

Review

Some Singular Spacetimes and Their Possible Alternatives [†]

Andrew DeBenedictis ^{1,2} ¹ Faculty of Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada; adebened@sfu.ca² The Pacific Institute for the Mathematical Sciences, S.F.U. Site, 8888 University Dr W, Burnaby, BC V5A 1S6, Canada[†] This paper is based on a talk given at the International Conference on Gravitation, Astrophysics and Cosmology 2024 (ICGAC-2024), Mathura, India, 14–16 June 2024.

Abstract: In this review, we begin with a historical survey of some singular solutions in the theory of gravitation, as well as a very brief discussion of how black holes could physically form. Some possible scenarios which could perhaps eliminate these singularities are then reviewed and discussed. Due to the vastness of the field, its coverage is not exhaustive; instead, the concentration is on a small subset of topics such as possible quantum gravity effects, non-commutative geometry, and gravastars. A simple singularity theorem is also reviewed. Although parts of the manuscript assume some familiarity with relativistic gravitation or differential geometry, the aim is for the broad picture to be accessible to non-specialists of other physical sciences and mathematics.

Keywords: curvature singularities; black holes; cosmology

PACS: 04.20.Dw; 02.40.Xx; 97.60.Lf; 98.80.Jk

1. Introduction

Despite its ubiquity, gravitation remains one of the most enigmatic phenomena in the physical sciences. There is, for example, its universal nature, which is not shared by the other fundamental forces. There is also its direct connection to causality and causal structure; gravitation determines what is the past, present, and future. In fact, our best theories of gravity thus far (of which general relativity still comes out on top even after over a century of scrutiny) indicate that gravity is not a force in the traditional sense at all. For much of this manuscript, it is assumed that gravitation, at least in the low-energy arena, is described by the theory of general relativity, which is governed by the Einstein equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (1)$$

For the non-gravitational specialist, in Equation (1), $R_{\mu\nu}$ represents the components of the Ricci curvature tensor, $R := R^\alpha_\alpha$ is its invariant trace (the Ricci scalar), $g_{\mu\nu}$ is the components of the metric (essentially the gravitational potentials), and $T_{\mu\nu}$ is the stress–energy tensor of the gravitating material. The signature of the metric is +2 (one negative metric eigenvalue and three positive values) and Greek indices (0 → 3) spanning the full spacetime, with the zeroth component assumed to be time-like (measuring time) and the 1, 2, 3 components being space-like (measuring space). Roman indices will denote only the spatial components of the tensors. Λ is the cosmological constant, which is generally assumed to be zero in this paper unless otherwise indicated. The Einstein tensor $G_{\mu\nu}$ is identically divergence-free such that $\nabla_\mu G^\mu_\nu \equiv 0$, which via Equation (1) implies the *conservation law*

$$\nabla_\mu T^\mu_\nu = 0. \quad (2)$$

In all expressions in this manuscript, the speed of light is set equal to one, as are the gravitational constant and Planck’s constant. We concentrate here on the theory of



Citation: DeBenedictis, A. Some Singular Spacetimes and Their Possible Alternatives. *Particles* **2024**, *7*, 899–917. <https://doi.org/10.3390/particles7040054>

Academic Editors: Kazuharu Bamba and Armen Sedrakian

Received: 12 September 2024

Revised: 2 October 2024

Accepted: 3 October 2024

Published: 14 October 2024



Copyright: © 2024 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

general relativity, and not all results in this manuscript hold if the theory of gravity is not general relativity.

The theory of general relativity has been highly successful. However, like many classical theories in physics, some of its solutions are singular, and these solutions, singularity aside, are not just academic curiosities but physically viable solutions to the equations of motion. It is desirable therefore to try to eliminate these singularities in some physically acceptable way. There is, of course, a long history to this arena of study, and here we only briefly describe a few of the many possible methods of singularity avoidance. It is hoped that this will give the reader some understanding of the problem and some of the methods which can be employed in attempting to rectify it, along with some of the difficulties encountered in the attempt. This manuscript is based on a seminar given at the International Conference on Gravitation, Astrophysics and Cosmology 2024, where the audience consisted of physical scientists and mathematicians. Some familiarity with the topic of relativistic gravity and differential geometry is assumed, but these details can just be accepted by non-specialists, and it is hoped that the overall picture will be accessible to a wider audience of scientists and mathematicians than just gravitational specialists.

This paper is laid out in two main sections. In Section 2, we briefly discuss the history of some important gravitational solutions which possess some sort of singularity. These include the famous Schwarzschild and Kerr solutions and Friedmann–Lemaître–Robertson–Walker metrics. We further briefly describe the physical process which could lead to the formation of a black hole. In the following section (Section 3), we discuss some possible ways that the singularities can be circumvented. An appendix is included which presents a simple singularity theorem illustrating how certain types of singularities can occur.

2. A History of Some Singular Solutions in Gravity

Here, we will present a brief history of some of the most important gravitational field solutions which possess some sort of singularity. The review here will be in chronological order, and therefore we will start with the result of Isaac Newton in his famous *Principia* [1]

$$V(r) = -G \frac{M}{r}, \tag{3}$$

where V represents the gravitational potential outside of a spherically symmetric body of a total mass M . The above potential leads to a gravitational force on an external mass m of the form

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{\mathbf{r}}. \tag{4}$$

From the expressions above, it is clear that there will be an infinite gravitational force at $r = 0$ in the case where M is a point mass.

Let us now move on to general relativity, governed by Equation (1). As in the Newtonian analysis above, let us concentrate on situations exhibiting spherical symmetry, and we will also assume that the potentials are time-independent. In this scenario, the metric tensor, which as mentioned above encodes the gravitational potentials, has the following components for spherical coordinates:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{\gamma(r)} dt^2 + e^{\alpha(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{5}$$

with $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. If we calculate the Ricci tensor and Ricci scalar with the metric components in the above line element and use them in the Einstein equation (Equation (1)), then the following equations result:

$$r^{-2} \cdot \left[1 - e^{-(\alpha+\gamma)} \cdot \partial_1 (r e^\gamma) \right] + 8\pi T^1_1(r) = 0, \tag{6}$$

$$e^{-\alpha} \cdot \left[- (1/2) \partial_1 \partial_1 \gamma - (1/4) (\partial_1 \gamma)^2 \right]$$

$$+ (1/2) r^{-1} (\partial_1 \alpha - \partial_1 \gamma) + (1/4) \partial_1 \alpha \cdot \partial_1 \gamma \Big] + 8\pi T^2_2(r) = 0, \tag{7}$$

$$r^{-2} \cdot [1 - \partial_1 (r e^{-\alpha})] + 8\pi T^0_0(r) = 0. \tag{8}$$

The G^3_3 equation in this coordinate system is identical to the G^2_2 equation and thus is not written.

The conservation of stress–energy law (Equation (2)) only has one non-trivial component in this scenario:

$$\begin{aligned} \frac{r}{2} \nabla_\mu T^\mu_1 &= -T^2_2(r) + (r/2) \cdot \partial_1 T^1_1 + [1 + (r/4) \cdot \partial_1 \gamma] T^1_1(r) \\ &\quad - (r/4) \cdot \partial_1 \gamma \cdot T^0_0(r) = 0. \end{aligned} \tag{9}$$

We will treat two of the components of the matter distribution, encoded in T^μ_ν , as being prescribable, since this is the level at which the system of equations is underdetermined. In this scenario, we can solve for the metric functions $\alpha(r)$ and $\gamma(r)$, and the remaining stress-energy components ($T^2_2 = T^3_3$) as follows:

1. Prescribe $T^0_0(r)$, which is “minus the energy density in the comoving frame” here, and solve Equation (8) for $e^{-\alpha(r)}$.
2. Now that $e^{-\alpha(r)}$ is known, prescribe $T^1_1(r)$, which is the “radial pressure in the comoving frame” here, and solve the equation made up of the linear combination of Equation (6) minus Equation (8) in order to find $e^{\gamma(r)}$.
3. The stress–energy tensor component T^2_2 (in this case, the transverse pressure in the comoving frame) will be *defined* by the conservation law in Equation (9).

