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Antimatter Gravity: Second Quantization and Lagrangian Formalism

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Abstract: The application of the CPT (charge-conjugation, parity, and time reversal) theorem to an apple falling on Earth leads to the description of an anti-apple falling on anti–Earth (not on Earth). On the microscopic level, the Dirac equation in curved space-time simultaneously describes spin-1/2 particles and their antiparticles coupled to the same curved space-time metric (e.g., the metric describing the gravitational field of the Earth). On the macroscopic level, the electromagnetically and gravitationally coupled Dirac equation therefore describes apples and anti-apples, falling on Earth, simultaneously. A particle-to-antiparticle transformation of the gravitationally coupled Dirac equation therefore yields information on the behavior of "anti-apples on Earth". However, the problem is exacerbated by the fact that the operation of charge conjugation is much more complicated in curved, as opposed to flat, space-time. Our treatment is based on second-quantized field operators and uses the Lagrangian formalism. As an additional helpful result, prerequisite to our calculations, we establish the general form of the Dirac adjoint in curved space-time. On the basis of a theorem, we refute the existence of tiny, but potentially important, particle-antiparticle symmetry breaking terms in which possible existence has been investigated in the literature. Consequences for antimatter gravity experiments are discussed.

Keywords: antimatter gravity; CPT symmetry; antimatter free-fall experiments; Lorentz violation; Dirac equation; curved space-time

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1. Introduction

It is common wisdom in atomic physics that the Dirac equation describes particles and antiparticles simultaneously, and that the negative-energy solutions of the Dirac equation have to be reinterpreted in terms of particles that carry the opposite charge as compared to particles, and in which numerical value of the energy E is equal to the negative value of the physically observed energy [1]. Based on the Dirac equation, the existence of the positron was predicted, followed by its experimental detection in 1933, by Anderson [2]. If we did not reinterpret the negative-energy solutions of the Dirac equation, then the helium atom would be unstable against decay into a state where the two electrons perform quantum jumps into continuum states [3]. This phenomenon is extremely well known in atomic physics as the "Brown–Ravnhall disease" and leads to actual, practical, numerical problems in so-called Multi–Configuration Dirac–Fock (MDCF) atomic structure codes, where considerable numerical and analytic effort has been invested into a resolution of said problems, with the help of projection operators [4–7]. If one did not invoke the positive-energy projection operators in MCDF codes, then, for a system as simple as helium, nonsensical results would be obtained. Namely, one of the electrons could undergo a quantum jump into the positive-energy continuum, the other, into the negative-energy continuum, with the sum of the energies of the two continuum states (final state of the two-electron nonradiative transition) being equal to the sum of the two bound-state energies of the helium atom from which the transition started [3,8,9]. The "Brown–Ravnhall disease" is of course addressed, on the theoretical level, by invoking the Dirac sea, or, alternatively, by invoking the reinterpretation principle and quantum field theory. Let us recall that the development of quantum field theory made it possible to reformulate the Dirac equation in a way that treats the antiparticles as "sea" particles rather than the absence of a particle from the "Dirac sea of negative-energy particles". Still, it is sometimes overlooked that the Dirac equation describes particles and antiparticles simultaneously, which constitutes a numerical manifestation being "Brown–Ravnhall disease" (see [3]).

The absolute necessity to reinterpret the negative-energy solutions of the Dirac equation as antiparticle wave functions, i.e., the necessity to interpret positive-energy and negative-energy solutions of one single equation as describing two distinct particles, hints at the possibility to use the Dirac equation as a bridge to the description of the gravitational interaction of antimatter. Namely, if the Dirac equation is being coupled to a gravitational field, then, since it describes particles and antiparticles simultaneously, the Dirac equation offers us an additional dividend: In addition to describing the gravitational interaction of particles, the Dirac equation automatically couples the antiparticle (the "anti-apple"), which is described by the same equation, to the gravitational field, too.

Corresponding investigations have been initiated in a series of recent publications [10–13]. One may ask whether the dynamics of particles and antiparticles differ in a central, static, gravitational field, but also, if there are any small higher-order effects breaking the particle-antiparticle symmetry under the gravitational interaction. For the central gravitational field, this question has been answered in [10–12], with the result being that the Dirac particle and antiparticle behave exactly the same in a central gravitational field, due to a perfect particle-antiparticle symmetry which extends to the relativistic and curved-space-time corrections to the equations of motion. For a de Sitter space-time, which has constant curvature, we refer to [14].

This is interesting because the transformation of the gravitational force under the particle-to-antiparticle transformation has been discussed controversially in the literature [15–18]. In [19], it was pointed out that the role of the CPT transformation in gravity needs to be considered with care: It relates the fall of an apple on Earth to the fall of an anti-apple on anti-Earth, but not on Earth. The Dirac equation, colloquially speaking, applies to both apples, as well as anti-apples, on Earth, i.e., to particles and antiparticles in the same space-time metric. One might have otherwise speculated about the existence of tiny violations of the particle-antiparticle symmetry, even on the level of the gravitationally coupled Dirac theory. For example, in [20], it was claimed that the Dirac Hamiltonian for a particle in a central gravitational field, after a Foldy–Wouthuysen transformation which disentangles the particle from the antiparticle degrees of freedom, contains the term (see the last term on the first line of the right-hand side of Equation (31)):

$$H \sim -\frac{\hbar}{2c}\vec{\Sigma}\cdot\vec{g}\,, \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{1}$$

We here explicitly include the reduced Planck constant $\hbar$ and the speed of light c in order to facilitate the comparison to [20]. Also, the vector of Pauli spin matrices is denoted as $\vec{\sigma}$. The term proportional to $\vec{\Sigma}\cdot\vec{g}$, where $\vec{g}$ is the acceleration due to gravity, would break parity, because $\vec{\Sigma}$ transforms as a pseudovector, while $\vec{g}$ transforms as a vector under parity. This aspect has given rise to discussion, based on the observation that an initially parity-even Hamiltonian (in a central field) should not give rise to parity-breaking terms after a disentangling of the effective Hamiltonians for particles and antiparticles [21,22].

