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On the Thermomechanics of Hadrons and Their Mass Spectrum

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Abstract: A little-known thermomechanical relation between entropy and action, originally discovered by Boltzmann in the classical domain, was later reconsidered by de Broglie in relation to the wave–particle duality in the free propagation of single particles. In this paper, we present a version adapted to the phenomenological description of the hadronization process. The substantial difference with respect to the original de Broglie scheme is represented by the universality of the temperature at which the process occurs; this, in fact, coincides with the Hagedorn temperature. The main results are as follows: (1) a clear connection between the universality of the temperature and the existence of a confinement radius of the color forces; (2) a lower bound on the hadronic mass, represented by the universal temperature, in agreement with experimental data; and (3) a scale invariance, which allows the reproduction of the well-known hadronic mass spectrum solution of the statistical bootstrap model. The approach therefore presents a heuristic interest connected to the study of the strong interaction.

Keywords: thermomechanics; hadronization; Hagedorn temperature; color confinement; scale invariance

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1. Introduction

In 1897, L. Boltzmann discovered an important relationship [1] between the variation in entropy of a system and its corresponding variation in mechanical action and, therefore, a direct relationship (not mediated by statistics) between a thermodynamic quantity and a mechanical quantity—in other words, a thermomechanical relationship. This is a relationship valid for systems with monoperiodic molecular motion, whose trajectories are subjected to a virtual variation between fixed extremes, separated by a time interval that is a multiple of the internal period τ. The variation in the trajectory involves both a variation δQ of the heat Q exchanged by the system along the natural trajectory in a period and a variation δA of the mechanical (Maupertuis) action A, relative to the same period. The relationship is as follows:

$$\delta Q = \nu\, \delta A \tag{1}$$

where $\nu = 1/\tau$ is the frequency of the monoperiodic motion. If the heat Q is exchanged reversibly at a constant absolute temperature T, Equation (1) takes the form of

$$T\delta S = \nu\, \delta A \tag{2}$$

where S is the exchanged entropy. Equation (2) remained substantially forgotten but was reconsidered by de Broglie many years later [1,2]. De Broglie observed that, from the relativistic point of view, T and ν transform as the reciprocal of the fourth component of a four-vector, while S and A are invariant. Therefore, (2) can be broken into two distinct relations ([2] and references cited in [3]) as follows:

$$kT = h\nu \tag{3}$$

$$\delta S/k = \delta A/h \tag{4}$$

where k and h are constants having the dimensions of an entropy and an action, respectively. De Broglie, respectively, identified them with the Boltzmann constant and the Planck constant. He applied Equations (3)–(4) to the study of wave–particle duality ([2] and references cited in [3]).

In a previous work [3], this scheme has been re-elaborated in a slightly modified form. According to this reformulation, relations (3)–(4) define a connection between the virtual processes of dissociation–recombination of a single particle and its propagation as a renormalized particle, physically observable as an asymptotic state. The particle process therefore presents two polarities. The first is that of the virtual phenomena of dissociation and recombination on the Compton scale, introduced by Quantum Field Theory (QFT) in its relativistic formulation [4], and described here phenomenologically as "thermal" phenomena. The second is the propagation of the renormalized particle, which represents the "mechanical" polarity of the process; only this second level is accessible to observation. The topic is directly connected to the problem of the propagation of the phase wave associated with the particle [2], and, more generally, to the phenomenon of the emergence of its proper time [3].

A question that arises spontaneously is whether the thermomechanical entropy–action relation can be applied not only to the propagation phase of an elementary particle but also to the phase of its creation or annihilation. In our opinion, the answer depends on the interaction that mediates the process of creation or annihilation. Excluding from our considerations, the gravitational interaction, on which there is not sufficient experimental information, the electroweak and strong interactions remain. To our best knowledge, there is no literature about the existence of a specific temperature associated with electroweak interactions, so it seems difficult to apply thermomechanical reasoning to them. It is well known, however, that in the context of strong interactions, the phenomena of hadronization and deconfinement occur at a well-defined temperature, experimentally well characterized, which is the Hagedorn temperature T_H [5–9]. In this paper, we intend to investigate the possibility of applying the entropy–action relation to the phenomena of the creation and annihilation of hadrons.

