} \subset \mathbb{F}$. For any $z \in [b,c]$, a FN $\mathcal{L}(a,b,c,d,L_L,R_L)$ is a formal model of linguistic variable "about z". Understanding the phrase "about z" depends on the applied pragmatics of the natural language. Let us note that FN may be replaced by generalized FN [72] which does not meet the condition (4).

In line with the identity (7), the unary minus operator "−" on $\mathbb{R}$ is extended to the minus operator $\ominus$ on $\mathbb{F}$ by the identity

$$\ominus \mathcal{L}(a,b,c,d,L_L,R_L) = \mathcal{L}\Big(-d,-c,-b,-a,R_L^{(-)},\ L_L^{(-)}\Big), \tag{11}$$

where

$$R_L^{(-)}(x) = R_L(-x), \tag{12}$$

$$L_L^{(-)}(x) = L_L(-x). \tag{13}$$

In further considerations, we will use the following concepts.

Definition 2. For *any upper semi-continuous non-decreasing function* $L \in [0,\ 1]^{[u,\ v]}$, *its cut-function* $L^\star \in [u,v]^{[0;1]}$ *is determined by the identity*

$$L^\star(\alpha) = \min\{x \in [u,v] :\ L(x) \ge \alpha\}. \tag{14}$$

Definition 3. *For any upper semi-continuous non-increasing function* $R \in [0,\ 1]^{[u,\ v]}$ *its cut-function* $R^\star \in [0,\ 1]^{[u,\ v]}$ *is determined by the identity*

$$R^\star(\alpha) = \max\{x \in [u,v] :\ R(x) \ge \alpha\}. \tag{15}$$

Definition 4. For *any bounded continuous and non-decreasing function* $l \in [l(0),\ l(1)]^{[0,1]}$ *its pseudo inverse* $l^\triangleleft \in [0,1]^{[l(0),\ l(1)]}$ *is determined by the identity*

$$l^\triangleleft(x) = \max\{\alpha \in [0,1] : l(\alpha) = x\}. \tag{16}$$

Definition 5. *For any bounded continuous and non-increasing function* $r \in [r(0),\ r(1)]^{[0,1]}$ *its pseudo inverse* $r^\triangleleft \in [0;1]^{[r(1),\ r(0)]}$ *of is determined by the identity*

$$r^\triangleleft(x) = \min\{\alpha \in [0,1] : r(\alpha) = x\}. \tag{17}$$

In reference [5], it is proved that FNs' sum $\oplus$ is given by the identity

$$\mathcal{L}\Big(a+e, b+f, c+g, d+h, L_J, R_J\Big) = \mathcal{L}(a,b,c,d,L_K,R_K) \oplus \mathcal{L}(e,f,g,h,L_M,R_M), \tag{18}$$

where

$$\forall_{\alpha\in[0,1]}\ l_J(\alpha) = L_K^\star(\alpha) + L_M^\star(\alpha) \tag{19}$$

$$\forall_{\alpha\in[0,1]}\ r_J(\alpha) = R_K^\star(\alpha) + R_M^\star(\alpha) \tag{20}$$

$$\forall_{x\in[a+e,b+f]}\ L_J(x) = l_J^\triangleleft(x), \tag{21}$$

$$\forall_{x\in[c+g,d+h]}\ R_J(x) = r_J^\triangleleft(x). \tag{22}$$

The difference $\ominus$ between FNs is determined in determined in the following way

$$\mathcal{L}(a,b,c,d,L_K,R_K)\ominus\mathcal{L}(e,f,g,h,L_M,R_M)=\mathcal{L}(a,b,c,d,L_K,R_K)\oplus(\ominus\mathcal{L}(e,f,g,h,L_M,R_M)). \tag{23}$$

Then identities (11)–(13) and (18)–(23) imply that

$$\mathcal{L}(a-h,b-g,c-f,d-e,L_W,R_W)=\mathcal{L}(a,b,c,d,L_K,R_K)\ominus\mathcal{L}(e,f,g,h,L_M,R_M), \tag{24}$$

where

$$\forall_{\alpha\in[0,1]}\ l_W(\alpha)=L_K^{\star}(\alpha)-R_M^{\star}(\alpha) \tag{25}$$

$$\forall_{\alpha\in[0,1]}\ r_W(\alpha)=R_K^{\star}(\alpha)-L_M^{\star}(\alpha) \tag{26}$$

$$\forall_{x\in[a-h,b-g]}\ L_W(x)=l_W^{\triangleleft}(x), \tag{27}$$

$$\forall_{x\in[c-f,d-e]}\ R_W(x)=r_W^{\triangleleft}(x). \tag{28}$$

The above arithmetic operators may be generalized to the case of intuitionistic FNs [73]. On the other hand, the dependencies (18)–(28) are not met for discrete FNs [74]. All above identities show a high complexity of arithmetic operations on the space $\mathbb{F}$. Due to that, in many practical applications researchers limit the use of FNs only to their kind distinguished below [75].

Definition 6. *For any non-decreasing sequence* $(a,b,c,d)\subset\mathbb{R}$*, a trapezoidal FN (TrFN) is the FN* $\mathcal{T}=Tr(a,b,c,d)\in\mathbb{F}$ *defined by its membership functions* $\mu_T\in[0,1]^{\mathbb{R}}$ *in the following way*

$$\mu_T(x)=\mu_{Tr}(x|a,b,c,d)=\begin{cases}0, & x\notin[a,d],\\ \frac{x-a}{b-a}, & x\in[a,b[,\\ 1, & x\in[b,c],\\ \frac{x-d}{c-d}, & x\in\,]c,d].\end{cases} \tag{29}$$

The space of all TrFNs is denoted by the symbol $\mathbb{F}_{Tr}$. For any TrFN we have

$$Tr(-d,-c,-b,-a)=\ominus Tr(a,b,c,d) \tag{30}$$

$$Tr(a+e,b+f,c+g,d+h)=Tr(a,b,c,d)\oplus Tr(e,f,g,h), \tag{31}$$

$$Tr(a-h,b-g,c-f,d-e)=Tr(a,b,c,d)\ominus Tr(e,f,g,h). \tag{32}$$

2.2. Ordered Fuzzy Numbers—Some Basic Facts

The notion of OFN is intuitively introduced by Kosiński [1–4], as such model of imprecise number that subtraction of OFNs is the inverse operator to addition of OFNs. Therefore, OFNs can contribute to specific problems concerning the solution of fuzzy linear equations of the form or help with the interpretation of specific improper fuzzy arithmetic results.

An important disadvantage of Kosiński's theory is that there exist such OFNs which are not linked to any membership function [4]. For this reason, the Kosiński's theory is revised in [6] where OFNs are defined as follows:

Definition 7 [6]. *For any monotonic sequence* $(a,b,c,d) \subset \mathbb{R}$*, the ordered fuzzy number OFN* $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L) = \overleftrightarrow{\mathcal{L}}$ *is the pair of orientation* $\overrightarrow{a,d} = (a,d)$ *and fuzzy set* $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ *described by membership function* $\mu_L(\cdot|a,b,c,d,S_L,E_L) \in [0,1]^{\mathbb{R}}$ *given by the identity*

$$\mu_L(x|a,b,c,d,S_L,E_L) = \begin{cases} 0, & x \notin [a,d] \equiv [d,a], \\ S_L(x), & x \in [a,b] \equiv [b,a], \\ 1, & x \in [b,c] \equiv [c,b], \\ E_L(x), & x \in [c,d] \equiv [d,c]. \end{cases} \tag{33}$$

where the starting function $S_L \in [0,1]^{[a,b]}$ *and the ending function* $E_L \in [0,1]^{[c,d]}$ *are upper semi-continuous monotonic ones meeting the conditions (6) and*

$$S_L(b) = E_L(c) = 1 \tag{34}$$

The identity (33) additionally describes such modified notation of numerical intervals which is applied in this work.