The absence of such parity-violating (and particle-antiparticle symmetry breaking) terms has meanwhile been confirmed in remarks following Equation (15) of [23], in the text following Equation (35) of [24], and also, in clarifying remarks given in the text following Equation (7.33) of [25]. Further clarifying analyses can be found in [26,27]. Related calculations have recently been considered in other contexts [25,28,29]. The question of whether such parity- and particle-antiparticle symmetry violating terms could exist in higher orders in the momentum expansion has been answered negatively in [11], but only for a static central gravitational field, and in [30], still negatively, for combined *static*, central gravitational, and electrostatic fields.

We should note that [20] was not the only place in the literature where the authors speculated about the existence of P (parity), and CP (charge-conjugation and parity) violating terms obtained after the identification of low-energy operators obtained from Dirac Hamiltonians in gravitational fields; For example, in Equation (46) of [31], spurious parity-violating, and CP-violating terms were obtained after a Foldy–Wouthuysen transformation; these terms would of course also violate particle-antiparticle symmetry.

In order to address the question of a general, dynamic space-time background, it is necessary to perform the full particle-to-antiparticle symmetry transformation of the Dirac formalism, in an arbitrary (possibly dynamic) curved-space-time-background. This transformation is most stringently carried out on the level of the Lagrangian formalism. A preliminary result has recently been published in [13,32], where a relationship was established between the positive-energy and negative-energy solutions of the Dirac equation in an arbitrary dynamics curved-space-time-background. However, the derivation in [13,32] is based on a first-quantized formalism, which lacks the unified description in terms of the field operator. The field operator comprises *all* (as opposed to *any*) solution of the gravitationally (and electromagnetically) coupled Dirac equation. In general, a satisfactory description of antiparticles, in the field-theoretical context, necessitates a description in terms of particle- and antiparticle creation and annihilation processes, and therefore, the introduction of a field operator. In consequence, the investigation [13,32] is augmented here on the basis of a transformation of the entire Lagrangian density, which can be expressed in terms of the charge-conjugated (antiparticle) bispinor wave function, and generalized to the level of second quantization. The origin [13] of a rather disturbing minus sign which otherwise appears in the Lagrangian formalism upon charge conjugation in first quantization will be addressed. The use of the Lagrangian formalism necessitates a definition of the Dirac adjoint in curved space-times. As a spin-off result of the augmented investigations reported here, we find the general form of the Dirac adjoint in curved space-times, in the Dirac representation of γ matrices.

According to [19], the role of the CPT transformation in gravity needs to be considered with care: *A priori*, a CPT transformation of a physical system consisting of an apple falling on Earth would describe the fall of an anti-apple on anti-Earth. Key to our considerations is the fact that, on the microscopic level, the Dirac equation applies (for one and the same space-time metric) to both particles and antiparticles simultaneously (this translates, on the macroscopic level, to "apples", as well as "anti-apples"). This paper is organized as follows: We investigate the general form of the Dirac adjoint in Section 2, present our theorem in Section 3, and, in Section 4, we provide an overview of connections to new forces and CPT violating parameters. Conclusions are reserved for Section 5.

2. Dirac Adjoint for Curved Space–Times

In order to properly write down the Lagrangian of a Dirac particle in a gravitational field, we first need to generalize the concept of the Dirac adjoint to curved space-times. We recall that the Dirac adjoint transforms with the inverse of the Lorentz transform as compared to the original Dirac spinor. A general, local, spinor Lorentz transformation $S(\Lambda)$ is given as follows (we here consider the transformation in curved, not flat, space) can be written as,

$$S(\Lambda(x)) = \exp\left(-\frac{\mathrm{i}}{4}\,\epsilon^{AB}(x)\,\sigma_{AB}\right), \qquad \sigma_{AB} = \frac{\mathrm{i}}{2}\left[\gamma^A, \gamma^B\right], \qquad A, B = 0, 1, 2, 3\,. \tag{2}$$

Here, S denotes the spinor representation of the Lorentz transformation Λ. Note that the generator parameters $\epsilon^{AB}(x) = -\epsilon^{BA}(x)$, for local Lorentz transformations, are in general coordinate-dependent. A decisive question is whether or not the Dirac adjoint, which should transform with the inverse of the local Lorentz transformation, changes its functional form when the Lorentz transformations are made local, instead of global, as is manifest in the coordinate-dependent generators $\epsilon^{AB}(x)$. This question has not been answered conclusively in the literature to the best of our knowledge, with a conjecture being formulated in Equation (14a) of [33]. Note that, in curved as opposed to flat space, the local Lorentz transformation $S(\Lambda) = S(\Lambda(x))$ also becomes coordinate-dependent. In the following, capital Roman letters $A, B, C, \cdots = 0, 1, 2, 3$ refer to Lorentz indices in a local freely falling coordinate system. The (flat-space) Dirac matrices γ^A are assumed to be taken in the Dirac representation [1],

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}, \qquad \gamma^A = \begin{pmatrix} 0 & \sigma^A \\ -\sigma^A & 0 \end{pmatrix}, \qquad A = 1, 2, 3. \tag{3}$$

Here, the vector of Pauli spin matrices is denoted as $\vec{\sigma}$, composed of the entries σ^A with $A = 1, 2, 3$. In consequence, the spin matrices σ_{AB} are the flat-space spin matrices. The spin matrices fulfill the commutation relations

$$[\tfrac{1}{2}\sigma^{CD}, \tfrac{1}{2}\sigma^{EF}] = \mathrm{i}\left(g^{CF}\tfrac{1}{2}\sigma^{DE} + g^{DE}\tfrac{1}{2}\sigma^{CF} - g^{CE}\tfrac{1}{2}\sigma^{DF} - g^{DF}\tfrac{1}{2}\sigma^{CE}\right). \tag{4}$$