To better understand our program, let us first consider, briefly, the propagation phase of a free hadron, specializing in the general scheme described in ref. [3] to the hadronic case. If M is the rest mass of the hadron and c is the limit speed, it is possible to define a temperature T through the following relation:

$$kT = Mc^2 \tag{5}$$

where k is the Boltzmann constant. The hypothesis advanced by de Broglie [2] is that the hadron, in its free propagation, is in contact with a "hidden" thermostat at the temperature T. In [3], this thermostat exchanges heat with the degrees of freedom of the virtual processes associated, according to the QFT description, with the free propagation of the hadron. The exchanged heat is a clock, which measures cyclic time. At the end of a cycle, the total heat released by the thermostat to these processes is Mc^2. In this interval, the relation between the entropy S_p exchanged by these processes with the thermostat and their mechanical action A_p is the following (p = propagation):

$$S_p/k = A_p/h \tag{6}$$

where h is the Planck constant. At the end of the interval is $S_p = k$, $A_p = h$; the entropy S_p is returned to the thermostat, while the action A_p is released to the renormalized particle. In other words, the action of the free, renormalized hadron is increased by h, while its quantum phase is increased by 2π and its proper time is increased by h/Mc^2. This process fuels the propagation of the hadronic phase wave, which can be experimentally detected by interferometric measurements (see, for example, [10], for a description of neutron interferometry procedures). Note that (6) connects the thermodynamic quantity S_p with the mechanical quantity A_p, directly and without statistical mediations. That is, it is a thermomechanical relation, as historically intended by Boltzmann and Helmoltz.

The question now is whether a similar scheme can be applied to hadron creation or annihilation, rather than to hadron propagation. Since the hadron formation process is clearly reversible (the inverse of creation is annihilation), we will focus here on hadron creation, maintaining that the same line of reasoning applies to the inverse process. In hadron creation processes, a certain amount of energy is released by the reaction between the particles entering the interaction vertex (e.g., e^+e^- annihilation); this energy is partly converted into the creation of outgoing hadrons.

A non-intuitive fact, first highlighted by Hagedorn in 1965 [5] and then extensively explored by other authors [11–15], is that the formation of new hadrons occurs at a defined "hadronization temperature" T_H, today commonly known as the Hagedorn temperature. This temperature is universal, i.e., independent of the specific hadron. The modern interpretationis that it is the deconfinement temperature of the quark–gluon plasma, which is the constituent of all hadrons [16,17]. In attempting to elaborate a thermomechanical scheme analogous to that of the propagation, one must therefore first consider that the thermostat involved in the formation phase is different from the one involved in the propagation phase. It is not constituted by the virtual processes of the non-renormalized hadron, which does not yet exist, but by the rapid thermalization—by the quark and gluonic degrees of freedom—of the energy made available to them by the reaction. As in the case of propagation, this energy is expressed, as far as the formation of a single hadron of mass M is concerned, by the rest energy Mc^2; this is in fact the energy required for the formation of a renormalized hadron (M is the *renormalized* mass, that is, the one experimentally measured). However, the temperature at which this energy is exchanged will no longer be the temperature T defined by (5); it will instead be T_H. The universality of T_H is connected with the rapid loss of memory of the state preceding the thermalization.

From the point of view of our idealized description, the process can be simplified in the following terms: the ideal thermostat with which the renormalized hadron in formation exchanges the heat $Q = Mc^2$ is the same for all hadrons, and it constitutes a thermal reservoir at the temperature T_H. Since, as we have seen, the exchange is reversible, the formation entropy of the hadron is $S_f = Q/T_H = Mc^2/T_H$. This relation can be put in the same form as Equation (6) as follows:

$$S_f/k = A_f/h \tag{7}$$

(f = formation) assuming that hadronic formation is associated with a change in action:

$$A_f = h \cdot (Mc^2)/(kT_H) \tag{8}$$

The time interval for the formation of the hadron is therefore the following:

$$T_0 \approx A_f/(Mc^2) = h/(kT_H) \tag{9}$$

independent of the hadron. This is an important difference with respect to propagation, because, in the latter, the duration h/Mc^2 of the cycle depends on the hadron. It is in fact possible to define a hadronization radius as follows:

$$R_0 = cT_0 \approx hc/(kT_H) \tag{10}$$

which is a property of the quark–gluon plasma, independent of the hadron. We can say that the property of the color charge appears only on scales smaller than R_0, while on larger scales, it disappears, leaving colorless hadronic matter.