The above procedure, thanks to J. L. Synge [2] and extended by A. Das, yields the following solution (see [3] for details, although this is a fairly straightforward manipulation):

$$e^{-\alpha(r)} = 1 + \frac{1}{r} \left[8\pi \int^r T^0_0(r') r'^2 dr' \right], \tag{10}$$

$$e^{\gamma(r)} = e^{-\alpha(r)} \cdot \exp \left\{ 8\pi \int^r [T^1_1(r') - T^0_0(r')] \cdot e^{\alpha(r')} \cdot r' dr' \right\}, \tag{11}$$

$$\begin{aligned} T^2_2(r) \equiv T^3_3(r) &:= (r/2) \cdot \partial_1 T^1_1 + [1 + (r/4) \cdot \partial_1 \gamma] \cdot T^1_1(r) \\ &\quad - (r/4) \cdot \partial_1 \gamma \cdot T^0_0(r). \end{aligned} \tag{12}$$

Now, let us look for the general relativistic analog of the Newtonian gravitational potential (Equation (3)). We recall that the Newtonian gravitational potential was due to a point mass M located at $r = 0$. Let us therefore use such a distribution in Equations (10)–(12); that is, let us prescribe

$$T^0_0 = -M \frac{\delta(r')}{4\pi r'^2}, \tag{13}$$

recalling that T^0_0 in this coordinate system corresponds to a value minus the energy density in the comoving frame of the gravitating material. The factor of $4\pi r'^2$ comes from the definition of the spherical coordinate delta function when the impulse is located at $r' = 0$, which is coincident with a boundary of the integration domain (It may seem that a factor of $\sqrt{|g_{11}|}$ is missing. This is due to the same factor being missing in the measure of the integral in Equation (10). A more physical explanation for these factors comes from the ADM mass integral of the spacetime. See, for example, [4]). If one instead uses another common definition where $\int_0^\epsilon \delta(r') dr' = 1/2$ instead of one, then the delta function is

written as $\delta(r)/(2\pi r'^2)$ (see, for example, [5,6]). Using Equation (13) in Equation (10) immediately yields

$$e^{-\alpha(r)} = 1 - \frac{2M}{r}. \tag{14}$$

Now, according to the solution scheme outlined above, we still need to prescribe the radial pressure T_1^1 . For this, we will appeal to the junction conditions. Namely, Synge’s junction condition in static spherical symmetry dictates that the radial pressure be continuous at all points in spacetime. Since away from the point mass, the radial pressure (and all stress–energy components) are zero, we will use Synge’s junction condition of radial pressure continuity to tell us that at the point mass, the radial pressure should be zero; that is, $T_1^1(0) = 0$. Admittedly though, it is somewhat dubious to rely on a junction condition at a singular point. With this choice of vanishing T_1^1 , we find from Equation (11) that

$$e^{\gamma(r)} = \left(1 - \frac{2M}{r}\right) e^{h_0}, \tag{15}$$

with h_0 as a constant. This constant can be absorbed in the rescaling of the t coordinate in Equation (5), where $e^{h_0/2}t \rightarrow t$. At this stage, we have the metric (gravitational potentials) in general relativity for a “point mass” at $r = 0$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{16}$$

The above is the famous Schwarzschild solution. Schwarzschild wrote the line element in a -2 signature and in a slightly different way in his original paper [7], and this is illustrated in Figure 1, which is taken directly from the original publication:

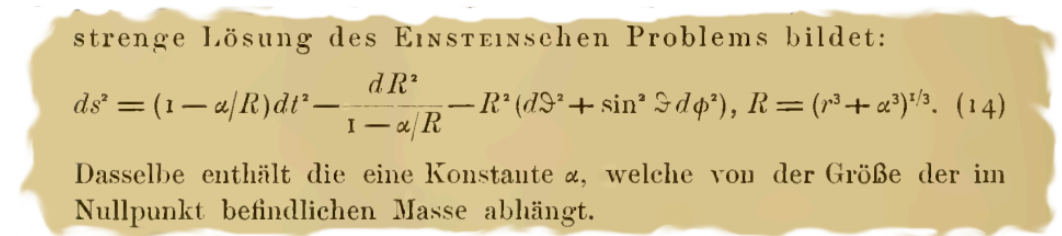


Figure 1. An excerpt from K. Schwarzschild’s original paper [7]. **Please note** that R in Schwarzschild’s paper is analogous to r in this manuscript. Note that negative values of r in this figure are allowed by extension, taking us down to $R = 0$ (the singularity). As we can see in this figure, there is a C^2 coordinate transformation between Schwarzschild’s R coordinate and Schwarzschild’s r coordinate ($\partial R/\partial r = r^2/(\alpha^3 + r^3)^{2/3}$, $\partial^2 R/(\partial r)^2 = 2\alpha^3 r/(\alpha^3 + r^3)^{5/3}$, $\partial r/\partial R = R^2/(R^3 - \alpha^3)^{2/3}$, $\partial^2 r/(\partial R)^2 = -2\alpha^3 R/(R^3 - \alpha^3)^{5/3}$), save for at the singular point $r = -\alpha$ ($R = 0$), which Schwarzschild did not consider. Figure included with the kind permission of the Berlin-Brandenburg Academy of Sciences and Humanities—Academy Library—Shelfmark: Z 350—19,161 (Akademiearchiv, Berlin-Brandenburgischen Akademie der Wissenschaften).

Related to the issue of singularities, we notice that the metric in Equation (16) has two singular points (not including the benign issue at the poles $\theta = 0, \pi$). These are located at $r = 2M$ and $r = 0$. In order to study these possible pathologies, we can look at the Riemann curvature tensor components, which essentially encode the “strength” of the gravitational field via the amount of spacetime curvature it produces. These components are not invariant though and are therefore subject to coordinate effects, meaning that pathologies found in the Riemann tensor components might be due to the coordinates utilized and not a true pathology. In order to alleviate this problem, we shall introduce an orthonormal tetrad, a set of four ($\hat{\alpha} = 0, 1, 2, 3$) orthonormal vectors whose components ($\mu = 0, 1, 2, 3$) will be denoted as $e^{\mu}_{\hat{\alpha}}$. The hatted indices represent orthonormal components, meaning that they

are raised and lowered by the orthonormal (i.e., Minkowski) metric. We shall choose for our set of vectors a set pointing in the coordinate directions; that is, we will pick

$$\begin{aligned}
 e^\mu_{\hat{r}} &= \delta^\mu_r \sqrt{|g^{11}|} = \delta^\mu_r \sqrt{\left|1 - \frac{2M}{r}\right|}, & e^\mu_{\hat{\theta}} &= \delta^\mu_\theta \sqrt{|g^{22}|} = \delta^\mu_\theta \frac{1}{r}, \\
 e^\mu_{\hat{\phi}} &= \delta^\mu_\phi \sqrt{|g^{33}|} = \delta^\mu_\phi \frac{1}{r \sin \theta}, & e^\mu_{\hat{t}} &= \delta^\mu_t \sqrt{|g^{00}|} = \delta^\mu_t \frac{1}{\sqrt{\left|1 - \frac{2M}{r}\right|}}.
 \end{aligned}
 \tag{17}$$

This set is orthonormal since it satisfies

$$\left[e^\mu_{\hat{\alpha}} e_{\mu\hat{\beta}} \right] = [\eta_{\hat{\alpha}\hat{\beta}}] = \text{diag}[-1, 1, 1, 1],$$

with the Minkowski metric $[\eta_{\hat{\alpha}\hat{\beta}}]$.

Projecting the Riemann curvature tensor into the orthonormal frame

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = R_{\mu\nu\rho\sigma} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} e^\rho_{\hat{\gamma}} e^\sigma_{\hat{\delta}},$$

yields the following singular components (plus those related by index symmetry and components that are finite):

$$\begin{aligned}
 R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= -\frac{2M}{r^3} = -R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} \\
 R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} &= \frac{M}{r^3} = R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}.
 \end{aligned}
 \tag{18}$$

We notice immediately from Equation (18) that $r = 0$ possesses a curvature singularity. However, the other potentially problematic surface, $r = 2M$, does not. As is well known, the event horizon ($r = 2M$) is non-singular.

Moving now into the 1920s and 1930s, scientists wished to apply the then-new theory of general relativity to the universe as a whole. A couple of assumptions went into this endeavor. Due to the difficulty of the problem, it was assumed that, on the absolute largest scales, the universe is homogeneous and isotropic. In the geometric language of general relativity, these assumptions mean that the spatial geometry of the universe is of a constant curvature; that is, at a fixed instant of time, the universe is constantly curved, and the choices for the constant curvature are of course zero curvature (flat), a constant positive curvature, or a constant negative curvature. There is a set of mathematical conditions that describe constant curvature spaces which lead almost directly to the metric compatible with these assumptions. We shall not describe them here, but interested readers may find them in [3]. The resulting metric is the one which yields the following line element:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - k r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right].
 \tag{19}$$

The above famous metric is known as the Friedmann–Lemaître–Robertson–Walker metric [8–11]. The constant k determines whether the $t = \text{const.}$ spatial slices are flat ($k = 0$), negatively curved ($k < 0$), or positively curved ($k > 0$). The function $a(t)$ is arguably the most important part of Equation (19), since its behavior tells us how the geometry of the spatial slices evolve over time.

An orthonormal tetrad may be constructed for this metric, and the Riemann curvature tensor’s components projected in the orthonormal frame can be computed. We shall not go into the same level of detail as with the Schwarzschild metric, but the important results

for us are summarized in the following two components, where the tetrad has been taken along the coordinate directions:

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{\ddot{a}(t)}{a(t)}, \quad R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \frac{\dot{a}^2(t) + k}{a^2(t)}. \tag{20}$$

We see that if $a(t)$ reaches zero, then we will have a curvature singularity, unless $\ddot{a}(t) \rightarrow 0$ and $\dot{a}^2(t) \rightarrow -k$ as $a(t) \rightarrow 0$. This is the famous Big Bang singularity. Of course, if these conditions can be met, or $a(t) = 0$ can be avoided, then we can circumvent the singularity. Unfortunately, when solving the Einstein field equations using physically reasonable T^μ_ν components, these conditions generally cannot be avoided, and a Big Bang singularity is predicted at some finite time in the past.