These commutation relations, we should note in passing, are completely analogous to those fulfilled by the matrices $\mathbb{M}_{AB}$ that generate (four-)vector local Lorentz transformations. As is well known, the latter have the components (denoted by indices C and D)

$$(\mathbb{M}_{AB})^C{}_D = g^C{}_A\, g_{DB} - g^C{}_B\, g_{DA}. \tag{5}$$

The (local) vector local Lorentz transformation Λ with components $\Lambda^C{}_D$ is obtained as the matrix exponential

$$\Lambda^C{}_D(x) = \left(\exp\left[\frac{1}{2}\,\epsilon^{AB}(x)\,\mathbb{M}_{AB}\right]\right)^C{}_D. \tag{6}$$

The algebra fulfilled by the $\mathbb{M}$ matrices is well known to be

$$[\mathbb{M}^{CD}, \mathbb{M}^{EF}] = g^{CF}\,\mathbb{M}^{DE} + g^{DE}\,\mathbb{M}^{CF} - g^{CE}\,\mathbb{M}^{DF} - g^{DF}\,\mathbb{M}^{CE}. \tag{7}$$

The two algebraic relations (4) and (7) are equivalent if one replaces

$$\mathbb{M}^{CD} \to -\frac{\mathrm{i}}{2}\,\sigma^{CD}, \tag{8}$$

which leads from Equation (2) to Equation (6).

Under a local Lorentz transformation, a Dirac spinor transforms as

$$\psi'(x') = S(\Lambda(x))\,\psi(x). \tag{9}$$

In order to write the Lagrangian, one needs to define the Dirac adjoint in curved space-time. In order to address this question, one has to remember that in flat-space-time, the Dirac adjoint $\overline{\psi}(x)$ is defined in such a way that is transforms with the inverse of the spinor Lorentz transform as compared to $\psi(x)$,

$$\overline{\psi}'(x') = \overline{\psi}(x)\,S(\Lambda^{-1}(x)) = \overline{\psi}(x)\,[S(\Lambda(x))]^{-1}. \tag{10}$$

The problem of the definition of $\overline{\psi}(x)$ in curved space-time is sometimes treated in the literature in a rather cursory fashion [34]. Let us see if in curved space-time, we can use the *ansatz*

$$\overline{\psi}(x) = \psi^+(x)\,\gamma^0, \tag{11}$$

with the same flat-space γ^0 as is used in the flat-space Dirac adjoint. (Here, $\psi^+(x)$ denotes the Hermitian adjoint. Throughout this article, we will denote the Hermitian adjoint of a vector or matrix by the superscript +.) In this case,

$$\overline{\psi}'(x') = \psi^+(x')\, S^+(\Lambda(x))\,\gamma^0 = \left(\psi^+(x')\,\gamma^0\right)\,\left[\gamma^0\, S^+(\Lambda(x))\,\gamma^0\right], \tag{12}$$

To first order in the Lorentz generators ϵ_{AB}, we have, indeed,

$$\gamma^0\, S^+(\Lambda(x))\,\gamma^0 = 1 + \frac{\mathrm{i}}{4}\,\epsilon^{AB}(x)\,\gamma^0\,\sigma^+_{AB}\,\gamma^0 = 1 + \frac{\mathrm{i}}{4}\,\epsilon^{AB}(x)\,\sigma_{AB} = [S(\Lambda(x))]^{-1}, \tag{13}$$

where we have used the identity

$$\begin{aligned}\sigma^+_{AB} &= -\frac{\mathrm{i}}{2}\,[\gamma^+_B, \gamma^+_A] = -\frac{\mathrm{i}}{2}\,\gamma^0\,[\gamma^0\gamma^+_B\gamma^0,\ \gamma^0\gamma^+_A\gamma^0]\,\gamma^0 \\ &= -\frac{\mathrm{i}}{2}\,\gamma^0\,[\gamma_B,\ \gamma_A]\,\gamma^0 = -\gamma^0\,\sigma_{BA}\,\gamma^0 = \gamma^0\,\sigma_{AB}\,\gamma^0 .\end{aligned} \tag{14}$$

It is easy to show that Equation (13) generalizes to all orders in the $\epsilon^{AB}(x)$ parameters, which justifies our *ansatz* given in Equation (11). The result is that the flat-space γ^0 matrix can be used in curved space, just like in flat space, in order to construct the Dirac adjoint. The Dirac adjoint spinor transforms with the inverse spinor representation of the Lorentz group (see Equation (10)).

Our findings, notably, Equation (11) ramify a suggestion originally made by Bargmann (see equation (14a) of [33]), who also proposed to define $\overline{\psi}(x)$ with the flat-space γ^0 matrix, rather than the curved-space $\overline{\gamma}^0$ [for the definition of the curved-space $\overline{\gamma}^\mu$ matrices, see Equation (16)]. The conjecture formulated in [33] has been mentioned around equation (61) of [32], with reference to the conservation of a correspondingly defined current. Here, we provide additional evidence for the validity of the *ansatz* given in Equation (11), by establishing the result that the Dirac adjoint, defined accordingly in curved space-time, transforms with the inverse of the local Lorentz transformation, as it should.

3. Lagrangian and Charge Conjugation

Equipped with an appropriate form of the Dirac adjoint in curved space-time, we start from the well-known Lagrangian density [13,32,34–45]

$$\begin{aligned}\mathcal{L}(x) &= \overline{\psi}(x)\,\left[\mathrm{i}\,\overline{\gamma}^\mu\,\mathcal{D}_\mu - m_I\right]\,\psi(x)\,, \\ \mathcal{D}_\mu &= \partial_\mu - \Gamma_\mu + \mathrm{i}e\,A_\mu\,, \\ \mathcal{S} &= \int \mathrm{d}^4x\,\sqrt{-\det g(x)}\,\mathcal{L}(x)\,.\end{aligned} \tag{15}$$