In summary, we propose that the Boltzmann–de Broglie entropy–action relation [1,2] can be applied to the hadronic creation or annihilation phase, assuming that, in this phase, the forming or dissolving hadron exchanges heat with a specific thermostat at the Hagedorn temperature. In the Materials and Methods section, we justify this proposal, supporting it with a simple general thermodynamical argument.In the Results section, we show how the present phenomenological scheme involves a scale invariance, leading to the hadronic mass spectrum, which is the solution of the statistical bootstrap equation. In the Discussion section, we examine a possible geometric justification of this scale invariance. In the Conclusions section, the connections with the general leitmotifs of the history of hadronic phenomenology are briefly stated.

2. Materials and Methods

Equations (6) and (7) can be seen as specializations of the well-known Bekenstein relation [18], which gives the maximum entropy content S of a system with energy content E enclosed in a volume of radius R, as follows:

$$S/k = (ER)/(hc) \tag{11}$$

Let us first consider the propagation Equation (6). It is comparable to (11) at the only instant in which it admits a space–time representation, that is, at the end of a thermal cycle [3]. In such conditions, as we have already mentioned, we have $S_p = k$, $A_p = h$. If the entropy S_p, returned to the thermostat, is substituted for S in Equation (11), we have $ER = hc$; that is, the action ER/c equals h, as required by (6). But the end of the thermal cycle corresponds to a completely renormalized particle, whose energy content E is Mc^2. This statement is true in the rest frame of the particle, but Equation (6) is relativistically invariant ([3] and references cited therein). It follows that the radius R, within which the virtual phenomena of dissociation and recombination involving the particle are confined, is given by $R = h/Mc$. In other words, the renormalization scale is the Compton scale. In the case of a hadron, which is a composite entity, the radius $r = h/Mc$ will not only represent the scale at which a hadronic dissociation is recovered by hadronic recombination; it will also be the confinement scale of the internal quark and gluonic processes, which are permanently virtual because they do not appear as separate asymptotic states with a separately defined proper mass. In this sense, we will allow ourselves to identify r as the "radius" of the hadron. This identification is compatible with experimental data ([19], Chapter 5).

Let us now turn to the formation Equation (7). Here, the entropy of formation is, as we have previously stated, $S_f = Mc^2/T_H$. We have also seen that it is, again, $E = Mc^2$. Substituting S_f for S in (11), we then have $R = hc/kT_H = R_0$. In other words, the interpretation of R_0 as a hadronization radius is confirmed. We can therefore affirm that (6) and (7) are consistent with known thermodynamics. Since the final phase of the hadronization process must consist ofthe production of the renormalized hadron, the radius r associated with the propagation of the latter must be contained within the radius R_0 that sizes the region of origin. That is, it must be $r \leq R_0$, a condition from which the inequality $Mc^2 \geq kT_H$ follows. This relationship is well verified experimentally. The different determinations of kT_H give values included in the range of 140–160 MeV [20,21], which are therefore superimposable on the rest energy of the lightest hadron, i.e., the pion, which is about 140 MeV.

We note that from the relations $S_f = Mc^2/T_H$, $Mc^2 \geq kT_H$, it follows that $S_f/k \geq 1$. This is another difference between the formation phase and the propagation phase, in which instead, as previously illustrated, we have $S_p/k = 1$ at the end of the thermal cycle. In both cases, however, the average entropy $S \cdot p(S)$ is ≤ 1 in k units. In the case of free propagation, in fact, the probability $p(S)$ of the state is identically 1 due to the absence of fluctuations in the entropy exchanged in a cycle, which is identically equal to 1 in units k; therefore, $S \cdot p(S) = 1 \cdot 1 = 1$. This implies an action exchanged in a cycle that is identically h, in accordance with the conservation principles that guarantee free propagation. In the case of formation, on the other hand, $S \cdot p(S) = S \cdot \exp(-S) \leq 1$, from which we obtain an average action associated with the formation process $\leq h$. This result is consistent with the virtual nature of the processes occurring at the interaction vertex where the hadron is generated.