Returning to the Schwarzschild solution now, recall that we derived this (Equation (16)) under the assumption that the source term was the stress–energy of a “point mass” at $r = 0$. The result was a singular spacetime. Nature, however, may very well not like true point masses. Instead, one should more realistically consider an extended body, like a star or nebula, which collapses under its own gravitational attraction and see what sort of metric results when Einstein’s equations are applied to this scenario. Such a study was first published in 1939 in the seminal paper by J. R. Oppenheimer and H. Snyder titled “On Continued Gravitational Contraction” [12].

Due to the extreme complication of the model, Oppenheimer and Snyder assumed that the gravitating material was a pressureless “dust”, although some analysis in the paper was performed for nonzero pressure. Outside of the spherically collapsing matter, in the vacuum region, the metric was taken to be that of Schwarzschild (Equation (16)). This is not really an assumption since it can be fairly easily shown that the Schwarzschild solution is unique in general relativity in describing a spherically symmetric vacuum. This is a result of the Birkhoff–Jebsen theorem [13,14]. Oppenheimer and Snyder found in their study that to an external observer, the matter collapses and asymptotically approaches the radius $r = 2M$, with the redshift factor approaching infinity as $r = 2M$ is approached. Also, photons emitted from the collapsing matter can escape to infinity over a narrower and narrower range of angles as the collapse progresses. Eventually, in finite time in the comoving frame of the matter, no photons from the matter could be emitted which would reach out to an observer beyond $r = 2M$; that is, the collapsing star will eventually appear to be “dark”. This is the formation of a black hole, although it was not called that at the time, with an event horizon at $r = 2M$. There was no real discussion of the singularity at $r = 0$, but its eventual formation under mild assumptions (the positivity of the energy density in the Oppenheimer–Snyder model) would be mathematically inevitable.

We now fast-forward to 1963. Roy Kerr studied a vacuum gravitational field which could possibly exist due to a rotating mass [15]. It was later discovered by B. Carter [16], R. Penrose [17], and R. Boyer and R. Lindquist [18] that this metric actually describes a spinning black hole, and furthermore, the solution is unique in the realm of pure vacuum gravity. In the Boyer–Lindquist coordinates, the Kerr line element is written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{a^2 \cos^2 \theta - 2Mr + r^2}{a^2 \cos^2 \theta + r^2} dt^2 + \frac{a^2 \cos^2 \theta + r^2}{a^2 - 2Mr + r^2} dr^2 + (a^2 \cos^2 \theta + r^2) d\theta^2 - \frac{4Mar \sin^2 \theta}{a^2 \cos^2 \theta + r^2} dt d\phi + \frac{\sin^2 \theta [(a^2 + r(r - 2M))a^2 \cos^2 \theta + r((r + 2M)a^2 + r^3)]}{a^2 \cos^2 \theta + r^2} d\phi^2. \tag{21}$$

Here, a represents the angular momentum per unit mass, a direct measure of the black hole’s spin. The above line element reduces to the Schwarzschild one in Equation (16) when $a = 0$.

Regarding the possible existence of curvature singularities, we could compute the Riemann tensor components and project them onto an orthonormal frame. However, the

results are unwieldy. Instead, we calculate the Kretschmann scalar of the Kerr metric (If the Kretschmann scalar is infinite, then there is an infinite pathology in the curvature. If it is finite, then there still might be a curvature singularity. In this sense, the Kretschmann scalar is a less robust measure of curvature singularities than studying the individual components of the Riemann tensor in an orthonormal frame):

$$K := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)}{(r^2 + a^2\cos^2\theta)^6}. \tag{22}$$

At $\theta = \pi/2$, we have

$$K = \frac{48M^2}{r^6}, \tag{23}$$

heralding a curvature singularity in the Kerr black hole at $r = 0$ and $\theta = \pi/2$.

Black hole-like objects were predicted, although not in their relativistic guise, by John Michell in a work written in 1783 and printed in 1784 [19] and by Laplace in 1796 in his *Exposition du Système du Monde* [20]. An excerpt from Michell’s work is shown in Figure 2.

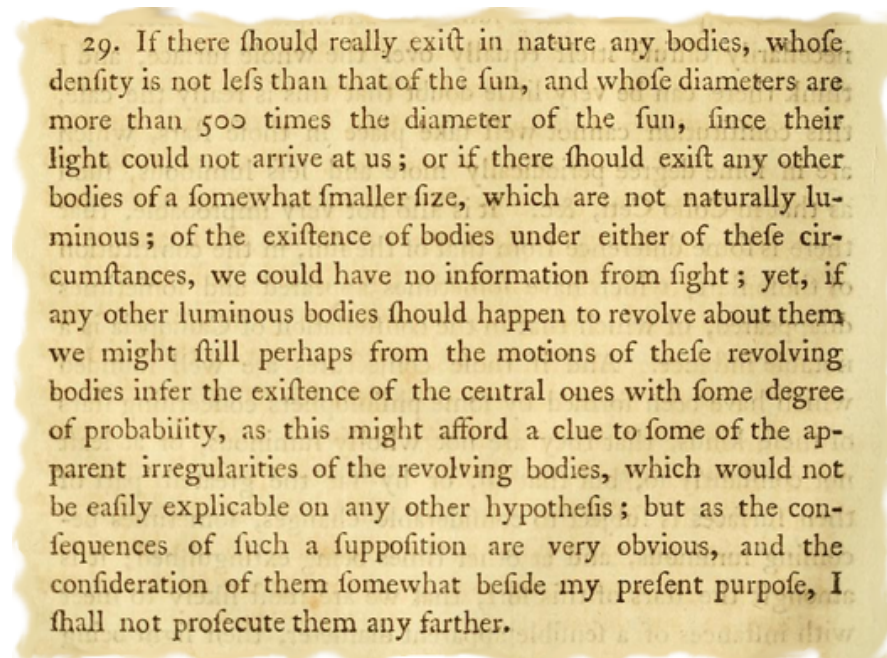


Figure 2. An excerpt from Michell’s original 1784 paper [19]. “If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, and the consideration of them somewhat beside my present purpose, I shall not prosecute them any farther.” Copyright status: Not in copyright.

Before ending this historical exposition, we mention here that the term black hole for these objects is often attributed to J. A. Wheeler circa 1967 or 1968. However, this phrase was in existence at least since 1964 in *Science News* [21]. The phrase was also allegedly used at the 1963 Texas Symposium on Relativistic Astrophysics [22], and Wheeler may have been present at the meeting. Thus, it still could have been Wheeler who coined the term black hole. Others attribute the phrase to R. Dicke [23].

Astrophysical Formation of Singular Spacetimes

The above discussion mainly described the mathematics of certain singular solutions. However, it is important to keep in mind that if the singularities discussed have no way of physically forming, then arguably, from a physics perspective, they are purely of an academic nature. For this, we will look at a thoroughly simplified version of the end of a main sequence star’s life, keeping in mind though that not all stars will follow this path.

A star will eventually run out of nuclear fuel and reach a state where its pressure can no longer counteract gravity to maintain dynamic equilibrium. The star will begin to contract under its own weight. Such a star may supernova, and this may leave behind a (non-singular) white dwarf, which is a body supported by degenerate electron matter. In some cases though, the remnant core may be massive enough that the degeneracy pressure is insufficient to support the core, and further contraction will occur. The mass threshold where this occurs is called the Chandrasekhar mass [24], which is approximately (with G , \hbar , and c not set to unity)

$$M_{Chand} \approx \frac{\alpha_0}{2(\mu_e m_H)^2} \sqrt{3\pi} \left(\frac{\hbar c}{G} \right)^{3/2} \tag{24}$$

Here, μ_e is the average molecular weight per electron, m_H is the mass of a hydrogen atom, and α_0 is a constant that arises via the Lane–Emden equation of astrophysics.

What happens beyond the Chandrasekhar limit is that protons and electrons combine via the reaction



forming neutrons and electron neutrinos. This combination allows further contraction to form a neutron star, now supported by neutron degeneracy. However, if the neutron star is massive enough, exceeding the Oppenheimer–Volkoff limit [25], then even this will not halt the collapse. If this happens, then other degeneracies could perhaps support the star (e.g., quark degeneracy), but still, above some mass limit, continued collapse will be inevitable. In general relativity, the problem is augmented by the fact that the pressure itself in the stress–energy tensor is a source of gravity and not simply the mass density, as in Newtonian gravity. Regardless of the scenario, it seems that above some pressure and energy density limit, black hole formation is inevitable unless some exotic physics comes into play.

3. Possible Resolutions to Singular Spacetimes

We will present in this section some possible alternatives to the types of singular spacetimes discussed in the previous section. As mentioned previously, this discussion and coverage are far from exhaustive.