Discussion about the terminology: We see above that the notion of “ordered fuzzy number” is defined without applying any ordering relation between FNs. In original Kosińki's works “ordered fuzzy number” is also defined without use of any ordering relation between FN. In each of these cases, “ordered fuzzy number” is defined as FN completed by orientation. Therefore, in my opinion term “ordered fuzzy number” should be replaced by the term “oriented fuzzy number”. The following premises support such a proposal for change:

- Any discussion about the ordering of “oriented fuzzy numbers” is clearer than a discussion of ordering of “ordered fuzzy numbers”.
- Professor Kosinski's mother language is Polish. In Polish OFNs is called “skierowana liczba rozmyta”. This term was proposed by Professor Kosiński. Against, the quoted Polish term is translated into English as “oriented fuzzy number” or “directed fuzzy number”. Moreover, the English term “ordered fuzzy number” is translated into Polish as “uporządkowana liczba rozmyta”. All this allows us to state that the meanings of the Polish term “skierowana liczba rozmyta” and the English term “ordered fuzzy number” are different.

“Ordered fuzzy numbers” are the most important work of life for Professor Kosiński. Therefore, the proposed change to the term OFN should be discussed with him. Because of Professor Kosiński passed away, this is not possible. Therefore, I agree with other scientists [76,77] that the OFN may be called the “Kosiński's number”. Future scientific discussion will allow us to choose a “oriented fuzzy number” or “Kosiński number” or another term. Today we still use the term “ordered fuzzy number”. No less in this work, the abbreviation OFN can be read “ordered fuzzy number” or “oriented fuzzy numbers”. The use of the second term makes easier to read the section on the ordering relationship between OFNs.

The symbol $\mathbb{K}$ denotes the space of all OFNs. Any OFN describes an imprecise number with additional information about the location of the approximated number. This information is given as orientation of OFN. If $a < d$ then OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ has the positive orientation $\overrightarrow{a,d}$. For any $z \in [b,c]$, the positively oriented OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ is a formal model of linguistic variable “about or slightly above z”. The symbol $\mathbb{K}^+$ denotes the space of all positively oriented OFN. If $a > d$, then OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ has the negative orientation $\overrightarrow{a,d}$. For any $z \in [c,b]$, the negatively oriented TrOFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ is a formal model of linguistic variable “about or slightly below z”. The symbol $\mathbb{K}^-$ denotes the space of all negatively oriented OFN. Understanding the phrases “about or slightly above

z" and "about or slightly below z" depend on the applied pragmatics of the natural language. If $a = d$, OFN $\overleftrightarrow{\mathcal{L}}(a,a,a,a,S_L,E_L) = [\![a]\!]$ describes unoriented number $a \in \mathbb{R}$. Summing up, we see that

$$\mathbb{K} = \mathbb{K}^+ \cup \mathbb{R} \cup \mathbb{K}^-. \tag{35}$$

The minus operator "–" on $\mathbb{R}$ is extended by Kosiński [4] to the minus operator $\boxminus$ on $\mathbb{K}$ by means of the identity

$$\boxminus \overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L) = \overleftrightarrow{\mathcal{L}}\Big(-a,-b,-c,-d,S_L^{(-)},\ E_L^{(-)}\Big), \tag{36}$$

where

$$S_L^{(-)}(x) = S_L(-x) \tag{37}$$

$$E_L^{(-)}(x) = E_L(-x) \tag{38}$$

Kosiński [1] defines the addition operator $\boxplus_K$ on $\mathbb{K}$ as the extension of operator $\oplus$ from $\mathbb{F}$ to $\mathbb{K}$. This extension is determined by extension the domain identities (18)–(22) from $\mathbb{F}$ to $\mathbb{K}$. In this way, Kosiński defines addition of OFNs as an extension of results obtained by Goetschel and Voxman [5] for addition of FNs. Moreover, Kosiński [4] have shown that there exist such OFNs that their sum $\boxplus_K$ does not exist. For this reason, Kosiński's operator $\boxplus_K$ is replaced by addition operator $\boxplus$ defined on $\mathbb{K}$ by the identity [6]

$$\overleftrightarrow{\mathcal{L}}(a_K,b_K,c_K,d_K,S_K,E_K) \boxplus \overleftrightarrow{\mathcal{L}}(a_M,b_M,c_M,d_M,S_M,E_M) = \overleftrightarrow{\mathcal{J}} = \overleftrightarrow{\mathcal{L}}\big(a_J,b_J,c_J,d_J,S_J,E_J\big), \tag{39}$$

where we have

$$\check{a}_J = a_K + a_M, \tag{40}$$

$$b_J = b_K + b_M, \tag{41}$$

$$c_J = c_K + c_M, \tag{42}$$

$$\check{d}_J = d_K + d_M, \tag{43}$$

$$a_J = \begin{cases} \min\{\check{a}_J, b_J\}, & (b_J < c_J) \vee (b_J = c_J \wedge \check{a}_J \le \check{d}_J), \\ \max\{\check{a}_J, b_J\}, & (b_J > c_J) \vee (b_J = c_J \wedge \check{a}_J > \check{d}_J), \end{cases} \tag{44}$$

$$d_J = \begin{cases} \max\{\check{d}_J, c_J\}, & (b_J < c_J) \vee (b_J = c_J \wedge \check{a}_J \le \check{d}_J), \\ \min\{\check{d}_J, c_J\}, & (b_J > c_J) \vee (b_J = c_J \wedge \check{a}_J > \check{d}_J), \end{cases} \tag{45}$$

$$\forall_{\alpha\in[0;1]}\ s_J(\alpha) = \begin{cases} S_K^\star(\alpha) + S_M^\star(\alpha), & a_J \ne b_J, \\ b_J, & a_J = b_J. \end{cases} \tag{46}$$

$$\forall_{\alpha\in[0;1]}\ e_J(\alpha) = \begin{cases} E_K^\star(\alpha) + E_M^\star(\alpha), & c_J \ne d_J, \\ c_J, & c_J = d_J. \end{cases} \tag{47}$$

$$\forall_{x\in[a_J,b_J]}\ S_J(x) = s_J^{\triangleleft}(x), \tag{48}$$

$$\forall_{x\in[c_J,d_J]}\ E_J(x) = e_J^{\triangleleft}(x). \tag{49}$$

In [6], the definition of addition operator $\boxplus$ is justified in detail. Then, difference $\boxminus$ between OFNs is given as follows

$$\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_K,E_K) \boxminus \overleftrightarrow{\mathcal{L}}(e,f,g,h,S_M,E_M) = \overleftrightarrow{\mathcal{L}}(a,b,c,d,L_K,R_K) \boxplus \Big(\boxminus\overleftrightarrow{\mathcal{L}}(e,f,g,h,S_M,E_M)\Big). \tag{50}$$

In [1,6], it is shown that for any $\overleftrightarrow{\mathcal{L}} \in \mathbb{K}$ we have

$$\overleftrightarrow{\mathcal{L}} \boxplus_K (\boxminus\overleftrightarrow{\mathcal{L}}) = [\![0]\!] = \overleftrightarrow{\mathcal{L}} \boxminus \overleftrightarrow{\mathcal{L}}. \tag{51}$$

We see that subtraction is inverse operator for both addition operators $\boxplus_K$ and $\boxplus$. We can say that OFNs meet the intuitive postulate put forward by Kosiński.