The narrative presented in the first two sections can be reversed by observing that strong interactions between hadrons are contact interactions. They are mediated by hadrons and can therefore always be reduced to interaction vertices into which hadrons enter and from which hadrons exit. This feature cannot be extended to electroweak or gravitational interactions, which are mediated by gauge quanta (such as the photon, the graviton, etc.) *distinct* from the particles they connect. The hadrons entering or leaving a strong interaction vertex will have distinct masses M and distinct renormalization radii $r = h/Mc$. It follows that the size of the interaction vertex is, approximately, expressed by the maximum value of r, i.e., by h/M_0c, where M_0 is the minimum hadronic mass. It is known that this is the mass of the pion. Thus, the strong charges within the quark–gluon plasma, the universal constituent of hadrons, become renormalized (by a generation of hadrons) on a scale $R_0 \approx hc/M_0c^2 = h/M_0c$, which corresponds to a temperature T_H defined by $M_0c^2 \approx kT_H$. R_0 and T_H are therefore properties of the plasma and, as such, independent ofthe mass M of a specific hadron.

The uncertainty relation associated with the actualization of a single outgoing hadron, $A_f \geq h$, is, by virtue of Equation (8), consistent with the assumption $M \geq M_0$, from which the relations $Mc^2 \geq kT_H$, $S_f/k \geq 1$ follow. The relation $A_f \geq h$ also expresses the independence of the hadronization radius R_0 and hadron formation time T_0 on the mass M [Equations (9) and (10)]. In other terms, the coalescence of the quark–gluon plasma into hadrons occurs at a temperature T_H independent ofthe values of M and $r = h/Mc$.

The fact that thermalization completely erases the organization of the quark–gluon plasma into hadrons translates into the behavior of this plasma, which is independent of M, and hence of $r = h/Mc$, that is, into a behavior that is (approximately) independent on the scale. Since, in the cooling plasma, there is nothing that fixes a scale of nucleation of the hadrons exiting from the vertex, in the distribution of the latter, all values of M (or of $r = h/Mc$) must be equally favored. In other words, if $n(M)$ is the reciprocal of the probability of selection of a mass fluctuation of amplitude M as the actual outgoing hadron, $n(M)$ must be independent of M. If the mass M given up to the actualization process of the outgoing hadron is seen as the absolute value of the difference $|M_1 - M_2|$ of the two masses M_1 and M_2, potentially given up to the same process, the correlation function $n(|M_1 - M_2|)$ is constant. In other words, the correlation length on the domain of M (and hence on the domain of r) is infinite; thus, we have a phenomenon entirely analogous to critical opalescence. In a later section, we will see how this independence from M is actually implemented in a scale-invariant fractionation scheme in the Lund string model.

The ready deletion, by thermalization, of the memory of hadrons entering a strong interaction vertex thus implies the approximate scale invariance of the interactions. The scale invariance is, however, approximate, because it is obviously broken at low energies, at the scale $Mc^2 = kT_{\mathrm{H}}$, where the finite value of R_0 sets an upper limit to the spatial size of nucleation. The idea of an approximate scale-invariant hadronic formation stimulates an application of the conceptual scheme proposed here, which is set out in the next section.

3. Results

If, in a strong interaction vertex, the annihilation of the incoming hadrons brings into play an energy $\geq Mc^2$, then a portion of Mc^2 of this energy can be converted into the creation of a hadron of mass M. In this case, if $n(M)$ is the number of virtual hadrons of that mass potentially exiting the vertex, one of them will be actualized as an actual outgoing hadron. However, we have seen, in the previous section, that these hadrons are associated with a renormalization/confinement "radius" $r = h/Mc$. The approximate scale invariance of the interaction means that, approximately, $n(M)$ must be independent of M. The approximation should be better the further away we are from the upper limit R_0 of the variable r, due to the foreseeable decrease in edge effects. In other words, the approximation should tend to the exact result in the limit $M \to \infty$.

From a phenomenological point of view, the function $n(M)$ can be constructed as the product of three factors. First, the same mass M can correspond to hadronic states of different quantum numbers; we will indicate with $N(M)$ the number of distinct hadronic states of identical mass M. We then observe that if the interaction volume has dimensions $\approx {R_0}^3$, as indicated in the previous sections, the number of hadrons of a specific state that can find place in this volume is given by $[R_0/(r(M)]^3$, where $r(M) = h/Mc$ is the "radius" of the hadron of mass M. Consequently, in the interaction volume, there can be $\approx [R_0/(h/Mc)]^3$ distinct virtual hadrons in the same state of mass M.