3.1. Quantum Gravity Effects

One issue with the field of gravitation is that practical experiments to measure gravitational effects are notoriously difficult to perform. The main reason for this is that gravity itself is so weak. One cannot set up an experiment in a laboratory to directly measure gravity’s strong field effects. The singularities we have been discussing here are heralded by an infinity in the (orthonormal) curvature, and at a large curvature, the gravitational field is strong. Now, the theory of general relativity has been thoroughly tested in the weak field limit, such as in the realm of the solar system [26]. It is also becoming fairly well tested in the stronger field regime [27]. However, when we are talking about singularities, we are in the realm of extremely strong gravity. It is possible that in this regime, general relativity is not the correct theory of gravity, and instead, some other theory takes over. An obvious candidate would be the theory of quantum gravitation. Unfortunately, to date, no such fully developed theory exists, although there are a number of candidates. The quantization of general relativity is fraught with problems. In the canonical 3 + 1 formulation, it is plagued by divergences which cannot be addressed in the same way as, say, the

theory of quantum electrodynamics. At the same time, the emergence of effective theories renders renormalizability unnecessary at least in the lower energy regime of quantum gravity effects.

One promising candidate theory of quantum gravity is loop quantum gravity (LQG) [28]. This theory is essentially a theory of quantum geometry and is a direct attempt at quantizing general relativity. At extremely high energies, perhaps near the Planck scale, it is expected that the effects of a quantum theory of gravity manifest themselves. However, while not having a fully workable theory at hand, one can look for an effective classical theory which encompasses some of the quantum effects.

We will be dealing within a symmetry reduced theory; that is, we will assume from the start that the metric has a form similar to Equation (16), but the metric functions will not be the same. We write the line element in Equation (16) in new variables as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 d\tau^2 + \frac{(E_2)^2}{E_3} dy^2 + E_3 d\varrho^2 + E_3 c_0 \sinh^2(\sqrt{d_0} \varrho) d\varphi^2, \quad (26)$$

where N , E_2 , and E_3 are functions of *time*. They are functions of time because, when examining Equation (16), the classical singularity is located in the domain $r < 2M$ inside the event horizon. In the coordinate chart of Equation (16), $g_{rr} < 0$ and $g_{tt} > 0$ for $r < 2M$ such that the r coordinate is actually time-like (measures time) and the t coordinate is space-like (measures distances). The Schwarzschild metric for $r < 2M$ is therefore actually time-dependent and homogeneous. Since we are going to be dealing with domains close to the singularity here, the metric functions are taken to be time-dependent, and we denote this time as τ .

The c_0 and d_0 factors in Equation (26) control the geometry of the $\tau = \text{const.}$ and $y = \text{const.}$ subspaces. These are the cases:

1. $c_0 = -1, d_0 = -1$: For this choice, the (ϱ, φ) sub-manifolds are spheres.
2. $\lim_{d_0 \rightarrow 0} c_0 = \frac{1}{d_0}, d_0 = 0$: In this case, the (ϱ, φ) sub-manifolds are toroidal (and the sub-manifolds for this case are intrinsically flat).
3. $c_0 = 1, d_0 = 1$: In this case, the (ϱ, φ) sub-manifolds are surfaces of a constant negative curvature of the genus $g > 1$. Such surfaces may be compact or not compact [29,30].

We will be dealing within the Hamiltonian formalism, and we omit the technical details save for the few that are required to understand the results. For the scenarios above, the gravitational Hamiltonian is given by

$$S = -\frac{N\sqrt{c_0}}{2\sqrt{d_0}E_3\gamma^2} \left[\left((a_2)^2 - \gamma^2 d_0 \right) E_2 + 2E_3 a_2 a_3 \right] + \frac{N}{2\sqrt{E_3}\gamma^2} \Lambda E_2 E_3. \quad (27)$$

Here, Λ is the cosmological constant (required for the toroidal and higher genus scenarios), and a_i represents functions of time which are components of an $su(2)$ connection and are conjugate to E_i in the same way that positions and canonical momenta are conjugate to each other in Hamiltonian mechanics. Also, γ is a number, known as the Barbero–Immirzi parameter, which can be set by studying black hole entropy (see, for example, [31]).

We will apply some quantum corrections inspired by LQG to the above Hamiltonian. Specifically, we will consider here what are known as holonomy corrections [32,33]. This amounts to making the following substitution in Equation (27):

$$a_i \rightarrow \frac{\sin(a_i \delta_i)}{\delta_i}. \quad (28)$$

There are several schemes in the literature for how to pick the quantities δ_i . In early studies, δ_i was taken to be constant [34,35]. Other schemes were later developed which improved the dynamics [34,36–38]. In this summary, we shall present the results in [39], where the δ_i values are given by those similar to the improved dynamics in [34]:

$$\delta_2 = \left(\frac{\Delta}{E_3}\right)^{1/2}, \quad \delta_3 = \frac{\sqrt{E_3 \Delta}}{E_2}, \tag{29}$$

where Δ is taken to be a constant (usually chosen to be roughly the Planck area).

Substituting Equation (29) into Equation (28) and then replacing the a_i in Equation (27) with that in Equation (28) yields the quantum-corrected Hamiltonian. It is then a matter of evolving the variables E_i and a_i according the Hamiltonian equations of motion $\dot{a}_2 = \{a_2, S\}$, etc. In particular, what is of interest is the variable E_3 , since $E_3 = 0$ would herald a curvature singularity analogous to the one in the Schwarzschild metric at $\tau = 0$ (which was $r = 0$ in our discussion of the Schwarzschild metric earlier). Qualitatively, what is happening when $E_3 = 0$ is the following. Note from the line element in Equation (26) that the volumes of the (ϱ, ϕ) sub-manifolds are given by

$$V_{(\varrho, \phi)} = E_3(\tau) \int \sqrt{c_0} \sinh(\sqrt{d_0} \varrho) d\varrho d\phi, \tag{30}$$

and if the pre-factor $E_3(\tau)$ trends toward zero, then the volume elements of this space, in a sense, “infinitely compress” to a volume of zero, resulting in singular behavior. More mathematically rigorous is that some of the components of the Riemann tensor in an orthonormal frame of the effective metric (Equation (26)) become infinite when $E_3(\tau) = 0$.

The time evolution of E_3 for the case of the loop quantum-corrected spherical scenario (with $\Lambda = 0.001$) is illustrated in Figure 3. The initial conditions away from the classical singularity were chosen to be the classical ones (i.e., the ones which satisfy the general relativity Schwarzschild solution as well as a required constraint condition $S = 0$). Also in Figure 3, the Hamiltonian evolution of E_3 for the non-quantum corrected Hamiltonian (i.e., pure general relativity) is illustrated. It can be seen that at $\tau = 0$, the uncorrected E_3 trended toward zero, and hence there was a curvature singularity. However, the quantum-corrected Hamiltonian evolution avoided $E_3 = 0$, thus avoiding the classical curvature singularity. The full results are discussed in [39].

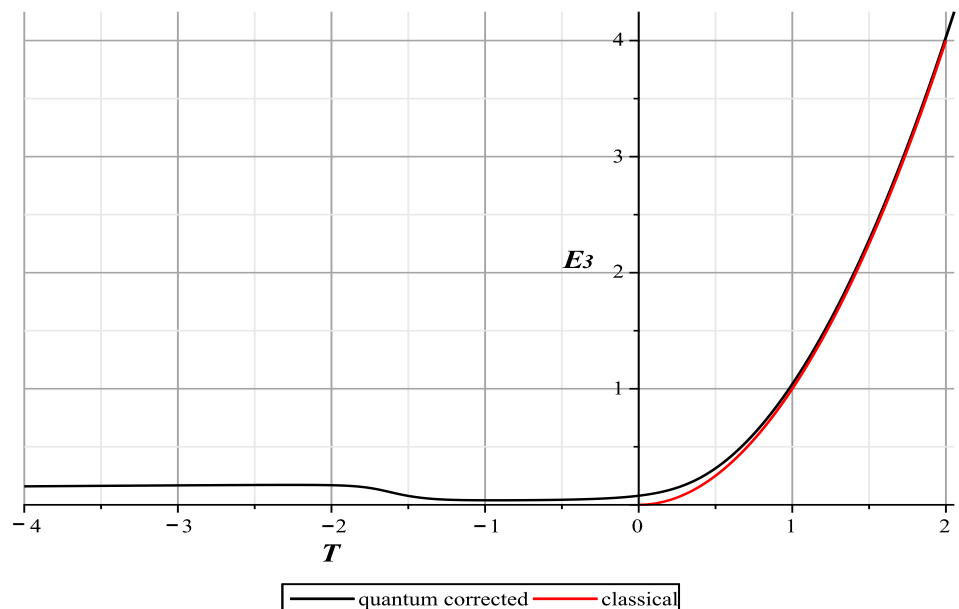


Figure 3. The Hamiltonian evolution of the E_3 factor in the classical and quantum corrected scenarios. The classical evolution (red), which is the Schwarzschild metric with a small cosmological constant, behaves as $E_3 = \tau^2$, and there is a singularity ($E_3 = 0$) at $\tau = 0$. The quantum-corrected evolution (black) avoids $E_3 = 0$ and has no corresponding singularity. Parameters: $\gamma = 0.274$, $M = 5$, and $\Lambda = 0.001$. The full results are discussed in [39].

We close this section by pointing out some drawbacks to this study. First, we should be clear that as of the writing of this manuscript, there is no actual fully developed theory of quantum gravity. The study here simply implements some low-order corrections from the theory of loop quantum gravity, and there are other corrections which can be applied [40]. Also, the scheme here is ambiguous in the step in Equation (29), and other schemes may be employed [36,38,41].