We recall that, throughout this article, capital Roman indices $A, B, C, \cdots = 0, 1, 2, 3$ refer to a freely falling coordinate system (a Lorentz index), while Greek indices $\mu, \nu, \rho, \cdots = 0, 1, 2, 3$ refer to an external coordinate system (an Einstein index). The partial derivative with respect to the coordinate x^μ is denoted as ∂_μ, and m_I is the inertial mass, while e is the electron charge. In Equation (15), $\mathcal{D}_\mu$ is the "double covariant" derivative describing both the gravitational interaction (via the spin connection Γ_μ), as well as the electromagnetic interaction (via the term i$e\,A_\mu$), where e denotes the electron charge. We carefully distinguish between the inertial mass m_I and the gravitational mass m_G; in Equation (15), it is the inertial mass which enters the formalism. The determinant $\det g(x)$ of the metric tensor $g_{\mu\nu}(x)$ enters the expression for the gauge-invariant action $\mathcal{S}$. We reemphasize that Equation (15) is well known and in agreement with the literature [13,32,34–45]. For a number of recent works, we refer to Equations (3.129) and (3.190) of [43], Equation (3.44) in Section 3.3.1 of [45], Equation (2) of [20], and, for a review, [44]; furthermore, the reader might consult [13] for a very recent paper. Note that several quantities which enter Equation (15) are coordinate-dependent, where, in the notation, we suppress the space-time coordinate dependence, as is customary in the literature [20,43–45]. In particular, one has $\overline{\gamma}^\mu = \overline{\gamma}^\mu(x)$, and $\Gamma_\mu = \Gamma_\mu(x)$. Because the underlying formalism is extremely well established and corresponding formulas have been discussed at length in the literature, we can restrict the discussion here to the essential formulas which define the spin connection (where we reserve the notation ω_μ^{AB} for the Ricci rotation coefficients),

$$\overline{\gamma}^\mu = e^\mu_A \, \gamma^A \,, \qquad \Gamma_\mu = \frac{\mathrm{i}}{4} \, \omega_\mu^{AB} \, \sigma_{AB} \,, \qquad \omega_\mu^{AB} = e^A_\nu \left(\partial_\mu e^{\nu B} + \Gamma^\nu_{\mu\rho} \, e^{\rho B} \right) . \tag{16}$$

Here, the e^μ_A are the *vielbein* (or *vierbein* with reference to the four-dimensional space-time) coefficients. The Ricci rotation coefficients are denoted as $\omega_\mu^{AB} = \omega_\mu^{AB}(x)$, while the $\Gamma^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho}(x)$ are the Christoffel symbols. Note that $\{\overline{\gamma}^\mu, \overline{\gamma}^\nu\} = 2\overline{g}^{\mu\nu} = 2\overline{g}^{\mu\nu}(x)$, where $\overline{g}^{\mu\nu}(x)$ is the curved-space metric, while we reserve $g^{AB} = \mathrm{diag}(1, -1, -1, -1)$ for the flat-space counterpart, following the conventions used in [10,13].

We shall attempt to derive the particle-antiparticle symmetry on the level of a transformation of the Lagrangian. In comparison to textbook treatments (see, e.g., pp. 89 ff. and 263 ff. of [46], p. 70 of [47], p. 66 of citeGa1975, pp. 89 ff. and 263 ff. of [48], p. 142 of [49], p. 218 of [50], p. 67 of [51], p. 116 of [52], p. 320 of [53], p. 153 of [1], and Chapter 7 of [54]), our derivation is much more involved in view of the appearance of the Γ_μ matrices which describe the gravitational coupling. In other words, we note that *none* of the mentioned standard textbooks of quantum field theory discuss the *gravitationally* coupled Dirac equation, and all cited descriptions are limited to the flat-space Dirac equation, where the role of the charge conjugation operation is much easier to analyze than in curved space.

From Equation (15), we recall that the double-covariant coupling to both the gravitational, as well as the electromagnetic, field is given as follows,

$$\mathcal{D}_\mu = \partial_\mu - \Gamma_\mu + \mathrm{i}e \, A_\mu = \nabla_\mu + \mathrm{i}e \, A_\mu \,, \tag{17}$$

where $\nabla_\mu = \partial_\mu - \Gamma_\mu$ is the gravitational covariant derivative.

As a side remark, we note that gravitational spin connections $\Gamma_\mu = \frac{\mathrm{i}}{4} \omega_\mu^{AB} \sigma_{AB}$ and other gauge-covariant couplings are unified in the so-called spin-charge family theory [55–59] which calls for a unification of all known interactions of nature in terms of an $SO(1,13)$ overarching symmetry group. (In the current article, we use the spin connection matrices purely in the gravitational context). The $SO(1,13)$ has a 25-dimensional Lie group, with 13 boosts and 12 rotations in the internal space. This provides for enough Lie algebra elements to describe the Standard Model interactions, and predict a fourth generation of particles. The spin-charge family theory is a significant generalization of Kaluza-Klein-type ideas [60,61].

In the context of the current investigations, though, we restrict ourselves to the gravitational spin connection matrices. In view of the (in general) nonvanishing space-time dependence of the Ricci rotation coefficients, we can describe the quantum dynamics of relativistic spin-1/2 particles on the basis of Equations (15) and (16). The σ_{AB} matrices defined in Equation (16) represent the six generators of the spin-1/2 representation of the Lorentz group.

The Lagrangian (15) is Hermitian, and so, we can transform the expression as follows:

$$\mathcal{L} = \mathcal{L}^+ = \psi^+(x) \left[\left\{ -\mathrm{i} \overleftarrow{\partial}_\mu - e \, A_\mu \right\} (\overline{\gamma}^\mu)^+ - (-\mathrm{i}) \left(\Gamma_\mu \right)^+ (\overline{\gamma}^\mu)^+ - m_I \right] \, \left[\overline{\psi}(x) \right]^+ . \tag{18}$$

Here, the left partial derivative, denoted as $\overleftarrow{\partial}_\mu$, acts on all expressions to the left of the derivative operator. We emphasize that the expression (18) is obtained as the plain Hermitian adjoint of the Lagrangian given in Equation (15); there is no partial integration necessary in order to go from Equation (15) to Equation (18). An insertion of γ^0 matrices under use of the identity $(\gamma^0)^2 = 1$ leads to the relation

$$\mathcal{L}^{+} = \psi^{+}(x)\,\gamma^0\,\left[\{-\mathrm{i}_\mu - e\,A_\mu\}\,\gamma^0\,(\overline{\gamma}^\mu)^{+}\,\gamma^0 + \mathrm{i}\left\{\gamma^0\,\left(\Gamma_\mu\right)^{+}\,\gamma^0\right\}\gamma^0\,(\overline{\gamma}^\mu)^{+}\,\gamma^0 - m_I\right]\,\gamma^0\,\left[\,\overline{\psi}(x)\,\right]^{+}\,. \tag{19}$$