The transfer of energy Mc^2 actualizes a virtual hadron which is a quantum fluctuation in energy of amplitude Mc^2, manifested in a thermostat at temperature T_H. As we know from statistical mechanics, the probability of such a fluctuation is $\exp(-Mc^2/kT_H)$. The exponent is nothing other than the entropy of formation S_f already discussed in the previous sections. Therefore, $n(M) = N(M) \cdot [R_0/(h/Mc)]^3 \cdot \exp(-Mc^2/kT_H)$. The requirement that $n(M)$ be constant then translates into the following relation:

$$N(M) \propto M^{-3} \exp(Mc^2/kT_H) \tag{12}$$

This approximate expression, which tends to the exact value in the limit $M \to \infty$, is nothing else than the well-known analytical solution [22] of the statistical bootstrap model [23,24]. It is demonstrated experimentally both by direct inspection of the systematics of hadronic states reported in the Particle Data Group publications, as was conducted historically by Frautschi and co-workers [25–27], and by the analysis of the distribution of transverse momenta in strong interactions between hadrons, as originally performed by Hagedorn [6]. Its present derivation without reference to the fractality (or self-similarity) of the hadronic generation mechanism, invoked by Hagedorn in his seminal work, is not surprising. In fact, fractality is connected to scale invariance [28,29], which is the starting point of the present derivation.

The conceptually relevant point is that the function $n(M)$ seems to continue to have a meaning even in the regime of full thermalization at $T = T_H$, so much so that it gives rise, with its approximate constancy, to the experimentally verified relation (12) (see Figure 1). In other words, although for temperatures $T > T_H$, there is an effective deconfinement and the concept of hadron is no longer applicable, for $T = T_H$, this concept is still applicable, but the coalescence of quarks and gluons in hadrons becomes independent of the hadronic "radius" r. For $T = T_H$, a sort of critical opalescence occurs, manifested in correspondence with a critical phase of the quark–gluon plasma in the process of cooling [30]. The process goes as if, in the regime $T > T_H$, the interaction energy wasconverted into the creation of strong charges (quarks and gluons), which homogeneously populate a region of size R_0; once this conversion is completed, this region fissions into smaller subregions, the hadrons, approximately, without preference for the size r of each subregion. This second phase occurs at the temperature T_H, defined by the geometric size R_0, according to (10). From an experimental point of view, these smaller regions are the well-known "fireballs" thatdecay, giving rise to hadrons exiting the vertex. The existence of these objects, which move away from the interaction vertex with different velocities, is well evidenced by the kinematic analysis of strong interactions between hadrons, both in cosmic rays and in accelerators. It is known that—as required by the statistical bootstrap model and as confirmed by the analysis of experimental data—these objects are nothing else than hadronic resonances. We refer the interested reader to ref. [24] for an in-depth discussion of the topic and an extensive bibliography.