3.2. Non-Commutative Geometry

Here, we discuss another possible theory which can avoid black hole singularities. It is an implementation of non-commutative geometry [42,43]. In quantum mechanics, non-commutative geometry refers to extending the usual commutator between positions and momenta ($[x^j, p_k] = i\delta_k^j$) to the positions

$$[x^j, x^k] = i\theta\epsilon^{jk}{}_{\ell}x^{\ell}. \tag{31}$$

Furthermore, this can be extended to non-commutativity between the momenta in an analogous manner:

$$[p_j, p_k] = i\beta\epsilon_{jk}{}^{\ell}p_{\ell}. \tag{32}$$

In Equations (31) and (32), the constants θ and β measure the amount of non-commutativity, and in quantum theory, this would be related to \hbar . The full technical details of the discussion in this section maybe found in [44].

Moving “backward” to classical Hamiltonian mechanics, we replace the commutator of quantum theory with the Poisson bracket of classical mechanics. For particle mechanics, this means that the analog of Equations (31) and (32) would be

$$\{p_a, x^b\} = \delta_a^b, \tag{33a}$$

$$\{x^a, x^b\} = \epsilon^{ab}{}_c\theta^c, \tag{33b}$$

$$\{p_a, p_b\} = \epsilon_{ab}{}^c\beta_c. \tag{33c}$$

Reviews of non-commutative mechanics may be found in [45,46] and the references therein. Specifically, the Poisson generalization to non-commutative mechanics is given by the Moyal product:

$$\{f, g\} := f \star g - g \star f, \tag{34}$$

where the Moyal product here is defined as

$$(f \star g)(v) := \exp\left[\frac{1}{2}w^{ab}\partial_a\tilde{\partial}_b\right]f(v)g(\tilde{v})|_{\tilde{v}=v}, \tag{35}$$

and operators with a tilde operate only on tilde coordinates and non-tilde operators operate on non-tilde coordinates. The two sets of coordinates are made coincident in the end. Here, w^{ab} represents the generalization of the symplectic form, whose matrix is given by

$$[w^{ab}] = \left[\begin{array}{c|c} \epsilon^{ab}{}_{\ell}\theta^{\ell} & -\delta^{ab} \\ \hline \delta^{ab} & \epsilon^{ab}{}_{\ell}\beta^{\ell} \end{array} \right]. \tag{36}$$

Next, we transition to field theories from the particle mechanics picture above. In field theories, the field variable is the configuration variable ($\psi^a(y)$), and associated with this is canonical field momentum ($\pi_a(x)$). Analogous brackets to Equations (33a)–(33c) are implemented:

$$\{\pi_a(x), \psi^b(y)\} \propto \delta_a^b\delta(x-y), \tag{37a}$$

$$\{\psi^a(x), \psi^b(y)\} \propto \epsilon^{ab}{}_{\ell}\theta^{\ell}\delta(x-y), \tag{37b}$$

$$\{\pi_a(x), \pi_b(y)\} \propto \epsilon_{ab}^c \beta_c \delta(x - y). \tag{37c}$$

In gravitation, this procedure is not necessarily straightforward [47–52]. To make the problem somewhat tractable, the symmetry is frozen so that the metric is fixed to the form in Equation (26). The system then mimics a mechanical model, and it can be argued that the Poisson algebra be augmented in a similar manner to Equations (37a)–(37c). For the variables in Equation (26), these become

$$\{E_i, a_j\} = \delta_{ij}, \quad \{a_i, a_j\} = \epsilon_{ij}\theta, \quad \{E_i, E_j\} = \epsilon_{ij}\beta, \tag{38}$$

with ϵ_{ij} being the two-dimensional antisymmetric symbol. The Hamiltonian in Equation (27) is then used to evolve the system $\dot{a}_2 = \{a_2, S\}$, etc., with the crucial difference from regular Hamiltonian evolution being that the brackets now include the extended non-commutativity of Equation (38). As in Section 3.1, the critical variable for singularity studies is E_3 , with the classical singularity corresponding to $E_3 = 0$ at $\tau = 0$. In Figure 4, one of many possible evolutions are illustrated for the case of a spherical black hole. The initial conditions are set in the same manner as in Section 3.1. We notice from this figure that the singularity at $\tau = 0$, where in the regular theory $E_3 \rightarrow 0$, is delayed indefinitely in the non-commutative mechanics.

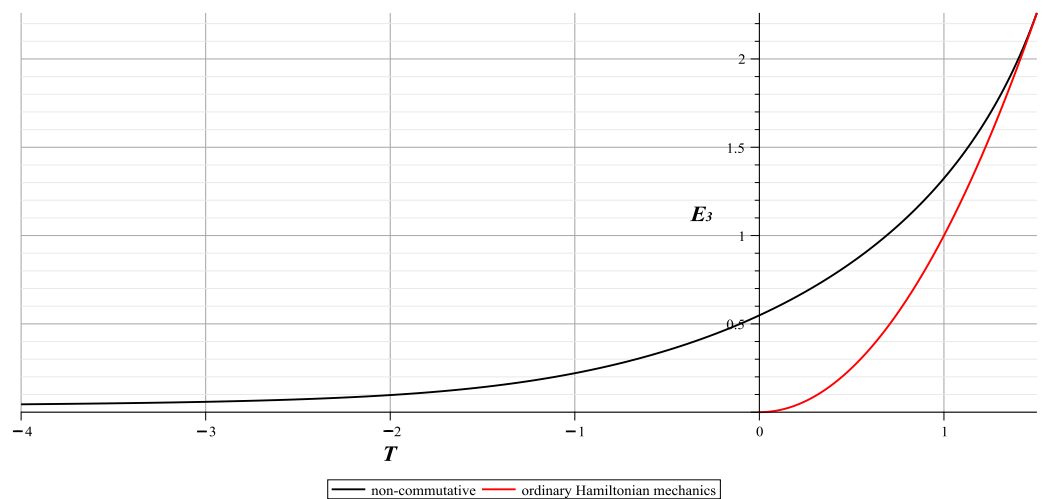


Figure 4. The Hamiltonian evolution of the E_3 factor in regular Hamiltonian mechanics (red) and the noncommutative theory (black) for a spherical black hole. The singularity in the regular scenario at $\tau = 0$ (signaled by $E_3 = 0$) is pushed indefinitely into negative τ values. Parameters: $M = 1$, $\Lambda = -0.1$, $\gamma = 0.274$, $\theta = -0.3$, and $\beta = -0.1$. Full results may be found in [44].

There are of course many scenarios which can be studied here, since not only can the geometry be spherical, toroidal, and higher-genus hyperbolic, but one can also choose θ and β to be zero or nonzero individually. We summarize the various results of [44] in Table 1.

A different non-commutative study using the same variables may be found in [53].

As in the previous section, we mention here some drawbacks to the scheme. First, from the results illustrated, the singularity is not necessarily removed in all scenarios. In particular, it seems non-commutativity in the canonical momenta is required in order to have singularity avoidance (i.e., $\beta \neq 0$). Non-commutativity in the configuration variables ($\theta \neq 0$) is insufficient. Also, some scenarios are inconclusive due to the formation of a second horizon. Also, adding the extra non-commutativity in the quantum mechanics of *free* particles yields wave equations whose solutions mimic those of particles coupled to an elec-

romagnetic field within regular quantum mechanics [54]. Therefore, non-commutativity seems to introduce some peculiar physical effects.

Table 1. Summary of the non-commutative evolution of various black holes.

1.	Spherical
	<ul style="list-style-type: none"> • $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ remove the singularity. • $\theta = 0, \beta \neq 0$: Singularity is removed. • $\theta \neq 0, \beta \neq 0$: Singularity is removed.
2.	Toroidal
	<ul style="list-style-type: none"> • $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ introduce a new horizon*. • $\theta = 0, \beta \neq 0$: Singularity is removed. • $\theta \neq 0, \beta \neq 0$: Singularity is removed for large θ values. For small θ a new horizon appears.
3.	Higher Genus
	<ul style="list-style-type: none"> • $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ introduce a new horizon. • $\theta = 0, \beta \neq 0$: Singularity is removed. • $\theta \neq 0, \beta \neq 0$: Singularity is removed for large θ values. For small θ values, a new horizon appears.

* The new horizon’s presence prevents the determination of a singularity beyond the second horizon since the numerical scheme cannot penetrate horizons.

3.3. Gravastars

One final alternative to singular spacetimes which we present in this brief review is an exotic object known as the gravastar (gravitational vacuum star). This model was first proposed by P. Mazur and E. Mottola [55], and the idea is to not only eliminate the black hole singularity but the horizon as well so that there is no black hole. The model is essentially an extension of what were previously known as de Sitter core black holes [56–60].