Due to high complexity of arithmetic operations of OFN, in many practical applications researchers limit the use of OFNs only to their kind distinguished below.

Definition 8 [6]. *For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$, the trapezoidal OFN (TrOFN) $\overleftrightarrow{Tr}(a, b, c, d) = \overleftrightarrow{\mathcal{T}}$ is the pair of the orientation $\overrightarrow{a, d} = (a, d)$ and fuzzy set $\mathcal{T} \in \mathcal{F}(\mathbb{R})$ determined explicitly by its membership functions $\mu_T \in [0, 1]^{\mathbb{R}}$ as follows*

$$\mu_T(x) = \mu_{Tr}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d] \equiv [d, a], \\ \frac{x-a}{b-a}, & x \in [a, b[\; \equiv \;]b, a], \\ 1, & x \in [b, c] \equiv [c, b], \\ \frac{x-d}{c-d}, & x \in \;]c, d] \equiv [d, c[. \end{cases} \tag{52}$$

The symbol $\mathbb{K}_{Tr}$ denotes the space of all TrOFNs. Identity (36) implies that the minus operator $\boxminus$ on $\mathbb{K}_{Tr}$ is given by the identity

$$\overleftrightarrow{Tr}(-a, -b, -c, -d) = \boxminus \overleftrightarrow{Tr}(a, b, c, d). \tag{53}$$

In line with (39), the sum of TrOFNs is determined as follows

$$\begin{aligned} &\overleftrightarrow{Tr}(a, b, c, d) \boxplus \overleftrightarrow{Tr}(p-a, q-b, r-c, s-d) = \\ &= \begin{cases} \overleftrightarrow{Tr}(\min\{p, q\}, q, r, \max\{r, s\}), & (q < r) \vee (q = r \wedge p \leq s), \\ \overleftrightarrow{Tr}(\max\{p, q\}, q, r, \min\{r, s\}), & (q > r) \vee (q = r \wedge p > s). \end{cases} \end{aligned} \tag{54}$$

Then the difference $\boxminus$ between TrOFNs is the TrOFN given as follows

$$\begin{aligned} &\overleftrightarrow{Tr}(a, b, c, d) \boxminus \overleftrightarrow{Tr}(a-p, b-q, c-r, d-s) = \\ &= \begin{cases} \overleftrightarrow{Tr}(\min\{p, q\}, q, r, \max\{r, s\}) & (q < r) \vee (q = r \wedge p \leq s) \\ \overleftrightarrow{Tr}(\max\{p, q\}, q, r, \min\{r, s\}) & (q > r) \vee (q = r \wedge p > s). \end{cases} \end{aligned} \tag{55}$$

2.3. Ordered Fuzzy Numbers vs. Fuzzy Numbers

For the case $a \geq d$ the membership function of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is equal to the membership function of FN $\mathcal{L}(a, b, c, d, S_L, E_L)$. This fact implies the existence of isomorphism $\Psi : (\mathbb{K}^+ \cup \mathbb{R}) \to \mathbb{F}$ given by the identity

$$\mathcal{L}(a, b, c, d, S_L, E_L) = \Psi\Big(\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)\Big). \tag{56}$$

This isomorphism may be extended to the space $\mathbb{K}$ by disorientation map $\overline{\overline{\Psi}} : \mathbb{K} \to \mathbb{F}$ given as follows

$$\overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{L}}) = \begin{cases} \Psi(\overleftrightarrow{\mathcal{L}}) & \overleftrightarrow{\mathcal{L}} \in \mathbb{K}^+ \cup \mathbb{R}, \\ \ominus\Psi(\boxminus\overleftrightarrow{\mathcal{L}}) & \overleftrightarrow{\mathcal{L}} \in \mathbb{K}^-. \end{cases} \tag{57}$$

Let us note, that the disorientation map $\overline{\overline{\Psi}} : \mathbb{K} \to \mathbb{F}$ may be equivalently defined by the identity

$$\overline{\overline{\Psi}}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) = \begin{cases} \mathcal{L}(a,b,c,d,S_L,E_L) & \overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L) \in \mathbb{K}^+ \cup \mathbb{R}, \\ \mathcal{L}(d,c,b,a,E_L,S_L) & \overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L) \in \mathbb{K}^-. \end{cases} \tag{58}$$

Example 1. *Let us consider the OFN* $\overleftrightarrow{\mathcal{X}} = \overleftrightarrow{\mathcal{L}}(12,14,18,20,S_X,E_X)$*, where*

$$\mu_X(x) = \begin{cases} 0, & x \notin [12,20], \\ S_X(x), & x \in [12,14], \\ 1, & x \in [14,18], \\ E_X(x), & x \in [18,20], \end{cases} = \begin{cases} 0, & x \notin [12,20], \\ \frac{2 \cdot x - 24}{x - 10}, & x \in [12,14], \\ 1, & x \in [14,18], \\ \frac{7 \cdot x - 140}{2 \cdot x - 50}, & x \in [18,20]. \end{cases} \tag{59}$$

and the OFN $\overleftrightarrow{\mathcal{Y}} = \overleftrightarrow{\mathcal{L}}(13,11,6,5,\mathrm{S_Y},\mathrm{E_Y})$*, where*

$$\mu_Y(x) = \begin{cases} 0, & x \notin [13,5], \\ S_Y(x), & x \in [13,11], \\ 1, & x \in [11,6], \\ E_Y(x), & x \in [6,5], \end{cases} = \begin{cases} 0, & x \notin [13,5], \\ \frac{6 \cdot x - 30}{x}, & x \in [13,11], \\ 1, & x \in [11,6], \\ \frac{2 \cdot x - 26}{x - 15}, & x \in [6,5]. \end{cases} \tag{60}$$

Since $\overleftrightarrow{\mathcal{X}} \in \mathbb{K}^+$*, using (57) we get*

$$\overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{X}}) = \overline{\overline{\Psi}}\left(\overleftrightarrow{\mathcal{L}}(12,14,18,20,S_X,E_X)\right) = \mathcal{L}(12,14,18,20,S_X,E_X) = \mathcal{L}(12,14,18,20,L_U,R_U) = \mathcal{U}, \tag{61}$$

where FN $\mathcal{U}$ *is explicitly determined by the following membership function*

$$\mu_U(x) = \begin{cases} 0, & x \notin [12,20], \\ L_U(x), & x \in [12,14], \\ 1, & x \in [14,18], \\ R_U(x), & x \in [18,20], \end{cases} = \begin{cases} 0, & x \notin [12,20], \\ \frac{2 \cdot x - 24}{x - 10}, & x \in [12,14], \\ 1, & x \in [14,18], \\ \frac{7 \cdot x - 140}{2 \cdot x - 50}, & x \in [18,20]. \end{cases} \tag{62}$$

Because $\overleftrightarrow{\mathcal{Y}} \in \mathbb{K}^-$*, using (57) we get*

$$\overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{Y}}) = \overline{\overline{\Psi}}\left(\overleftrightarrow{\mathcal{L}}(13,11,6,5,S_Y,E_Y)\right) = \mathcal{L}(5,6,11,13,E_Y,S_Y) = \mathcal{L}(5,6,11,13,L_V,R_V) = \mathcal{V}, \tag{63}$$

where FN $\mathcal{V}$ *is described by the membership function*

$$\mu_V(x) = \begin{cases} 0, & x \notin [5,13], \\ L_V(x), & x \in [5,6], \\ 1, & x \in [6,11], \\ R_V(x), & x \in [11,13], \end{cases} = \begin{cases} 0, & x \notin [5,13], \\ \frac{6 \cdot x - 30}{x}, & x \in [5,6], \\ 1, & x \in [6,11], \\ \frac{2 \cdot x - 26}{x - 15}, & x \in [11,13]. \end{cases} \tag{64}$$

The above example shows that the disorientation map is a simple transformation the space $\mathbb{K}$ on the space $\mathbb{F}$. This simplicity is apparent. It follows from the fact that the arithmetic operations on $\mathbb{F}$ are consistent with the Zadeh's Extension Principle when the arithmetic operations on $\mathbb{K}$ are not consistent with this principle. The main difficulties arise from the difference between the definition (11)–(13) of minus operator $\ominus$ on $\mathbb{F}$ and the definition (36)–(38) of minus operator $\boxminus$ on $\mathbb{K}$.