In addition, we recall that $\gamma^0\,(\Gamma_\mu)^{+}\,\gamma^0 = -\Gamma_\mu$, because

$$\Gamma^{+}_\mu = -\frac{\mathrm{i}}{4}\,\omega^{AB}_\mu\,\sigma^{+}_{AB} = -\frac{\mathrm{i}}{4}\,\omega^{AB}_\mu\,\gamma^0\,\sigma_{AB}\,\gamma^0 = -\gamma^0\,\Gamma_\mu\gamma^0\,. \tag{20}$$

So, the adjoint of the Lagrangian is

$$\mathcal{L}^{+} = \psi^{+}(x)\,\gamma^0\,\left[\left\{-\mathrm{i}\overleftarrow{\partial}_\mu - e\,A_\mu\right\}\overline{\gamma}^\mu - \mathrm{i}\,\Gamma_\mu\,\overline{\gamma}^\mu - m_I\right]\,\gamma^0\,\left[\,\overline{\psi}(x)\,\right]^{+}\,. \tag{21}$$

Now, we use the relations $\psi^{+}(x)\,\gamma^0 = \overline{\psi}(x)$ and $\gamma^0\,\left[\,\overline{\psi}(x)\,\right]^{+} = \psi(x)$ and arrive at the form

$$\mathcal{L}^{+} = \overline{\psi}(x)\,\left[\left\{-\mathrm{i}\overleftarrow{\partial}_\mu - e\,A_\mu\right\}\overline{\gamma}^\mu - \mathrm{i}\,\Gamma_\mu\,\overline{\gamma}^\mu - m_I\right]\,\psi(x)\,. \tag{22}$$

Because $\mathcal{L}$ is a scalar, a transposition again does not change the Lagrangian, and we have (the left derivative again becomes a right derivative)

$$\left(\mathcal{L}^{+}\right)^{\mathrm{T}} = \psi^{\mathrm{T}}(x)\,\left[(\overline{\gamma}^\mu)^{\mathrm{T}}\left\{-\mathrm{i}\overrightarrow{\partial}_\mu - e\,A_\mu\right\} - \mathrm{i}\,(\overline{\gamma}^\mu)^{\mathrm{T}}\,\left(\Gamma_\mu\right)^{\mathrm{T}} - m_I\right]\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,. \tag{23}$$

An insertion of the charge conjugation matrix $C = \mathrm{i}\gamma^2\,\gamma^0$ (with the flat-space γ^2 and γ^0) leads to

$$\begin{aligned}\left(\mathcal{L}^{+}\right)^{\mathrm{T}} = \psi^{\mathrm{T}}(x)\,C^{-1}\,\Big[C\,(\overline{\gamma}^\mu)^{\mathrm{T}}\,C^{-1}&\left\{-\mathrm{i}\overrightarrow{\partial}_\mu - e\,A_\mu\right\}\\ &-\mathrm{i}\,C\,(\overline{\gamma}^\mu)^{\mathrm{T}}\,C^{-1}\,C\,\Gamma^{\mathrm{T}}_\mu\,C^{-1} - m_I\Big]\,C\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,.\end{aligned} \tag{24}$$

We use the identities $C\,(\overline{\gamma}^\mu)^{\mathrm{T}}\,C^{-1} = -\overline{\gamma}^\mu$, and $C\,\left(\Gamma_\mu\right)^{\mathrm{T}}\,C^{-1} = -\Gamma_\mu$. The latter of these can be shown as follows:

$$C\,\Gamma^{\mathrm{T}}_\mu\,C^{-1} = \frac{\mathrm{i}}{4}\left\{\frac{\mathrm{i}}{2}\omega^{AB}_\mu\,C\,\left[\gamma^{\mathrm{T}}_B,\,\gamma^{\mathrm{T}}_A\right]\,C^{-1}\right\} = \frac{\mathrm{i}}{4}\left\{\frac{\mathrm{i}}{2}\omega^{AB}_\mu\,\left[-\gamma_B,\,-\gamma_A\right]\right\} = -\Gamma_\mu\,. \tag{25}$$

The result is the expression

$$\left(\mathcal{L}^{+}\right)^{\mathrm{T}} = \psi^{\mathrm{T}}(x)\,C^{-1}\,\left[(-\overline{\gamma}^\mu)\left\{-\mathrm{i}\overrightarrow{\partial}_\mu - e\,A_\mu\right\} - \mathrm{i}\,(-\overline{\gamma}^\mu)\,\left(-\Gamma_\mu\right) - m_I\right]\,C\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,. \tag{26}$$

Now, we express the result in terms of the charge-conjugate spinor $\psi^{\mathcal{C}}(x)$ and its adjoint $\overline{\psi^{\mathcal{C}}(x)}$ (further remarks on this point are presented in Appendix B),

$$\psi^{\mathcal{C}}(x) = C\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,,\qquad \overline{\psi^{\mathcal{C}}(x)} = -\psi^{\mathrm{T}}(x)\,C^{-1}\,, \tag{27}$$

where we use the identity $C^{-1} = -C$ (see also Appendix A). The Lagrangian becomes

$$\begin{aligned}\mathcal{L} = \left(\mathcal{L}^{+}\right)^{\mathrm{T}} &= -\overline{\psi^{\mathcal{C}}(x)}\,\left[\overline{\gamma}^\mu\left\{\mathrm{i}\overrightarrow{\partial}_\mu + e\,A_\mu\right\} - \mathrm{i}\,\overline{\gamma}^\mu\,\Gamma_\mu - m_I\right]\,\psi^{\mathcal{C}}(x)\\ &= -\,\overline{\psi^{\mathcal{C}}(x)}\,\left[\overline{\gamma}^\mu\left\{\mathrm{i}(\partial_\mu - \Gamma_\mu) + e\,A_\mu\right\} - m_I\right]\,\psi^{\mathcal{C}}(x)\,.\end{aligned} \tag{28}$$

The Lagrangian given in Equation (28) differs from Equation (18) only with respect to the sign of electric charge, as is to be expected, and with respect to the replacement of the Dirac spinor $\psi(x)$ by its charge conjugation $\psi^{\mathcal{C}}(x)$. The overall minus sign is physically irrelevant as it does not influence the variational equations derived from the Lagrangian; besides, it finds a natural explanation in terms of the reinterpretation principle, if we interpret $\psi(x)$ as a Dirac wave function in first quantization.