The basic idea behind the gravastar is the following. Under extreme gravitational conditions, such as those found near the singularity of a black hole, the vacuum undergoes a phase transition, and the spacetime locally becomes one of constant positive curvature. This is de Sitter spacetime, where the effective pressure equals minus the effective energy density. Note that this violates the inequality in Equation (A3) and therefore circumvents the singularity theorem of Appendix A. In the coordinate system of Equation (5), the de Sitter spacetime possesses the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (39)$$

with the cosmological constant $\Lambda > 0$. The Riemann tensor components in the orthonormal frame, whose defining vectors $e^{\hat{\mu}}_{\hat{\alpha}}$ point in the coordinate directions, are constant with a magnitude $\Lambda/3$. Therefore, since in the vicinity of the black hole singularity ($r = 0$ here) the spacetime is replaced by (39), the curvature singularity does not exist in the gravastar model. If one moves the cosmological constant term in Equation (1) to the right-hand side of the equals sign, one can interpret the cosmological constant term in Einstein’s equations as a stress–energy of the vacuum (Since if $T_{\mu\nu}$, the stress–energy tensor of the material vanishes, there is still a source term on the right-hand side of the field equations); that is, in this interpretation the stress–energy of the vacuum is given by

$$T_{\mu\nu}^{\text{vac}} = -\frac{\Lambda}{8\pi}g_{\mu\nu}. \quad (40)$$

Therefore, in the de Sitter scenario, which corresponds to $\Lambda > 0$, the energy density $-T^0_0$ is positive, and the pressures T^i_i (no sum) are negative.

The occurrence of such a phase transition described above is motivated by condensed matter physics, where a stable low-energy state (analogous to the gravitational vacuum here) phase transitions to another state at a critical temperature. Further discussion can be found in [61], and an excellent review of gravastars in general may be found in [62].

As mentioned above, the gravastar proposal not only aims at the elimination of the singularity but also the event horizon. However, there is now good observational evidence for black holes or at least objects that closely resemble black holes. It is therefore postulated that the phase transition occurs when gravitational conditions are such that a horizon is just about to form, and outside the gravastar, the gravitational field is almost indistinguishable from the spacetime of the corresponding black hole that the gravastar replaces. Qualitatively, the process would be something similar to the following. A star’s nuclear fuel diminishes near the end of its life, and as described in the section on Astrophysical Formation of Singular Spacetimes, gravitational contraction ensues. Assuming that nothing stops the contraction, as the material collapses in on itself, the gravitational field at some points in spacetime approaches closer and closer to that of an event horizon. When the conditions for event horizon formation are almost satisfied, the gravitational phase transition takes over such that the would-be black hole now evolves to a gravastar configuration. Therefore, just outside of where the horizon would have formed, the spacetime is almost exactly that of the corresponding black hole, making it difficult to distinguish a gravastar from a black hole using exterior observations. However, it is not impossible to distinguish between the two scenarios [63].

In the case of spherical symmetry, the interior de Sitter spacetime (Equation (39)) needs to transition to an essentially Schwarzschild spacetime (Equation (16)) at somewhere near but outside of the event horizon $r = 2M$. In order for this to happen continuously, the radial pressure T^1_1 and the tangential pressures T^2_2 and T^3_3 must differ in some region along the transition zone [64,65]. This makes the gravastar anisotropic in parts of its interior region. This anisotropy is demanded by the Einstein equation (Equation (1)) governing the gravitational field. Furthermore, it is physically desirable that the outermost layer of the gravastar possess non-exotic physics. This means that the outer layer, which we call the atmosphere of the gravastar, should have a decreasing radial pressure and energy density, just like one expects in an ordinary star. The qualitative behavior of the continuous radial pressure which transitions from de Sitter to Schwarzschild, as discussed here, is illustrated in Figure 5. Continuity of the radial pressure in static spherical symmetry is demanded by the Sygne junction condition [2]:

$$\left[T^{\mu}_{\nu} \hat{n}^{\nu} \right]_{-} - \left[T^{\mu}_{\nu} \hat{n}^{\nu} \right]_{+} = 0, \tag{41}$$

when taking the unit normal \hat{n}^{ν} as pointing radially outward from an $r = \text{const.}$ junction surface. The + and – subscripts indicate whether one is approaching the junction surface from above or below. Note that since $T^{\mu}_{\nu} = 0$ for the Schwarzschild metric, Equation (41) implies that at the stellar surface, where the material terminates, the radial pressure must be zero. This mathematically defines the boundary of the star.

There has been much work conducted on the gravastar alternative to black holes, and a comprehensive review may be found in [62]. Of particular importance in astrophysics are models with rotation. Perturbative rotating models were studied in [66–72], and a class of exact models was constructed in [73], where an exactly de Sitter interior was smoothly patched to an exactly Kerr (Equation (21)) exterior with a physically acceptable atmosphere region. Rotating models are much more complicated to analyze, and thus we omitted a detailed discussion of them here.

Again, we should close with some drawbacks to this model of singularity avoidance. One potential issue is that it is not clear why such a phase transition, as required for gravastar formation, should occur, nor exactly what the physics behind it would be. It

is also not clear why the phase transition, even if it is feasible, should occur at the onset of horizon formation. It is true though that there are arguments that quantum phase transition effects are not only strong field effects but can occur under relatively weak conditions [61], and thus the weakness of the gravitational field in the vicinity of a horizon does not necessarily preclude some sort of gravitationally induced quantum effects from occurring there.

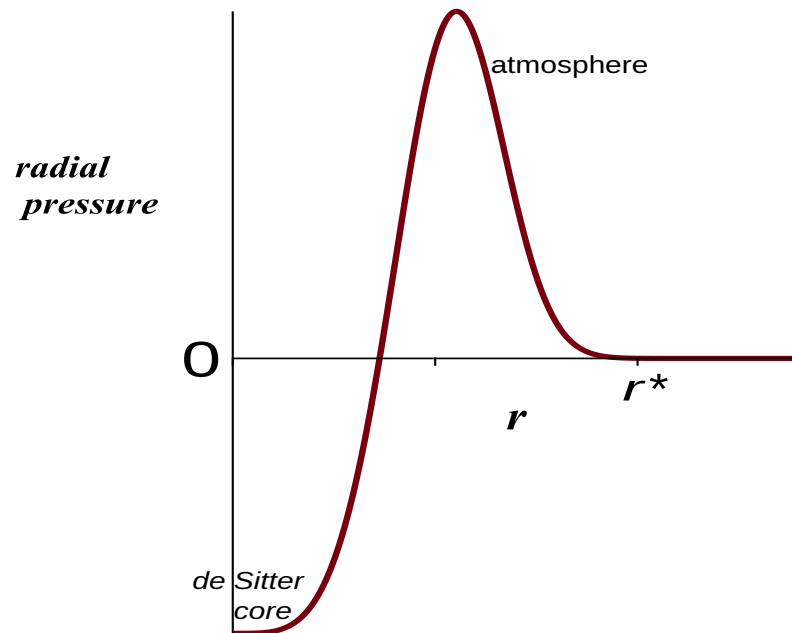


Figure 5. The radial pressure T^1_1 for a gravastar which is locally de Sitter at $r = 0$ and essentially Schwarzschild at $r > r^*$, taken as the boundary of the star. Since Schwarzschild is vacuum, this means the pressure must smoothly vanish at r^* . As stated in the main text, it is demanded that in the atmosphere this pressure is monotonically decreasing. Full results may be found in [64,65].

4. Conclusions

We presented here a brief summary of the history of singular spacetimes, mainly within the theory of general relativity. It has long been desired to find an acceptable way to remove the singularities in these spacetimes and replace them with something more benign. This often requires the invocation of new or exotic physics, and every alternative has its pros and cons. We presented here only a few possibilities, such as certain possible quantum gravity effects, non-commutative geometry effects, which were also inspired by quantum mechanics, and the gravastar alternative to black holes. All of these are admittedly speculative at the moment, but the topic of gravitation is extremely difficult to test experimentally in a strong field regime, and thus a certain degree of speculation is inevitable in the field. As mentioned in the main text, this is not an exhaustive review but will hopefully provide the reader with an understanding of the problem and possible avenues for a resolution.

Funding: This research received no external funding.

Acknowledgments: The author would like to thank S. Ray of the Centre for Cosmology, Astrophysics and Space Science (CCASS) at GLA University in Mathura, India for the invitation to participate in the conference associated with this manuscript. The author would also like to thank the anonymous referees for their suggestions, which improved the manuscript.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. A Simple Singularity Theorem

Here, we present a simple singularity theorem which illustrates how certain types of singularities can occur. The understanding of this theorem is not critical to the rest of the manuscript and may be skipped without loss of continuity. A more in-depth analysis of such singularity theorems may be found in the classic book by Hawking and Ellis [74]. We should note though that certain singularity theorems do not actually guarantee that a singularity will occur [75].

Before presenting the theorem and its proof, we state the strong energy condition. The strong energy condition asserts that

$$T_{\mu\nu}V^\mu V^\nu \geq \frac{1}{2}T V^\mu V_\mu \quad \forall \text{timelike } V^\mu, \tag{A1}$$

where $T_{\mu\nu}$ is the stress–energy tensor of the gravitating material and $T := T^\alpha_\alpha$ is its invariant trace. If the stress–energy tensor is diagonalizable, and the “zeroth” coordinate is time-like while the other three are space-like (Hawking–Ellis type one [74]), then the stress–energy tensor can be represented by

$$\left[T^\mu_\nu \right] = \text{diag}[-\rho, p_1, p_2, p_3], \tag{A2}$$

where ρ represents the energy density of the material in its rest frame and p_i is its principal pressures (i.e., momentum flux). For this structure of stress–energy tensors, the strong energy condition (Equation (A1)) simplifies to

$$\rho + p_i \geq 0 \quad \text{and} \quad \rho + \sum_{i=1}^3 p_i \geq 0. \tag{A3}$$

Now, if the Einstein field in Equation (1) with $\Lambda = 0$ holds, then it can easily be shown by tracing the Einstein equations that $T = -R/8\pi$ and, with this result, that

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} \right). \tag{A4}$$

Therefore, in terms of the Ricci tensor, the strong energy condition in Equation (A1) can be recast as

$$R_{\mu\nu}V^\mu V^\nu \geq 0. \tag{A5}$$

This statement of non-negative Ricci curvature physically implies that in general relativity, matter should not create a repulsive gravitational effect for time-like test particles.