Let us compare the semigroups $\langle \mathbb{F}, \oplus \rangle$ and $\langle \mathbb{K}, \boxplus \rangle$. The identities (18)–(22) and (39)–(46) imply that the number $[\![0]\!]$ is the identity element in both these semigroups.

In [6], it is shown that addition $\boxplus$ is not associative. It implies that semigroup $\langle \mathbb{K}, \boxplus \rangle$ is not group. Moreover, the identity (51) implies that subtraction $\boxminus$ is the inverse operator to addition $\boxplus$.

The identity (18–22) implies that the addition $\oplus$ is associative. On the other hand, for any TrFN $\mathcal{T} = Tr(a,b,c,d) \in (\mathbb{F}_{Tr} \backslash \mathbb{R}) \subset \mathbb{F}$ we have

$$\mathcal{T} \ominus \mathcal{T} = Tr(a-d, b-c, c-b, d-a) \neq [\![0]\!]. \tag{65}$$

It shows that subtraction $\ominus$ is not inverse operator to addition $\oplus$. It proves that semigroup $\langle \mathbb{F}, \oplus \rangle$ is not group.

All above simple conclusions imply that:

- additive semigroup $\langle \mathbb{F}, \oplus \rangle$ and additive semigroup $\langle \mathbb{K}, \boxplus \rangle$ cannot be considered as homomorphic algebraic structures;
- any theorems on FNs cannot automatically extended to the case of OFNs.

3. Relation "Greater than or Equal to" for Fuzzy Numbers

We consider the pair $(\mathcal{K}, \mathcal{L}) \in \mathbb{F}^2$ of FNs determined by membership functions $\mu_K, \mu_L \in [0,1]^{\mathbb{R}}$. On the space $\mathbb{F}$, we can consider the relation $\mathcal{K}.GE.\mathcal{L}$, which reads:

$$\text{“}FN\ \mathcal{K}\ is\ greater\ than\ or\ equal\ to\ FN\ \mathcal{L}.\text{”} \tag{66}$$

Orlovsky [61] shows that in agreement with the Zadeh's Extension Principle, this relation is a fuzzy preorder $[GE] \in \mathcal{F}\left(\mathbb{F}^2\right)$ described by membership function $\nu_{[GE]} \in [0,1]^{\mathbb{F}^2}$ determined as follows

$$\nu_{[GE]}(\mathcal{K}, \mathcal{L}) = \sup\{\min\{\mu_K(x), \mu_L(y)\} : x \geq y\}. \tag{67}$$

From the multivalued logic point of view, the value $\nu_{[GE]}(\mathcal{K}, \mathcal{L})$ is considered as a truth-value of the sentence (66). It means that we have

$$\nu_{[GE]}(\mathcal{K}, \mathcal{L}) = tv(\text{“}\mathcal{K}.GE.\mathcal{L}\text{”}). \tag{68}$$

We prove that the fuzzy preorder $[GE] \in \mathcal{F}\left(\mathbb{F}^2\right)$ fulfils the following well-known properties.

Theorem 2. *For any pair $(\mathcal{K}, \mathcal{L}) \in \mathbb{F}^2$, we have:*

$$\nu_{[GE]}(\mathcal{K}, \mathcal{L}) = \nu_{[GE]}(\ominus\mathcal{L}, \ominus\mathcal{K}), \tag{69}$$

$$\nu_{[GE]}(\mathcal{K}, \mathcal{L}) = \nu_{[GE]}(\mathcal{K} \ominus \mathcal{L}, [\![0]\!]). \tag{70}$$

Proof. Take into account the quadruple $(\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}) \in \mathbb{F}^4$ of FNs represented respectively by their membership functions $\mu_K, \mu_L, \mu_M, \mu_N \in [0,1]^{\mathbb{R}}$.

Let us assume that $\mathcal{M} = \ominus\mathcal{K}$ and $\mathcal{N} = \ominus\mathcal{L}$. Using the identities (11), (12), and (13) we obtain:

$$\mu_M(y) = \mu_K(-y) \text{ and } \mu_N(x) = \mu_L(-x).$$

Then the identity (67) implies

$$\begin{gathered}\nu_{[GE]}(\ominus\mathcal{L}, \ominus\mathcal{K}) = \nu_{[GE]}(\mathcal{N}, \mathcal{M}) = \sup\{\min\{\mu_N(x), \mu_M(y)\} : x \geq y\} = \\ = \sup\{\min\{\mu_L(-x), \mu_K(-y)\} : -x \leq -y\} = \sup\{\min\{\mu_L(u), \mu_K(v)\} : u \leq v\} = \nu_{[GE]}(\mathcal{K}, \mathcal{L}).\end{gathered}$$

Let us assume now that $\mathcal{M} = \mathcal{K} \ominus \mathcal{L}$. Using the identity (7) we obtain:

$$\mu_M(z) = \sup\{\min\{\mu_K(x), \mu_L(y)\} : z = x - y, (x, y) \in \mathbb{R}\}.$$

Then the identity (67) implies

$$\begin{gathered} \nu_{[GE]}(\mathcal{K} \ominus \mathcal{L}, [\![0]\!]) = \nu_{[GE]}(\mathcal{M}, [\![0]\!]) = \sup\{\mu_M(z) : z \geq 0\} = \\ = \sup\{\sup\{\min\{\mu_K(x), \mu_L(y)\} : z = x - y, (x, y) \in \mathbb{R}\} : z \geq 0\} = \\ = \sup\{\min\{\mu_K(x), \mu_L(y)\} : x - y \geq 0\} = \nu_{[GE]}(\mathcal{K}, \mathcal{L}). \text{ QED} \end{gathered}$$

Theorem 3. *For any FNs $\mathcal{L}(a, b, c, d, L_K, R_K)$, $\mathcal{L}(e, f, g, h, L_M, R_M) \in \mathbb{F}$ we have*

$$\nu_{[GE]}(\mathcal{L}(a, b, c, d, L_K, R_K),\ \mathcal{L}(e, f, g, h, L_M, R_M)) = \begin{cases} 0 & 0 < d - e, \\ R_W(0) & d - e \leq 0 < c - f, \\ 1 & 0 \leq c - f, \end{cases} \tag{71}$$

where the function $R_W : [d - e, c - f] \to [0, 1]$ is given by identity (28).