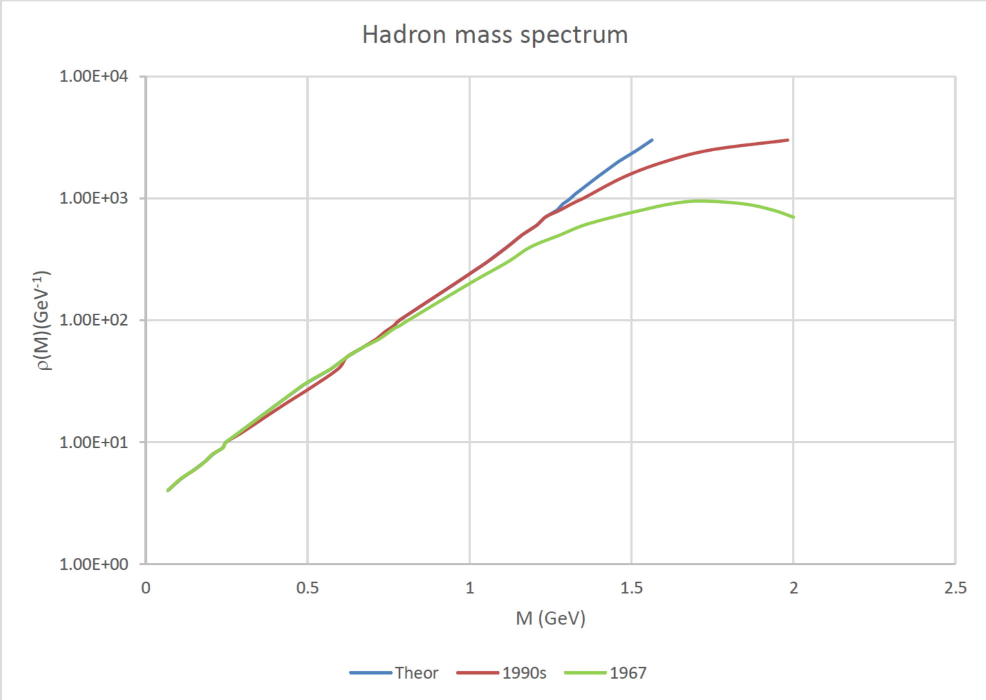

Figure 1. The smoothed mass spectrum of hadronic states as a function of mass. Experimental data: *green line* with the 1411 states known in 1967; *red line* with the 4627 states of mid-1990s. The *blue line* represents the exponential fit with Equation (12) yielding T_H = 158 MeV. Adapted from Rafelski and Ericson ([24], Ch. 6, Figure 6.2).

While we do not dwell here on the old question of whether the geometric constraint R_0 is derivable from QFT or is additional to it, referring the reader to the substantial literature on the subject (see, for example, [31–37]), in the next section we show a simple geometric model of hadronization functional to a qualitative illustration of the two main concepts emerging from these considerations: the finite confinement radius and scale invariance.

4. Discussion

The confinement, i.e., the finite value of R_0, is connected with the stability of the vacuum with respect to the creation of quark–antiquark pairs of "up" and "down" flavor; in other words, the separation of such pairs has a finite energetic cost, plausibly originated from the symmetry breakings that give mass to quarks. Since pions of all charge states originate from separation processes of this type, they have positive mass and this fact, as we have seen, implies a finite value of R_0. In other works ([38,39] and a work in progress [40]), we have proposed a geometric description of confinement, which can be linked to non-perturbative aspects of quantum chromodynamics, along a line of reasoning conceptually similar to that followed by Feynman for the derivation of the curved space of general relativity from the non-linear terms of quantum gravity on a flat space [41]. We will not go into the description of this proposal in depth, referring the interested reader to the articles in which it is exposed, since what we need in the context of this article is only an intuitive model that accounts for the dynamics of hadronization that emerges from the argumentation exposed in the previous sections.

It is enough to say that, according to this proposal, a position eigenstate of a hadron corresponds to a de Sitter space (dS) tangent to spacetime at the pointevent relative to that eigenstate. The constant-time section of the dS space is a four-dimensional sphere of radius $r = h/Mc$ (M = hadron mass), tangent to ordinary space. The strong charges (quarks and gluons) are assumed to be confined on the surface of this 4-sphere, while their spacetime positions are given by the vertical projection, onto spacetime, of their positions on the 4-sphere. Since the 4-spherical space is closed, Gauss' theorem imposes a condition of white global color on the 4-sphere [42], while the spatial distance between two charges cannot exceed $2r$.

A 4-sphere on which, say, the valence quarks A, B, and Care confined can then split into two 4-spheres on which the quarks A′, D′, and C′ and D″ and B′are confined, respectively. Here, A′, B′, and C′ are, respectively, the projections of A, B, and C on the two new 4-spheres and have the same color as the original quarks, while D′ and D″ have opposite colors (D′ has the same color as B) and are the projection, on the two new 4-spheres, of a point of the original 4-sphere that was neutral (Figure 2). Naturally, in addition to the process of fission of a hadron into two or more hadrons, the inverse process of fusion of several hadrons is also imaginable, with the annihilation of charges of opposite colors. Let us remember, however, that the radius r is limited above by R_0.