Namely, there is a connection of the spatial integrals of the mass term, proportional to

$$J = \int \mathrm{d}^3 r \, \overline{\psi}(x)\psi(x) = \int \mathrm{d}^3 r \, \overline{\psi}(t,\vec{r}) \, \psi(t,\vec{r}) = \int \mathrm{d}^3 r \, \psi^{+}(t,\vec{r})\gamma^0\psi(t,\vec{r}) \,, \tag{29}$$

and the charge conjugate,

$$J^{\mathcal{C}} = \int \mathrm{d}^3 r \, \overline{\psi}^{\mathcal{C}}(x)\psi^{\mathcal{C}}(x) = \int \mathrm{d}^3 r \, \left(\psi^{\mathcal{C}}(t,\vec{r})\right)^{+} \gamma^0\psi(t,\vec{r}) \,. \tag{30}$$

Both of the above integrals connect to the energy eigenvalue of the Dirac equation in the limit of time-independent fields (see Appendix A and B). One can show that the energy eigenvalues of Dirac eigenstates ψ, in the limit of weak potentials and states composed of small momentum components, exactly correspond to the integrals J and $J^{\mathcal{C}}$ (up to a factor m_I). In turn, the dominant term in the Lagrangian in this limit is

$$\mathcal{L} \to -\overline{\psi}(x) \, m_I \, \psi(x) = +\overline{\psi^{\mathcal{C}}(x)} \, m_I \, \psi^{\mathcal{C}}(x) \,. \tag{31}$$

Because the integral $\int \mathrm{d}^3 r \, \mathcal{L}$ equals $-J$ (or $+J^{\mathcal{C}}$), the sign change becomes evident: it is due to the fact that the states $\psi^{\mathcal{C}}$ describe antiparticle wave functions where the sign of the energy flips in comparison to particles. The matching of m_I to the gravitational mass can be performed in a central, static field [10,13], and results in the identification $m_I = m_G$, where m_G is the gravitational mass. The gravitational covariant derivative $\partial_\mu - \Gamma_\mu$ has retained its form in going from Equation (18) to Equation (28), in agreement with the perfect particle-antiparticle symmetry of the gravitational interaction. Because the above demonstration is general and holds for arbitrary (possibly dynamic) space-time background Γ, there is no room for a deviation of the gravitational interactions of antiparticles (antimatter) to deviate from those of matter. This has been demonstrated here on the basis of Lagrangian methods, supplementing a recent preliminary result [13].

In order to fully clarify the origin of the minus sign introduced upon charge conjugation, one consults Chapters 2 and 3 of [1] and Chapter 7 of [54]. Namely, in second quantization, there is an additional minus sign incurred upon the charge conjugation, which restores the original sign pattern of the Lagrangian. According to Equations (2.107) and (3.157) of [1], we can write the expansion of the free Dirac field operator as

$$\hat{\psi}(x) = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{m}{E} \, \left[a_s(\vec{p}) \, u_s(\vec{p}) \, \mathrm{e}^{-\mathrm{i}p\cdot x} + \mathrm{e}^{\mathrm{i}p\cdot x} \, v_s(\vec{p}) \, b_s^{+}(\vec{p}) \right] \,. \tag{32}$$

The field operator is denoted by a hat in order to differentiate it from the Dirac wave function. The four-momentum is $p^\mu = (E, \vec{p})$, where $E = \sqrt{\vec{p}^2 + m^2}$ is the free Dirac energy, and $u_s(\vec{p})$ and $v_s(\vec{p})$ are the positive-energy and negative-energy spinors with spin projection s (onto the z axis). Furthermore, the particle annihilation operator $a_s(\vec{p})$ and the antiparticle creation operator $b_s^{+}(\vec{p})$, and their Hermitian adjoints, fulfill the commutation relations given in Equation (3.161) of [1],

$$\left\{ a_s(\vec{p}), a_{s'}^{+}(\vec{p}) \right\} = \frac{E}{m} \, (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{p}\,') \, \delta_{ss'} \,, \tag{33}$$

$$\left\{ b_s(\vec{p}), b_{s'}^{+}(\vec{p}\,') \right\} = \frac{E}{m} \, (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{p}\,') \, \delta_{ss'} \,. \tag{34}$$

Here, $\delta_{ss'}$ denotes the Kronecker delta. The spinors are normalized according to equation (2.43a) of [1], i.e., they fulfill the relation $u_s^{+}(\vec{p}) \, u_s(\vec{p}) = v_s^{+}(\vec{p}) \, v_s(\vec{p}) = E/m$. For the charge conjugation in the second-quantized theory, it is essential that an additional minus sign is incurred in view of the anticommutativity of the field operators. Namely, without considering the interchange of the field operators, one would have, under charge conjugation, $J^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) = \overline{\psi}^{\mathcal{C}}(x)\gamma^\mu\psi^{\mathcal{C}}(x) = J^{\mathcal{C}\mu}(x)$, i.e., the current would not change under charge conjugation which is intuitively inconsistent (see the remark following Equation (4.618) of [54]). However, for the field operator current (from here on, we denote field operators with a hat), we have $\hat{J}^\mu(x) = \hat{\overline{\psi}}(x)\gamma^\mu\hat{\psi}(x) = -\hat{\overline{\psi}}^{\mathcal{C}}(x)\gamma^\mu\hat{\psi}^{\mathcal{C}}(x) = -\hat{J}^{\mathcal{C}\mu}(x)$, because one has incurred an additional minus sign due to the restoration of the field operators into their canonical order after charge conjugation (see the remark following Equation (7.309) of [54]).