Proof. For $e > d$, using (67) and (8) we get

$$\begin{gathered} \nu_{[GE]}(\mathcal{L}(a, b, c, d, L_K, R_K),\ \mathcal{L}(e, f, g, h, L_M, R_M)) = \sup\{\min\{\mu_K(x), \mu_M(y)\} : x \geq y\} = \\ = \max\{\sup\{\min\{0, \mu_M(x)\} : x \geq y \geq e\}, \sup\{\min\{\mu_K(x), 0\} : x \geq y \in]e, -\infty[\}\} = 0. \end{gathered} \tag{72}$$

For $c \geq f$ we have

$$\begin{gathered} 1 \geq \nu_{[GE]}(\mathcal{L}(a, b, c, d, L_K, R_K),\ \mathcal{L}(e, f, g, h, L_M, R_M)) = \sup\{\min\{\mu_K(x), \mu_M(y)\} : x \geq y\} \geq \\ \geq \sup\{\min\{\mu_K(x), \mu_M(y)\} : c \geq x \geq y \geq f\} = \sup\{\min\{1, 1\}\} = 1. \end{gathered} \tag{73}$$

For $d \leq e$ and $f < c$ we have $d - e \leq 0 < c - f$. Then from (24), (67) and (70) we obtain

$$\begin{gathered} \nu_{[GE]}(\mathcal{L}(a, b, c, d, L_K, R_K),\ \mathcal{L}(e, f, g, h, L_M, R_M)) = \nu_{[GE]}(\mathcal{L}(a - h, b - g, c - f, d - e, L_W, R_W), [\![0]\!]) = \\ = R_W(0). \text{ QED} \end{gathered} \tag{74}$$

Example 2. *Let us take into account the FNs $\mathcal{U} = \mathcal{L}(12, 14, 18, 20, L_U, R_U)$ and $\mathcal{V} = \mathcal{L}(5, 6, 11, 13, L_V, R_V)$ respectively determined by identities (62) and (64). We compare these FNs with using of fuzzy preorder $[GE] \in \mathcal{F}\left(\mathbb{F}^2\right)$. We have here*

$$-1 = 12 - 13 \leq 0 \leq 14 - 11 = 3.$$

Therefore, we should establish the variability of the function $R_W \in [0, 1]^{[-1,3]}$ determined by the identity (28). First, by using identities (14) and (15), we assign functions

$$L_U^\star(\alpha) = \min\{x \in [12, 14] :\ L_U(x) \geq \alpha\} = L_U^{-1}(\alpha) = \frac{10\alpha - 24}{\alpha - 2}, \tag{75}$$

$$R_V^\star(\alpha) = \max\{x \in [11, 15] :\ R_V(x) \geq \alpha\} = R_V^{-1}(\alpha) = \frac{15\alpha - 26}{\alpha - 2}. \tag{76}$$

In the next step, applying (25) and (27), we obtain

$$r_W(\alpha) = R_V^\star(\alpha) - L_U^\star(\alpha) = \frac{15\alpha - 26}{\alpha - 2} - \frac{10\alpha - 24}{\alpha - 2} = \frac{5\alpha - 2}{\alpha - 2}, \tag{77}$$

$$R_W(x) = r_W^{\triangleleft}(x) = \min\{\alpha \in [0; 1] : l_W(\alpha) = x\} = l_W^{-1}(x) = \frac{2 \cdot (x - 1)}{x - 5}. \tag{78}$$

Finally, using identity (71), we get

$$\nu_{[GE]}(\mathcal{V},\mathcal{U}) = R_W(0) = \frac{2}{5}. \tag{79}$$

The above example together with Theorem 3 shows that fuzzy preorder $[\mathrm{GE}] \in \mathcal{F}\left(\mathbb{F}^2\right)$ depends only on the interaction between the right reference function of the first compared FN and the left reference function of the second compared FN.

Moreover, Theorem 3 immediately implies that for any TrFNs we have:

Theorem 4. *For any TrFNs $Tr(a,b,c,d),\ Tr(e,f,g,h) \in \mathbb{F}_{Tr}$ we have*

$$\nu_{[GE]}(Tr(a,b,c,d),\ Tr(e,f,g,h)) = \begin{cases} 0, & 0 < d-e, \\ \frac{d-e}{d+f-c-e}, & d-e \le 0 < c-f, \\ 1, & 0 \le c-f. \end{cases} \tag{80}$$

4. Relation "Greater than or Equal to" for Ordered Fuzzy Numbers

Let us consider the pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in \mathbb{K}^2$ represented by the pair $(\mu_K,\mu_L) \in \left([0,1]^{\mathbb{R}}\right)^2$ of their membership functions. On the space $\mathbb{K}$, we introduce the relation $\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\overleftrightarrow{\mathcal{L}}$, which reads:

$$\text{“OFN } \overleftrightarrow{\mathcal{K}} \text{ is greater than or equal to OFN } \overleftrightarrow{\mathcal{L}}.\text{”} \tag{81}$$

This relation is a fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ defined by its membership function $\nu_{GE} \in [0,1]^{\mathbb{K}^2}$. From the point view of the multivalued logic, the value $\nu_{GE}(\mathcal{K},\mathcal{L})$ is considered as a truth-value of the sentence (81). It means that we have

$$\nu_{GE}(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \boldsymbol{tv}(\text{“}\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\overleftrightarrow{\mathcal{L}}\text{”}). \tag{82}$$

The fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ cannot be determined with use of the Zadeh's Extension Principle because of this principle is not valid for OFNs. Therefore, we additionally assume that any membership function $\nu_{GE} \in [0,1]^{\mathbb{K}^2}$ meets the following well-known conditions:

- for any pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R})^2$ the extension principle

$$\nu_{GE}(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \nu_{[GE]}(\Psi(\overleftrightarrow{\mathcal{K}}),\Psi(\overleftrightarrow{\mathcal{L}})), \tag{83}$$

- for any pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in (\mathbb{K}^- \cup \mathbb{R})^2$ the sign exchange law

$$\nu_{GE}(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \nu_{GE}(\boxminus \overleftrightarrow{\mathcal{L}},\boxminus \overleftrightarrow{\mathcal{K}}), \tag{84}$$

- for any pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R}) \times (\mathbb{K}^- \cup \mathbb{R})$ the law of subtraction of parties

$$\nu_{GE}(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \nu_{GE}(\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}}, [\![0]\!]). \tag{85}$$

Among other things, we prove here:

Lemma 1. *Any pair* $(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R}) \times (\mathbb{K}^- \cup \mathbb{R})$ *satisfies the condition*

$$\Psi(\overset{\leftrightarrow}{\mathcal{K}} \boxminus \overset{\leftrightarrow}{\mathcal{L}}) = \Psi(\overset{\leftrightarrow}{\mathcal{K}}) \ominus (\ominus\, \Psi(\boxminus \overset{\leftrightarrow}{\mathcal{L}})) \tag{86}$$

Proof. Let $\overset{\leftrightarrow}{\mathcal{K}} = \overset{\leftrightarrow}{\mathcal{L}}(a, b, c, d, S_K, E_K) \in \mathbb{K}^+ \cup \mathbb{R}$ and $\overset{\leftrightarrow}{\mathcal{L}} = \overset{\leftrightarrow}{\mathcal{L}}(e, f, g, h, S_L, E_L) \in \mathbb{K}^- \cup \mathbb{R}$. Then, we have $\boxminus \overset{\leftrightarrow}{\mathcal{L}}$, $\overset{\leftrightarrow}{\mathcal{K}} \boxminus \overset{\leftrightarrow}{\mathcal{L}} \in \mathbb{K}^+$ because of the sequences $(-e, -f, -g, -h)$ and $(a-e, b-f, c-g, d-h)$ are nondecreasing. Then, from (39), (50), and (58) we get

$$\Psi(\overset{\leftrightarrow}{\mathcal{K}} \boxminus \overset{\leftrightarrow}{\mathcal{L}}) = \Psi\Big(\overset{\leftrightarrow}{\mathcal{L}}(a-e, b-f, c-g, d-h, S_M, E_M)\Big) = \mathcal{L}(a-e, b-f, c-g, d-h, S_M, E_M), \tag{87}$$

where

$$\forall_{\alpha \in [0;1]}\ s_M(\alpha) = \begin{cases} S_L^{\star}(\alpha) + S_L^{\star}(\alpha), & a-e \neq b-f, \\ b-f, & a-e = b-f. \end{cases} \tag{88}$$