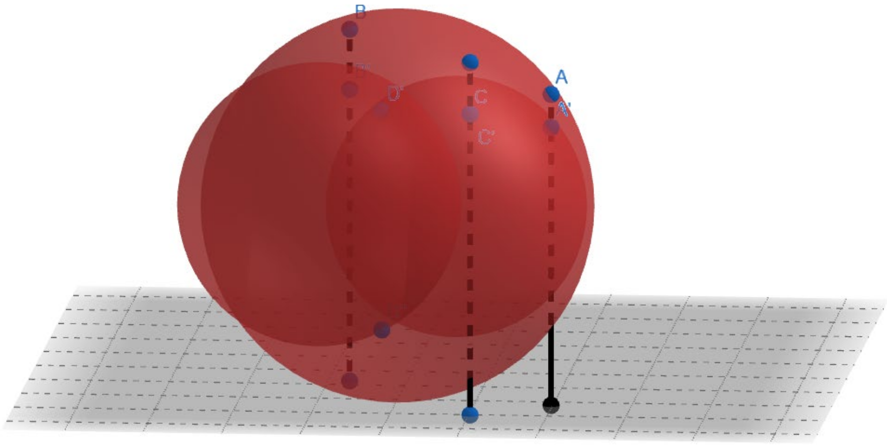

Figure 2. Visual representation, using three-dimensional spheres, of the fission of a hadron into two hadrons, with the creation of a pair of opposite colored charges. All the spheres are tangent to ordinary space. For clarity, the relativistic contraction of the radii of the spheres is not shown.The additional coordinate, perpendicular to spacetime, is only an aid in visualizing the intrinsic curvature of the positional constraint on strong charges and has no physical relevance in itself.

It is now possible to hazard a geometric description of what happens in a vertex of strong interaction. The hadrons entering the vertex are melted if the constituent charges thermalize rapidly at a temperature $T > T_H$, and the result is the quantum delocalization of the individual strong charges along each spatial coordinate, including the fifth perpendicular to spacetime. The average extent of this delocalization is R_0 and it corresponds, by the uncertainty principle, to a minimum energy kT_H, connected to R_0 by (10). When cooling brings the plasma to the minimum temperature T_H, the color charges are completely delocalized in the space delimited by the projection of the 4-sphere of radius R_0 on spacetime. The dynamics of their interaction aretherefore approximately independent of the scale. At this point, the formation of the outgoing hadrons begins, which will be conditioned by this approximate invariance, and by the erasure of the memory of the previous states due to thermalization. Under such conditions, as discussed in previous sections, the mass spectrum of the outgoing hadrons will satisfy the relation (12).

For a correct appreciation of this construction, it should be noted that, for all hadrons, the hadronization time T_0 is orders of magnitude smaller than the decay time; only in the case of decays mediated by the strong interaction are the two times approximately superimposable [43]. The only exception is the top quark, whose lifetime (~5×10^{-25} s) is one order of magnitude smaller than T_0; for this reason, the top quark does not form hadrons. According to the proposed description, a top–antitop pair generated in an associated production process remains delocalized on a 4-sphere of radius R_0 and it decays before forming hadrons.

If, in the splitting process exemplified in Figure 2, the space–time positions of the quarks are arranged in a straight line, the phenomenon described by the breaking of a

Lund string [44,45] is obtained. According to this one-dimensional description, the linear hadronic string A–C–B (along which the flow of color is assumed to be confined) breaks into the two linear strings A–C–D and anti-D–B. The breaking point is intermediate between the C and B quarks, and from it emerges the D and anti-D quarks that become the ends of the outgoing strings. The "breaking" of the string at this point corresponds, in Figure 2, to the appearance of the two quarks D′ and D″ on the spheres of the two final hadrons in a space–time position that did not correspond to any quark in the initial hadron. At the moment of their appearance, the two quarks are separated in the "fifth dimension". Such separation obviously does not appear in the Lund string model, which describes the appearance of the D–anti-D pair in terms of their exit from a spatial point belonging to the parent string. This output is modeled as a tunnel effect. What is important to note here is the following form of the chosen tunneling factor [45]:

$$\exp(-\pi\, {m_T}^2/\kappa) = \exp(-\pi\, m^2/\kappa)\, \exp(-\pi\, {p_T}^2/\kappa) \quad (13)$$

where m is the mass of the quark D, m_T is its transverse mass, and p_T is its transverse momentum; κ is the string tension. It is evident that (13) does not contain the mass of the hadron undergoing fragmentation, nor the masses of the hadrons produced by the fragmentation (the string tension is assumed to be universal). The fragmentation process is therefore independent ofthese parameters, consistent with the scale invariance previously discussed.