In our derivation above, when one transforms to a second-quantized Dirac field (but keeps classical background electromagnetic field and a classical non-quantized curved-space-time metric), one starts from Equation (22) as an equivalent, alternative formulation of Equation (15). One observes that in going from Equations (22) to (23), one has actually changed the order of the field operators in relation to the Dirac spinors. Restoring the original order, much in the spirit of equation (7.309) of [54], one incurs an additional minus sign which ensures that

$$\begin{aligned}\hat{\mathcal{L}} &= \hat{\overline{\psi}}(x)\left[\overline{\gamma}^\mu\left\{\mathrm{i}\left(\partial_\mu - \Gamma_\mu\right) - e\,A_\mu\right\} - m_I\right]\,\hat{\psi}(x) \\ &= \hat{\overline{\psi}}^{\mathcal{C}}(x)\,\left[\overline{\gamma}^\mu\left\{\mathrm{i}(\partial_\mu - \Gamma_\mu) + e\,A_\mu\right\} - m_I\right]\,\hat{\psi}^{\mathcal{C}}(x)\,,\end{aligned} \tag{35}$$

exhibiting the effect of charge conjugation in the second-quantized theory—and restoring the overall sign of the Lagrangian. The theorem (35) shows that particles and antiparticles behave exactly the same in gravitational fields. It also demonstrates the sign change in the charge of the antimatter particle, when compared to matter ("antimatter electromagnetism"). However, it does not automatically imply the equality of the inertial and gravitational masses, which is a point to be discussed in Equations (36) and (37).

The result (35) confirms and expands results previously obtained in [13,32], regarding the behavior of the gravitationally coupled Dirac equation under charge conjugation. Superficially, one might say that the result is somewhat trivial, because it only shows that the *masses* (which ones? the inertial or the gravitational ones?) of particles and antiparticles are the same. However, this is far from the truth. The point is that this trivialization would overlook the distinction between the gravitational and the inertial mass. Let us be clear about this point: *A priori*, the mass term which enters the electromagnetically coupled Dirac equation is the inertial mass, not the gravitational mass (which is why it has been denoted as m_I, not m_G). The above result (35) confirms that the inertial masses of particles and antiparticles remain exactly the same under gravitational coupling. But it does much more. It also allows us to also make statements about the relation of the inertial and gravitational masses for both particles, as well as antiparticles, because combined particle-antiparticle equation (the Dirac equation) has been coupled to a curved space-time. The above result thus tells that the gravitational coupling of particles and antiparticles must be exactly the same, i.e., that if, say, the inertial mass term m_I relevant for particles can be shown to be related to the gravitational mass as $m_G = f(m_I)$, with a given functional relationship $f(\cdot)$, then the gravitational and inertial masses of antiparticles have to be related by the same functional relationship. All that remains is to find f. For this last step, we can resort to a particular rather than general problem, e.g., the central-field problem analyzed in [13]. There, in Equations (33a) and (33b) in Section 3.1 of [13], it was shown, by considering the central-field problem as an *anchor point*, that

$$m_G = m_I \qquad \text{(particles)}\,, \tag{36}$$

implying that $f(\cdot)$ is the identity transformation, and, therefore, per our above considerations, we have

$$m_G = m_I \qquad \text{(anti-particles)}\,. \tag{37}$$

The ramification of this result on the basis of the second-quantized formalism is a main result of the current paper. Note that the distinction between the gravitational and the inertial mass has been absent from a number of previous investigations (e.g., [32]), in the context of the gravitationally coupled Dirac equation.

One should, at this stage, remember that experimental evidence, to the extent possible, supports the above symmetry relation (35). The only direct experimental result on antimatter and gravity comes, somewhat surprisingly, from the Supernova 1987A. Originating from the Large Magellanic Cloud, the originating neutrinos and antineutrinos eventually were detected on Earth. In view of their travel time of about 160,000 years, they were bent from a "straight line" by the gravity from our own galaxy. The gravitational bending changed the time needed to reach Earth by about 5 mon. Yet, both neutrinos and antineutrinos reached Earth within the same 12 s interval, shows that neutrinos and antineutrinos fall similarly, to a precision of about 1 part in a million [62,63]. In view of the exceedingly small rest mass of neutrinos, the influence of the mass term (even a conceivable tachyonic mass term) on the trajectory is negligible [64]. Yet, it is reassuring that experimental evidence, at this time, is consistent with Equation (35).

4. Alternative Interpretations of Free–Fall Antimatter Gravity Experiments

In view of the symmetry relations derived in this article for the gravitationally and electromagnetically coupled Dirac equation, it is certain justified to ask about an adequate interpretation of antimatter gravity experiments. We have shown that canonical gravity cannot account for any deviations of gravitational interactions of matter versus antimatter. How could tests of antimatter "gravity" be interpreted otherwise? The answer to that question involves clarification of the question which "new" interactions could possibly mimic gravity. The criteria are as follows: (i) The "new" interaction would need to violate CPT symmetry. (ii) The "new" interaction would have to be a long-range interaction, mediated by a massless virtual particle.