$$\forall_{\alpha \in [0;1]}\ e_M(\alpha) = \begin{cases} E_K^{\star}(\alpha) + E_L^{\star}(\alpha), & c-g \neq d-h, \\ c-g, & c-g = d-h. \end{cases} \tag{89}$$

$$\forall_{x \in [a-e, b-f]}\ S_M(x) = s_M^{\triangleleft}(x), \tag{90}$$

$$\forall_{x \in [c-g, d-h]}\ E_M(x) = e_M^{\triangleleft}(x). \tag{91}$$

On the other hand, successively from (36), (57), (11), and (22), we obtain

$$\begin{aligned} &\Psi(\overset{\leftrightarrow}{\mathcal{K}}) \ominus (\ominus\, \Psi\Big(\boxminus \overset{\leftrightarrow}{\mathcal{L}}\Big)) = \Psi\Big(\overset{\leftrightarrow}{\mathcal{L}}(a, b, c, d, S_K, E_K)\Big) \ominus \Big(\ominus\, \Psi\Big(\boxminus \overset{\leftrightarrow}{\mathcal{L}}(e, f, g, h, S_L, E_L)\Big)\Big) = \\ &= \mathcal{L}(a, b, c, d, S_K, E_K) \ominus \Big(\ominus\, \Psi\Big(\overset{\leftrightarrow}{\mathcal{L}}(-e, -f, -g, -h, S_L^{(-)}, E_L^{(-)})\Big)\Big) = \\ &= \mathcal{L}(a, b, c, d, S_K, E_K) \ominus \Big(\ominus\, \mathcal{L}(-e, -f, -g, -h, S_L^{(-)}, E_L^{(-)})\Big) = \\ &= \mathcal{L}(a, b, c, d, S_K, E_K) \ominus \mathcal{L}(h, g, f, e, E_L^{(-)}, S_L^{(-)}) = \mathcal{L}(a-e, b-f, c-g, d-h, S_M, E_M). \text{ QED} \end{aligned} \tag{92}$$

The conjunction of assumptions (83)–(85) is a sufficient condition for the formulation of the following theorem:

Theorem 5. *For any pair* $(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) \in \mathbb{K}^2$ *we have*

$$\nu_{GE}(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) = \nu_{[GE]}(\overset{=}{\Psi}(\overset{\leftrightarrow}{\mathcal{K}}), \overset{=}{\Psi}(\overset{\leftrightarrow}{\mathcal{L}})). \tag{93}$$

Proof. For any pair $(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R})^2$ the identity (93) is obvious.

Let us assume that $(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) \in (\mathbb{K}^- \cup \mathbb{R})^2$. Then, $(\boxminus\overset{\leftrightarrow}{\mathcal{K}}, \boxminus\overset{\leftrightarrow}{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R})^2$ and successively from (84), (83), (69) and (56), we get

$$\begin{aligned} &\nu_{GE}(\overset{\leftrightarrow}{\mathcal{K}}, \overset{\leftrightarrow}{\mathcal{L}}) = \nu_{GE}(\boxminus \overset{\leftrightarrow}{\mathcal{L}}, \boxminus \overset{\leftrightarrow}{\mathcal{K}}) = \nu_{[GE]}(\Psi(\boxminus \overset{\leftrightarrow}{\mathcal{L}}), \Psi(\boxminus \overset{\leftrightarrow}{\mathcal{K}})) = \nu_{[GE]}(\ominus\, \Psi(\boxminus \overset{\leftrightarrow}{\mathcal{K}}), \ominus\, \Psi(\boxminus \overset{\leftrightarrow}{\mathcal{L}})) = \\ &= \nu_{[GE]}(\overset{=}{\Psi}(\overset{\leftrightarrow}{\mathcal{K}}), \overset{=}{\Psi}(\overset{\leftrightarrow}{\mathcal{L}})). \end{aligned} \tag{94}$$

Let us assume now that $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in (\mathbb{K}^+ \cup \mathbb{R}) \times (\mathbb{K}^- \cup \mathbb{R})$. Then $\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}} \in \mathbb{K}^+$ and successively from (85), (83), (86), (70) and (57), we get

$$\begin{aligned} &\nu_{GE}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \nu_{GE}(\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}}, [\![0]\!]) = \nu_{[GE]}(\Psi(\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}}), [\![0]\!]) = \nu_{[GE]}(\Psi(\overleftrightarrow{\mathcal{K}}) \ominus (\ominus \Psi(\boxminus \overleftrightarrow{\mathcal{L}})), [\![0]\!]) = \\ &= \nu_{[GE]}(\Psi(\overleftrightarrow{\mathcal{K}}), \ominus \Psi(\boxminus \overleftrightarrow{\mathcal{L}})) = \nu_{[GE]}(\overset{=}{\Psi}(\overleftrightarrow{\mathcal{K}}), \overset{=}{\Psi}(\overleftrightarrow{\mathcal{L}})). \end{aligned} \tag{95}$$

Let us assume now that $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in (\mathbb{K}^- \cup \mathbb{R}) \times (\mathbb{K}^+ \cup \mathbb{R})$. Then $\overleftrightarrow{\mathcal{L}} \boxminus \overleftrightarrow{\mathcal{K}} \in \mathbb{K}^+$ and successively from (85), (84), (83), (86), (69), (70), and (57), we get

$$\begin{aligned} &\nu_{GE}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \nu_{GE}(\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}}, [\![0]\!]) = \nu_{GE}([\![0]\!], \overleftrightarrow{\mathcal{L}} \boxminus \overleftrightarrow{\mathcal{K}}) = \nu_{[GE]}([\![0]\!], \Psi(\overleftrightarrow{\mathcal{L}} \boxminus \overleftrightarrow{\mathcal{K}})) = \\ &= \nu_{[GE]}([\![0]\!], \Psi(\overleftrightarrow{\mathcal{L}}) \ominus (\ominus \Psi(\boxminus \overleftrightarrow{\mathcal{K}}))) = \nu_{[GE]}(\ominus \Psi(\boxminus \overleftrightarrow{\mathcal{K}}) \ominus \Psi(\overleftrightarrow{\mathcal{L}}), [\![0]\!]) = \\ &\nu_{[GE]}(\ominus \Psi(\boxminus \overleftrightarrow{\mathcal{K}}), \Psi(\overleftrightarrow{\mathcal{L}})) = \nu_{[GE]}(\overset{=}{\Psi}(\overleftrightarrow{\mathcal{K}}), \overset{=}{\Psi}(\overleftrightarrow{\mathcal{L}})). \text{ QED} \end{aligned} \tag{96}$$

Example 3. *Let us compare the OFN* $\overleftrightarrow{\mathcal{X}} = \overleftrightarrow{\mathcal{L}}(12, 14, 18, 20, S_X, E_X)$ *determined by (59) and the OFN* $\overleftrightarrow{\mathcal{Y}} = \overleftrightarrow{\mathcal{L}}(13, 11, 6, 5, S_Y, E_Y)$ *determined by (60). Using (93), (62), (64), and (41), we get*

$$\begin{aligned} &\nu_{GE}(\overleftrightarrow{\mathcal{Y}}, \overleftrightarrow{\mathcal{X}}) = \nu_{[GE]}(\overset{=}{\Psi}(\overleftrightarrow{\mathcal{Y}}), \overset{=}{\Psi}(\overleftrightarrow{\mathcal{X}})) = \nu_{[GE]}(\mathcal{L}(5, 6, 11, 13, E_Y, S_Y),\ \mathcal{L}(12, 14, 18, 20, S_X, E_X)) = \\ &= \nu_{[GE]}(\mathcal{L}(5, 6, 11, 13, L_V, R_Y), \mathcal{L}(12, 14, 18, 20, L_U, R_U)) = \nu_{[GE]}(\mathcal{V}, \mathcal{U}) = \tfrac{2}{5}. \end{aligned} \tag{97}$$

The simplicity of the calculations in the above example is apparent. In fact, Example 3 together with Theorem 5 shows that:

- if compared OFNs are both positively oriented then the fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ depends only on the interaction between the ending function of the first compared OFN and the starting function of the second compared OFN;
- if the first compared OFN is positively oriented and the second compared OFN is negatively oriented then the fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ depends only on the interaction between the ending functions of compared OFN;
- if the first compared OFN is negatively oriented and the second compared OFN is positively oriented then the fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ depends only on the interaction between the starting functions of compared OFN;
- if compared OFNs are both negatively oriented, then the fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ depends only on the interaction between the starting function of the first compared OFN and the ending function of the second compared OFN.