The singularity theorem is stated as follows [3]

Theorem A1. Assume that the following field equations hold in a domain $D \subset \mathbb{R}^4$:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{A6}$$

Furthermore, assume that $T_{\mu\nu}$ is diagonalizable, as in Equation (A2). Let the strong energy condition in Equation (A1) hold. Also assume that the vorticity tensor vanishes, and further suppose that a congruence of time-like velocities V^μ possesses a negative expansion scalar $\theta(x) < 0$ in a neighborhood of D . Then, each of the time-like geodesics $x = \chi(s)$ will terminate at a negative expansion singularity at $x^* = \chi(s^*) \subset \partial D$ after a finite proper time.

Proof. Consider the Raychaudhuri equation with vanishing vorticity:

$$\frac{d\theta(s)}{ds} + \frac{1}{3}\theta^2(s) = -\left\{ \sigma^{\mu\nu}\sigma_{\mu\nu} + R_{\mu\nu}V^\mu V^\nu \right\}_{|x=\chi(s)}, \tag{A7}$$

where $\sigma_{\mu\nu}$ is the shear tensor whose square, $\sigma^{\mu\nu}\sigma_{\mu\nu}$, can be shown to be nonnegative [76]. The field equation (Equation (A4)) holds, and thus, via Equation (A2), the Raychaudhuri equation can be written as

$$\frac{d\theta(s)}{ds} + \frac{1}{3}\theta^2(s) = - \left\{ \sigma^{\mu\nu}\sigma_{\mu\nu} + 4\pi \left(\rho + \sum_i p_i \right) \right\}_{|x=\chi(s)}. \tag{A8}$$

If the strong energy condition holds, then the right-hand side of the above is less than or equal to zero such that

$$- \left\{ \sigma^{\mu\nu}\sigma_{\mu\nu} + 4\pi \left(\rho + \sum_i p_i \right) \right\}_{|x=\chi(s)} =: -[f(s)]^2 \leq 0, \tag{A9}$$

and multiplying Equation (A8) by $\theta^{-2}(s)$ allows us to write it as

$$\frac{d}{ds}\theta^{-1}(s) = \frac{1}{3} + [f(s)]^2[\theta(s)]^{-2} \geq 0. \tag{A10}$$

Now, assume that there is an initial convergence $\theta(s_0) < 0$. The solution to the differential in Equation (A10) is

$$\theta^{-1}(s) = \theta^{-1}(s_0) + \frac{1}{3}(s - s_0) + \int_{s_0}^s [f(s')]^2[\theta(s')]^{-2} ds'. \tag{A11}$$

While assuming continuity and differentiability, from the monotonicity of Equation (A11) there will be some finite $s = s^* > s_0$ where $\theta^{-1}(s^*) = 0$ or $\theta(s^*) \rightarrow -\infty$, and there will be infinitely strong contraction. \square

References

1. Newton, I. *Mathematical Principles of Natural Philosophy*; Phoenix Classics Ebooks: East Rutherford, NJ, USA, 1687.
2. Synge, J.L. *Relativity: The General Theory*; North-Holland Publishing: Amsterdam, The Netherlands, 1966.
3. Das, A.; DeBenedictis, A. *The General Theory of Relativity: A Mathematical Exposition*; Springer: New York, NY, USA, 2012.
4. Gourgoulhon, G. 3 + 1 formalism and bases of numerical relativity. *arXiv* **2007**, arXiv:gr-qc/0703035. [[CrossRef](#)]
5. Amaku, M.; Coutinho, F.A.B.; Éboli, O.; Massad, E. Some problems with the use of the Dirac delta function I: What is the value of $\int_0^\infty \delta(x) dx$? *Rev. Brasil. Ens. Fís.* **2021**, *43*, e20210132. [[CrossRef](#)]
6. Bracewell, R.N. *The Fourier Transform and Its Application*; McGraw-Hill: Boston, MA, USA, 2000.
7. Schwarzschild, K. Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie. *Sitz. Preuss. Akad. Wiss.* **1916**, *1*, 189–196.
8. Friedmann, A. Über die Krümmung des Raumes. *Z. Phys.* **1922**, *10*, 377. [[CrossRef](#)]
9. Lemaître, G. Un Univers homogène de Masse Constante et de Rayon Croissant Rendant Compte de la Vitesse Radiale des Nébuleuses Extra-Galactiques. *Ann. Soc. Sci. Brux.* **1927**, *47*, 49–59.
10. Robertson, H.P. Relativistic Cosmology. *Rev. Mod. Phys.* **1933**, *5*, 62. [[CrossRef](#)]
11. Walker, A.G. Completely Symmetric Spaces. *J. Lond. Math. Soc.* **1944**, *19*, 219–226. [[CrossRef](#)]
12. Oppenheimer, J.R.; Snyder, H. On Continued Gravitational Contraction. *Phys. Rev.* **1939**, *56*, 455. [[CrossRef](#)]
13. Jebsen, J.T. On the General Spherically Symmetric Solutions of Einstein’s Gravitational Equations in Vacuo. *Ark. Mat. Ast. Fys.* **1921**, *15*, 18. [[CrossRef](#)]
14. Birkhoff, G.D. *Relativity and Modern Physics*; Harvard University Press: Cambridge, MA, USA, 1923.
15. Kerr, R.P. Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. *Phys. Rev. Lett.* **1963**, *11*, 237. [[CrossRef](#)]
16. Carter, B. Complete Analytic Extension of the Symmetry Axis of Kerr’s Solution of Einstein’s Equations. *Phys. Rev.* **1966**, *141*, 1242. [[CrossRef](#)]
17. Penrose, R. Gravitational Collapse and Space-Time Singularities. *Phys. Rev. Lett.* **1964**, *14*, 57. [[CrossRef](#)]
18. Boyer, R.H.; Lindquist, R.W. Maximal Analytic Extension of the Kerr Metric. *J. Math. Phys.* **1967**, *8*, 265–281. [[CrossRef](#)]
19. Michell, J. On the Means of Discovering the Distance, Magnitude, &c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should be Found to Take Place in any of Them, and Such Other Data Should be Procured from Observations, as Would be Farther Necessary for That Purpose. *Philos. Trans. R. Soc. Lond.* **1784**, *74*, 35–57.
20. LaPlace, P.S. *Exposition du Système du Monde*; L’imprimerie du Cercle-Social: Paris, France, 1796.