The coherence between the scenario represented in Figure 2 and the Lund model is probably even deeper. Each point of the string is a potential breaking point and can be seen as the limit of an infinitesimal interval dl of string to which an energy dE is associated. There exists therefore a string tension $\kappa = dE/dl$. Now, this limit, which from the space–time point of view is a dimensionless point, is instead provided with a finite extension $2R_0$ in the fifth dimension, as is evident in Figure 2. The product $2R_0\kappa$ is the thermal energy associated with the separation of color charges along the fifth dimension, that is, precisely the breaking of the string when the temperature is T_H. This splitting originates a quark–antiquark pair that can manifest itself in six different quark flavors (u, d, s, b, c, t) and three different colors (R, G, B) for a total of 18 different possible outputs. In the (purely virtual) point limit, the difference between the Compton wavelengths (and therefore the tunneling factors (13)) of the different quark flavors is not relevant, and therefore the equipartition of the energy can be assumed. The thermal energy associated with each "degree of freedom" of the breaking is therefore kT_H and the total energy is $18kT_H$. From the equation $2R_0\kappa = 18kT_H$, we obtain $\kappa = 9kT_H/R_0 = 9hc/{R_0}^2$. The energy $2R_0\kappa$ is distributed over the surface, in five-dimensional space, having as sides the extension $2R_0$ of the breaking point along the fifth dimension, and the maximum spatial extension $2R_0$ of the string, along which the breaking point is (prior to breaking) delocalized. Dividing the energy $2R_0\kappa$ by the extension $2R_0$ of this surface along the ordinary space gives the energy density along the space element dl: $(2R_0\kappa)/2R_0 = \kappa$. The energy dE of the string element dl is therefore κdl, as required.

Since, as we have seen, hc/R_0 is approximately equal to the pion rest energy (137 MeV), from the relation $\kappa = 9hc/{R_0}^2$,we have $\kappa \approx 0.88$ GeV/fm. This value is in agreement with both the experimental data relating to the slopes of the Regge trajectories [46] and with the lattice computations [47]. It should be emphasized that the tension κ thus estimated does not depend on the point chosen on the string, nor on the length of the string.

5. Conclusions

Equation (12) represents, as is known, the exact analytical solution [22] of the statistical bootstrap model equation [48]. This model was formulated in an era in which hadrons were considered to have no internal structure, and it envisaged a distribution of the energy, released at the interaction vertex, of a fractal type. Hadrons, in other words, had to be composed of other hadrons, the latter of still others, and so on [48].

This vision merged with that of the bootstrap in a wider sense, proposed by Chew and Frautschi [49], or was at least parallel to it. The bootstrap interpreted hadrons at the same time as elementary particles and as mediators of (strong) force between them. The role played by the hadron in a specific reaction, interacting particle or interaction mediator could be exchanged through the operation of crossing the interaction diagrams. It was assumed that the mediation of the interaction occurred through rapid thermalization (a hypothesis we have revisited), which made the weight of the interaction channel proportional to the phase space factor [27]. Neglecting, on the basis of heuristic considerations, the correlation terms, the exchange was then described by a summation of contributions from distinct hadronic states. A duality property of the strong interactions thus emerged.

What we are interested in noting here is that this duality seems to emerge, as an approximation, from (12) and, then, retracing our journey backward, from (7); that is, in a seemingly unexpected way, from a simple adaptation of the de Broglie thermal model to the hadronic formation phase. In this adaptation, a peculiar role is played by the presence of a confinement radius R_0 of the strong force, which, on the one hand, originates a universal hadronization temperature through Equation (9), and, on the other hand, determines the fractal nature of the process through Equation (12). This role of R_0 can be clarified by simple geometric–topological modeling, regardless of the—however possible—justification of such models in terms of an underlying gauge theory. In this sense, it seems to us that the proposed adaptation presents at least a heuristic value in the phenomenological description of the non-perturbative regime of the strong interaction.

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