5. Relations "Greater Than" and "Equal to" for Ordered Fuzzy Numbers

In the last section, we explicitly define the preorder "greater than or equal to" $\widetilde{GE}$ on the space $\mathbb{K}$ of all OFNs. This relation may be applied as start point for determining other basic relations on $\mathbb{K}$.

Let us consider any pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K}^2$. On the space $\mathbb{K}$ we introduce the relation $\overleftrightarrow{\mathcal{K}}.\widetilde{GT}.\overleftrightarrow{\mathcal{L}}$, which reads:

$$\text{"OFN } \overleftrightarrow{\mathcal{K}} \text{ is greater than OFN } \overleftrightarrow{\mathcal{L}}.\text{"} \tag{98}$$

This relation is a fuzzy strict order $\widetilde{GT} \in \mathcal{F}(\mathbb{K}^2)$ defined by its membership function $\nu_{GT} \in [0,1]^{\mathbb{K}^2}$. From the point view of the multivalued logic, the value $\nu_{GT}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}})$ is considered as a truth-value of the sentence (98) which is equivalent to the sentence:

$$\text{“OFN } \overleftrightarrow{\mathcal{L}} \text{ is not greater than or equal to OFN } \overleftrightarrow{\mathcal{K}}.\text{”} \tag{99}$$

It means that we have

$$\nu_{GT}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = tv(\text{“}\overleftrightarrow{\mathcal{K}}.\widetilde{GT}.\overleftrightarrow{\mathcal{L}}\text{”}) = tv(\text{“}\neg\overleftrightarrow{\mathcal{L}}.\widetilde{GE}.\overleftrightarrow{\mathcal{K}}\text{”}) = 1 - tv(\text{“}\overleftrightarrow{\mathcal{L}}.\widetilde{GE}.\overleftrightarrow{\mathcal{K}}\text{”}). \tag{100}$$

Therefore, the membership function $\nu_{GT} \in [0,1]^{\mathbb{K}^2}$ is determined by the identity

$$\nu_{GT}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = 1 - \nu_{GE}(\overleftrightarrow{\mathcal{L}}, \overleftrightarrow{\mathcal{K}}). \tag{101}$$

Moreover, on the space $\mathbb{K}$ we introduce the relation $\overleftrightarrow{\mathcal{K}}.\widetilde{EQ}.\overleftrightarrow{\mathcal{L}}$, which reads:

$$\text{“OFN } \overleftrightarrow{\mathcal{K}} \text{ is equal to OFN } \overleftrightarrow{\mathcal{L}}.\text{”} \tag{102}$$

The relation $\widetilde{EQ} \in \mathcal{F}(\mathbb{K}^2)$ is fuzzy equivalence determined by membership function $\nu_{EQ} \in [0,1]^{\mathbb{K}^2}$. From the point view of the multivalued logic, the value $\nu_{EQ}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}})$ is considered as a truth-value of the sentence (102) which is equivalent to the sentence:

$$\text{“OFN } \overleftrightarrow{\mathcal{K}} \text{ is greater than or equal to OFN } \overleftrightarrow{\mathcal{L}} \text{ and OFN } \overleftrightarrow{\mathcal{L}} \text{ is greater than or equal to OFN } \overleftrightarrow{\mathcal{K}}\text{”} \tag{103}$$

It means that we have

$$\begin{aligned} \nu_{EQ}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) &= tv(\text{“}\overleftrightarrow{\mathcal{K}}.\widetilde{EQ}.\overleftrightarrow{\mathcal{L}}\text{”}) = tv(\text{“}\overleftrightarrow{\mathcal{K}}.\widetilde{EQ}.\overleftrightarrow{\mathcal{L}}\text{”} \wedge \text{“}\overleftrightarrow{\mathcal{L}}.\widetilde{GE}.\overleftrightarrow{\mathcal{K}}\text{”}) = \\ &= \min\{tv(\text{“}\overleftrightarrow{\mathcal{K}}.\widetilde{EQ}.\overleftrightarrow{\mathcal{L}}\text{”}), tv(\text{“}\overleftrightarrow{\mathcal{L}}.\widetilde{GE}.\overleftrightarrow{\mathcal{K}}\text{”})\}. \end{aligned} \tag{104}$$

Therefore, the membership function $\nu_{EQ} \in [0,1]^{\mathbb{K}^2}$ is determined by the identity

$$\nu_{EQ}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \min\{\nu_{GE}(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}), \nu_{GE}(\overleftrightarrow{\mathcal{L}}, \overleftrightarrow{\mathcal{K}})\}. \tag{105}$$

For any finite set $A = \{\overleftrightarrow{\mathcal{K}}_1, \overleftrightarrow{\mathcal{K}}_2, \ldots, \overleftrightarrow{\mathcal{K}}_n\} \subset \mathbb{K}_{Tr}$ we can distinguish set of maximal elements $\mathrm{Max}\{A\} \in \mathcal{F}(A)$ which is described by membership function $\mu_{\mathrm{Max}\{A\}} \in [0,1]^A$ determined in the following way [62]

$$\mu_{\mathrm{Max}\{A\}}(\overleftrightarrow{\mathcal{K}}_i) = \min\{\nu_{GE}(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{K}}_j) : \overleftrightarrow{\mathcal{K}}_j \in A\}. \tag{106}$$

This set may be applied as solution of optimization tasks using OFNs. Moreover, let us note, that the set $\mathrm{Max}\{A\}$ of maximal elements may be used as a fuzzy choice function [78].

In [17], the relation $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2_{Tr})$ is applied for ordering negotiation packages [79]. The considered case study is fully described there. Moreover, let us look on a short case study of applying the relation $\widetilde{GE} \in \mathcal{F}(\mathbb{K}^2)$ for financial effectivity analysis.

6. Financial Effectivity Determined by Imprecise Return—A Numerical Example

Let any financial security $\mathcal{Z} \in \mathbb{Z}$ be represented by the pair (R_z, σ_Z^2), where $R_z \in \mathbb{R}$ is an expected return rate from this security and $\sigma_Z^2 \in \mathbb{R}$ is a variance of its return rate. The symbol $\mathbb{Z}$ denotes the family of all considered securities.