21. Ewing, A. “Black Holes” in Space. *Sci. News Lett.* **1964**, *85*, 39. [[CrossRef](#)]
22. Siegfried, T. 50 Years Later, It’s Hard to Say Who Named Black Holes. Available online: <https://www.sciencenews.org/blog/context/50-years-later-its-hard-say-who-named-black-holes> (accessed on 1 October 2024).
23. Herdeiro, C.A.R.; Lemos, J.P.S. O Buraco Negro Cinquenta Anos Depois: A Génese do Nome. *Gazeta de Fisica* **2018**, *41*, 2.
24. Chandrasekhar, S. The Maximum Mass of Ideal White Dwarfs. *Astrophys. J.* **1931**, *74*, 81. [[CrossRef](#)]
25. Oppenheimer, R.J.; Volkoff, G.M. On Massive Neutron Cores. *Phys. Rev.* **1939**, *55*, 374. [[CrossRef](#)]
26. Shapiro, I.I. Solar system tests of general relativity: Recent results and present plans. In *General Relativity and Gravitation, 1989: Proceedings of the 12th International Conference on General Relativity and Gravitation*; Ashby, N., Bartlett, D.F., Wyss, W., Eds.; Cambridge University Press: Cambridge, UK, 1990; pp. 313–330.
27. The Event Horizon Telescope Collaboration; Akiyama, K.; Alberdi, A.; Alef, W.; Asada, K.; Azulay, R.; Baczkó, A.K.; Ball, D.; Baloković, M.; Barrett, J.; et al. First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole. *Astrophys. J. Lett.* **2019**, *875*, L6.
28. Frisoni, P. Introduction to Loop Quantum Gravity: Rovelli’s lectures on LQG. *arXiv* **2023**, arXiv:2305.12215. [[CrossRef](#)]
29. Smith, W.L.; Mann, R.B. Formation of Topological Black Holes from Gravitational Collapse. *Phys. Rev. D* **1997**, *56*, 4942. [[CrossRef](#)]
30. Nakahara, M. *Geometry, Topology, and Physics*, 2nd ed.; I.O.P. Publishing: Bristol, UK; Philadelphia, PA, USA, 2003.
31. Ashtekar, A.; Baez, J.; Corichi, A.; Krasnov, K. Quantum Geometry and Black Hole Entropy. *Phys. Rev. Lett.* **1998**, *80*, 904. [[CrossRef](#)]
32. Ashtekar, A.; Bojowald, M. Quantum Geometry and the Schwarzschild Singularity. *Class. Quant. Grav.* **2006**, *23*, 391. [[CrossRef](#)]
33. Modesto, L. Loop Quantum Black Hole. *Class. Quant. Grav.* **2006**, *23*, 5587. [[CrossRef](#)]
34. Boehmer, C.G.; Vandersloot, K. Loop Quantum Dynamics of the Schwarzschild Interior. *Phys. Rev. D* **2007**, *76*, 104030. [[CrossRef](#)]
35. Modesto, L. Semiclassical Loop Quantum Black Hole. *Int. J. Theor. Phys.* **2010**, *49*, 1649–1683. [[CrossRef](#)]
36. Ashtekar, A.; Pawłowski, T.; Singh, P. Quantum Nature of the Big Bang: Improved Dynamics. *Phys. Rev. D* **2006**, *74*, 084003. [[CrossRef](#)]
37. Chiou, D.W. Phenomenological Loop Quantum Geometry of the Schwarzschild Black Hole. *Phys. Rev. D* **2008**, *78*, 064040. [[CrossRef](#)]
38. Gan, W.-C.; Wang, A. A New Quantization Scheme of Black Holes in Effective Loop Quantum Gravity. *arXiv* **2024**, arXiv:2408.04494. [[CrossRef](#)]
39. Brannlund, J.; Kloster, S.; DeBenedictis, A. The Evolution of Λ Black Holes in the Mini-Superspace Approximation of Loop Quantum Gravity. *Phys. Rev. D* **2009**, *79*, 084023. [[CrossRef](#)]
40. Morales-Técotl, H.A.; Rastgoo, S.; Ruelas, J.C. Effective Dynamics of the Schwarzschild Black Hole Interior With Inverse Triad Corrections. *Ann. Phy.* **2021**, *426*, 168401. [[CrossRef](#)]
41. Zhang, H.; Gan, W.-C.; Gong, Y.; Wang, A. On the Improved Dynamics Approach in Loop Quantum Black Holes. *Commun. Theor. Phys.* **2024**, *76*, 035401. [[CrossRef](#)]
42. Connes, A. *Noncommutative Geometry*; Academic Press: San Diego, CA, USA, 1990.
43. Madore, J. An Introduction to Noncommutative Geometry. In *Geometry and Quantum Physics*; Gausterer, H., Pittner, L., Grosse, H., Eds.; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 2000; Volume 543, pp. 231–273.
44. Schneider, M.; DeBenedictis, A. Noncommutative Black Holes of Various Genera in the Connection Formalism. *Phys. Rev. D* **2020**, *102*, 024030. [[CrossRef](#)]
45. Djemai, A.E.F. Noncommutative Classical Mechanics. *Int. J. Theor. Phys.* **2004**, *43*, 299–314. [[CrossRef](#)]
46. Gouba, L. A Comparative Review of Four Formulations of Noncommutative Quantum Mechanics. *Int. J. Mod. Phys.* **2016**, *A31*, 1630025. [[CrossRef](#)]
47. Chamseddine, A.H. Deforming Einstein’s Gravity. *Phys. Lett. B* **2001**, *504*, 33–37. [[CrossRef](#)]
48. Mukherjee, P.; Saha, A. Note on the Noncommutative Correction to Gravity. *Phys. Rev. D* **2006**, *74*, 027702. [[CrossRef](#)]
49. Calmet, X.; Kobakhidze, A. Noncommutative general relativity. *Phys. Rev. D* **2005**, *72*, 045010. [[CrossRef](#)]
50. Calmet, X.; Kobakhidze, A. Second Order Noncommutative Corrections to Gravity. *Phys. Rev. D* **2006**, *74*, 047702. [[CrossRef](#)]
51. Fucci, G.; Avramidi, I.G. Noncommutative Einstein Equations. *Class. Quant. Grav.* **2008**, *25*, 025005. [[CrossRef](#)]
52. Ashieria, P.; Di Grezia, E.; Esposito, G. Non-commutative Einstein Equations and Seiberg–Witten Map. *Int. J. Mod. Phys. Conf. Ser.* **2011**, *03*, 143–149. [[CrossRef](#)]
53. Espinoza-García, A.; Torres-Lomas, E.; Pérez-Payán, S.; Díaz-Barrón, L.R. Noncommutativity in Effective Loop Quantum Cosmology. *Adv. High Energy Phys.* **2019**, *2019*, 9080218. [[CrossRef](#)]
54. Dąbrowski, L.; Parashar, P. A Free Particle in Noncommutative Space-time. *Czech. J. Phys.* **1996**, *46*, 1211–1215. [[CrossRef](#)]
55. Mazur, P.O.; Mottola, E. Gravitational Condensate Stars: An Alternative to Black Holes. *Universe* **2003**, *9*, 88. Original paper arXiv:gr-qc/0109035 (2001). [[CrossRef](#)]
56. Gliner, É.B. Algebraic Properties of the Energy-momentum Tensor and Vacuum-like States o^+ Matter. *Sov. J. Exp. Theor. Phys.* **1966**, *22*, 378.
57. Starobinsky, A.A. A New Type of Isotropic Cosmological Models without Singularity. *Phys. Lett. B* **1980**, *91*, 99–102. [[CrossRef](#)]
58. Markov, M.A. Problems of a Perpetually Oscillating Universe. *Ann. Phys.* **1984**, *155*, 333–357. [[CrossRef](#)]
59. Shen, W.; Zhu, S. Junction Conditions on Null Hypersurface. *Phys. Lett. A* **1988**, *126*, 229–232.
60. Poisson, E.; Israel, W. Structure of the Black Hole Nucleus. *Class. Quant. Grav.* **1988**, *5*, L201. [[CrossRef](#)]

61. Mazur, P.O.; Mottola, E. Gravitational Vacuum Condensate Stars. *Proc. Natl. Acad. Sci. USA* **2004**, *101*, 9545–9550. [[CrossRef](#)]
62. Ray, S.; Sengupta, R.; Nimesh, H. Gravastar: An Alternative to Black Hole. *Int. J. Mod. Phys. D* **2020**, *29*, 2030004. [[CrossRef](#)]
63. Chirenti, C.B.M.H.; Rezzolla, L. How to Tell a Gravastar from a Black Hole. *Class. Quant. Grav.* **2007**, *24*, 4191. [[CrossRef](#)]
64. Cattoen, C.; Faber, T.; Visser, M. Gravastars Must Have Anisotropic Pressures. *Class. Quant. Grav.* **2005**, *22*, 4189. [[CrossRef](#)]
65. DeBenedictis, A.; Horvat, D.; Ilijic, S.; Kloster, S.; Viswanathan, K.S. Gravastar Solutions with Continuous Pressures and Equation of State. *Class. Quant. Grav.* **2006**, *23*, 2303. [[CrossRef](#)]
66. Cardoso, V.; Pani, P.; Cadoni, M.; Cavaglia, M. Ergoregion Instability of Ultracompact Astrophysical Objects. *Phys. Rev. D* **2008**, *77*, 124044. [[CrossRef](#)]
67. Uchikata, N.; Yoshida, S.; Pani, P. Tidal Deformability and I-Love-Q Relations for Gravastars with Polytopic Thin Shells. *Phys. Rev. D* **2016**, *94*, 064015. [[CrossRef](#)]
68. Uchikata, N.; Yoshida, S. Slowly Rotating Thin Shell Gravastars. *Class. Quant. Grav.* **2016**, *33*, 025005. [[CrossRef](#)]
69. Posada, C. Tidal Deformability of Ultracompact Schwarzschild Stars and Their Approach to the Black Hole Limit. *Mon. Not. R. Astr. Soc.* **2017**, *468*, 2139.
70. Beltracchi, P.; Gondolo, P.; Mottola, E. Slowly Rotating Gravastars. *Phys. Rev. D* **2022**, *105*, 024002. [[CrossRef](#)]
71. Rutkowski, M.; Rostworowski, A. Ultracompact Rotating Gravastars and the Problem of Matching with Kerr Spacetime. *Phys. Rev. D* **2021**, *104*, 084041. [[CrossRef](#)]
72. Chirenti, C.; Rezzolla, L. Did GW150914 Produce a Rotating Gravastar? *Phys. Rev. D* **2016**, *94*, 084016. [[CrossRef](#)]
73. Ilijic, S.; DeBenedictis, A. On a Class of Exact Arbitrarily Differentiable de Sitter Cores with Kerr Exteriors: Possible Gravastar or Regular Black Hole Mimickers. *arXiv* **2023**, arXiv:2309.00551. [[CrossRef](#)]
74. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 1973.
75. Kerr, R.P. Do Black Holes have Singularities? *arXiv* **2023**, arXiv:2312.00841. [[CrossRef](#)]
76. Blau, M. Lecture Notes on General Relativity. Available online: <http://www.blau.itp.unibe.ch/GRLecturenotes.html> (accessed on 1 October 2024).

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.