We introduce the relation $\mathcal{P}.NLE.\mathcal{Q}$ which reads

$$\text{“The security } \mathcal{P} \in \mathbb{Z} \text{ is no less effective than the security } \mathcal{Q} \in \mathbb{Z}\text{”.} \tag{107}$$

In financial practice, this relation is defined by the equivalence

$$\mathcal{P}.NLE.\mathcal{Q} \Leftrightarrow (R_P \geq R_Q \vee \sigma_P^2 \leq \sigma_Q^2). \tag{108}$$

In [15], it is justified that return rate may be evaluated OFN. In this case, any financial security $\mathcal{Z}$ be represented by the pair $(\overleftrightarrow{R}_z, \sigma_Z^2)$, where $\overleftrightarrow{R}_z \in \mathbb{K}$ is an expected return rate evaluated by OFN. Therefore, the relation $\mathcal{P}.NLE.\mathcal{Q}$ should be replaced by the relation $\mathcal{P}.\widetilde{NLE}.\mathcal{Q}$ defined by the equivalency

$$\mathcal{P}.\widetilde{NLE}.\mathcal{Q} \Leftrightarrow (\overleftrightarrow{R}_P.\widetilde{GE}.\overleftrightarrow{R}_Q \vee \sigma_P^2 \leq \sigma_Q^2). \tag{109}$$

The relation $\mathcal{P}.\widetilde{NLE}.\mathcal{Q}$ also reads as the sentence (107). The relation $\widetilde{NLE} \in \mathcal{F}(\mathbb{K}^2)$ is fuzzy one determined by membership function $\nu_{NLE} \in [0,1]^{\mathbb{Z}^2}$. From the point view of the multivalued logic, the value $\nu_{NLE}(\mathcal{P},\mathcal{Q})$ is considered as a truth-value of the sentence (105). It means that we have

$$\begin{aligned} \nu_{NLE}(\mathcal{P},\mathcal{Q}) &= tv(\text{“}\overleftrightarrow{R}_P.\widetilde{GE}.\overleftrightarrow{R}_Q \vee \sigma_P^2 \leq \sigma_Q^2\text{”}) = \max\left\{tv(\text{“}\overleftrightarrow{R}_P.\widetilde{GE}.\overleftrightarrow{R}_Q\text{”}), tv(\text{“}\sigma_P^2 \leq \sigma_Q^2\text{”})\right\} = \\ &= \begin{cases} tv(\text{“}\overleftrightarrow{R}_P.\widetilde{GE}.\overleftrightarrow{R}_Q\text{”})\ \sigma_P^2 > \sigma_Q^2, \\ 1\ \sigma_P^2 \leq \sigma_Q^2. \end{cases} = \begin{cases} \nu_{GE}(\overleftrightarrow{R}_P, \overleftrightarrow{R}_Q)\ \sigma_P^2 > \sigma_Q^2, \\ 1\ \sigma_P^2 \leq \sigma_Q^2. \end{cases} . \end{aligned} \tag{110}$$

In order to increase the transparency of the considerations, we restrict our future considerations to the case of return rate evaluated by TrOFNs. We consider the securities $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively represented by the pairs $(\overleftrightarrow{R}_P, \sigma_P^2) = (\overleftrightarrow{Tr}(0.010, 0.010, 0.035, 0.040), 0.00023)$, $(\overleftrightarrow{R}_Q, \sigma_Q^2) = (\overleftrightarrow{Tr}(0.020, 0.025, 0.030, 0.045), 0.00024)$ and $(\overleftrightarrow{R}_R, \sigma_R^2) = (\overleftrightarrow{Tr}(0.065, 0.055, 0.050, 0.035), 0.00012)$. The return rates $\overleftrightarrow{R}_P$ and $\overleftrightarrow{R}_Q$ are positively oriented TrOFNs. Therefore, we can anticipate an increase in the rates of return from the securities $\mathcal{P}$ and $\mathcal{Q}$. Moreover, we can predict a decrease in the rate of return from the security $\mathcal{R}$ because of the return rate $\overleftrightarrow{R}_R$ is negatively oriented TrOFN. For these reasons, we consider two investment decisions:

(A) We sell the security $\mathcal{R}$ and for the funds obtained we buy the security $\mathcal{P}$,
(B) We sell the security $\mathcal{R}$ and for the funds obtained we buy the security $\mathcal{Q}$.

Let us compare a financial effectivity of considered securities $\mathcal{P}$ and $\mathcal{R}$. In line with (108), (93), (58) and (71), we get

$$\begin{aligned} \nu_{NLE}(\mathcal{P},\mathcal{R}) &= \nu_{GE}\left(\overleftrightarrow{Tr}(0.010, 0.010, 0.035, 0.040), \overleftrightarrow{Tr}(0.065, 0.055, 0.050, 0.035)\right) = \\ &\nu_{[GE]}\left(\overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(0.010, 0.010, 0.035, 0.040)\right), \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(0.065, 0.055, 0.050, 0.035)\right)\right) = \\ &= \nu_{[GE]}(Tr(0.010, 0.010, 0.035, 0.040), Tr(0.035, 0.050, 0.055, 0.065)) = \\ &= \frac{0.040-0.035}{0.040+0.050-0.035-0.035} = \frac{1}{4}. \end{aligned} \tag{111}$$

In the same way, we can compare a financial effectivity of considered securities $\mathcal{Q}$ and $\mathcal{R}$. We obtain

$$\begin{aligned} \nu_{NLE}(\mathcal{Q},\mathcal{R}) &= \nu_{GE}\left(\overleftrightarrow{Tr}(0.020, 0.025, 0.030, 0.045), \overleftrightarrow{Tr}(0.065, 0.055, 0.050, 0.035)\right) = \\ &\nu_{[GE]}\left(\overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(0.020, 0.025, 0.030, 0.045)\right), \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(0.065, 0.055, 0.050, 0.035)\right)\right) = \\ &= \nu_{[GE]}(Tr(0.020, 0.025, 0.030, 0.045), Tr(0.035, 0.050, 0.055, 0.065)) = \\ &= \frac{0.045-0.035}{0.045+0.050-0.030-0.035} = \frac{1}{3}. \end{aligned} \tag{112}$$

Therefore, we can say that the investment decisions (A) and (B) are both partially justified. Because of $\nu_{NLE}(\mathcal{Q},\mathcal{R}) > \nu_{NLE}(\mathcal{P},\mathcal{R})$, we ultimately recommend the investment decision (B).

7. Final Remarks

Relation "greater than or equal to" $\widetilde{GE}$ is explicitly defined on the space of all OFNs. In my best knowledge, it will be the first fuzzy order determined for OFNs. Determined relation $\widetilde{GE}$ compares OFNs without losing information about the imprecision and orientation of evaluated OFNs. From the point-view of application needs, this approach is desirable. Nevertheless, I proved that the relation $\widetilde{GE}$ is independent of the orientation of the numbers being compared. This conclusion may be applied for simplification of many OFN applications.

The first application of relation $\widetilde{GE}$ is cited in Section 5. The next application is described in Section 6. Meanwhile, we will employ the proposed relation to model some imprecision decision making problems from some concrete applied fields, such as medical decision making, behavioural economic [11], management [15,16], telecommunication, and financial assessment [7,9–14]. Then these relations may be used for decision making problems with scoring function evaluated by OFNs. In [15,16], such evaluation of scoring function follows from the fact that partial ratings are evaluated by OFNs. Moreover, studying multi criterial group decision making problems, we should take into account some imprecise weights of criteria [80]. Then these weights may be evaluated by OFNs what implies that also the scoring function is evaluated by OFNs. In general, the relation $\widetilde{GE}$ can be applied in any such quantitative model of the real world that a comparison of imprecise numbers is used.

In Section 2.2, I point out some terminology problems connected with the notion of OFN. I believe that this is a very important problem from an ethical point of view. I invite people of science to discuss this topic.

For any OFN we can determine the family of oriented α-cuts defined as a pair of usual α-cut and OFN orientation. An important direction for further development is to propose such fuzzy order of OFNs which is determined by the family of all α-cuts for FNs. At present, the oriented α-cuts theory is unknown.

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