One example of such an interaction would be induced if hydrogen atoms were to acquire, in addition to the electric charges of the constituents (electrons and protons), an additional "charge" $\eta\, e$, where e is the elementary charge, while antihydrogen atoms would acquire a charge $-\eta\, e$, where η is a small parameter. One could conjecture, somewhat ad hoc, the existence of a small, CPT-violating "mass-equivalent charge" $\eta e/2$ for electrons, protons, and neutrons, while positrons and antiprotons, and antineutrons, would carry a "mass-equivalent charge" $-\eta e/2$. We will refer to this concept as the "η force" in the following. The difference in the gravitational force (acceleration due to the Earth's field) felt by a hydrogen versus an antihydrogen atom is

$$F^{\eta}_{\overline{\mathrm{H}}} - F^{\eta}_{\mathrm{H}} = 2\,\eta\,\left[\frac{\eta}{2}(N_p + N_n + N_e)\right]\,\frac{e^2}{4\pi\epsilon\, R_\oplus^2}\,. \tag{38}$$

Here, $R_\oplus$ is the radius of the Earth, while N_p, N_n, and N_e are the numbers of protons, neutrons, and electrons in the Earth. The gravitational force on a falling antihydrogen atom is

$$F^{G}_{\overline{\mathrm{H}}} = G\,\frac{m_p\,M_\oplus}{R_\oplus^2}\,. \tag{39}$$

Let us assume that an experiment establishes that $|F^{\eta}_{\overline{\mathrm{H}}} - F^{\eta}_{\mathrm{H}}| < \chi\, F^{G}_{\overline{\mathrm{H}}}$, where χ is a measure of the deviation of the acceleration due to gravity+"η"-force for antihydrogen versus hydrogen. A quick calculation shows that this translates into a bound

$$\eta < 7.3 \times 10^{-19}\,\sqrt{\chi}\,. \tag{40}$$

Antimatter gravity tests thus limit the available parameter space for η and could be interpreted in terms of corresponding limits on the maximum allowed value of η.

Functionally, for nonrelativistic antimatter free-fall experiments, the "η force" has the same phenomenological consequences as the model recently formulated in Section 3.3.1 of [45], where a conceivable alteration of the gravitational interaction of antiparticles is formulated in terms of a_0 and c_{00} coefficients, implicitly defined in equation (3.44) of [45], which multiply additional terms of the form $a_\mu \overline{\psi}\overline{\gamma}^\mu \psi$ and $c_{\mu\nu}\,\mathrm{i}\,\overline{\psi}\,\overline{\gamma}^\mu(\partial_\nu - \Gamma_\mu + \mathrm{i}eA_\nu)\,\psi$ added to the Lagrangian (15), with constant a_μ and $c_{\mu\nu}$ coefficients. Our ad hoc model has the advantage that we avoid the necessity of fine-tuning the

CPT-violating a_0 coefficient to the c_{00} coefficient (in the notation of [45]). This fine-tuning, in the form of the postulate $a_0 = -c_{00}/3$ (again, using the notation of [45]), is given in the third paragraph following equation (3.47) of [45].

5. Conclusions

In the current paper, we analyzed the particle-antiparticle symmetry of the gravitationally (and electromagnetically) coupled Dirac equation and come to the conclusion that a symmetry exists, for the second-quantized formulation, which precludes the existence particle-antiparticle symmetry breaking terms on the level of Dirac theory. In a nutshell, one might say the following: Just as much as the electromagnetically coupled Dirac equation predicts that antiparticles have the opposite charge as compared to particles (but otherwise behave exactly the same under electromagnetic interactions), the gravitationally coupled Dirac equation predicts that particles and antiparticles follow exactly the same dynamics in curved space-time, i.e., with respect to gravitational fields (in particular, they have the same gravitational mass, and there is no sign change in the gravitational coupling). In the derivation of our theorem (35), we use the second-quantized Dirac formalism, in the Lagrangian formulation. Our general result for the Dirac adjoint, communicated in Section 2, paves the way for the Lagrangian of the gravitationally coupled field, and its explicit form is an essential ingredient of our considerations.

Let us also represent the advances reported here, in regard to the existing literature. First, the question regarding the absence of particle-antiparticle symmetry breaking terms for general, dynamic space-time backgrounds has not been answered conclusively in the literature up to this point [20,31], to the best of our knowledge, because of the lack of a careful distinction between the gravitational and inertial masses under charge conjugation operations. This has been the task of the current paper. In particular, our results imply a no-go theorem regarding the possible emergence of particle-antiparticle-symmetry breaking gravitational, and combined electromagnetic-gravitational terms in general static and dynamic curved-space-time backgrounds. Furthermore, the equality of the inertial and gravitational mass for particles and antiparticles, i.e., the validity of the (weak) equivalence principle, is established for antiparticles (Equations (35) and (37)). Any speculation [20,31] about the re-emergence of such terms in a dynamic space-time background can thus be laid to rest. Concomitantly, we demonstrate that there are no "overlap" or "interference" terms generated in the particle-antiparticle transformation, between the gauge groups, namely the $SO(1,3)$ gauge group of the local Lorentz transformations, and the $U(1)$ gauge group of the electromagnetic theory.

In passing, we also firmly establish the validity of a conjecture [33] (see Equation (11)) regarding the explicit form of the Dirac adjoint in curved space-time, by calculating its transformation properties under local Lorentz transformations. We also show in Equation (35) that a somewhat disturbing minus sign in the transformation of the Lagrangian density disappears in the second-quantized formalism.

This result implies both progress and, unfortunately, some disappointment, because the emergence of matter-antimatter symmetry violating terms would have been fascinating and would have opened up, quite possibly, interesting experimental opportunities. In our opinion, free-fall antimatter gravity experiments should be interpreted in terms of limits on CPT-violating parameters, such as the η parameter introduced in Section 4. This may be somewhat less exciting than a "probe of the equivalence principle for antiparticles" but, still, of utmost value for the scientific community.

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Appendix A. Sign Change of $\overline{\psi}\,\psi$ under Charge Conjugation

With the charge conjugation matrix $C = \mathrm{i}\gamma^2\gamma^0$ (superscripts denote Cartesian indices) and the Dirac adjoint $\overline{\psi} = \psi^+\,\gamma^0$, we have

$$\psi^{\mathcal{C}} = C\,\overline{\psi}^{\mathrm{T}} = \mathrm{i}\gamma^2\,\gamma^0\,\gamma^0\,\psi^* = \mathrm{i}\gamma^2\,\psi^*\,. \tag{A1}$$

We recall that the γ^2 (contravariant index, no square) matrix in the Dirac representation matrix is

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \left(\sigma^2\right)^+ = \sigma^2\,, \tag{A2}$$

which implies that $\left(\gamma^2\right)^+ = -\gamma^2$. The Pauli matrices are denoted as σ^i with $i = 1,2,3$. The Dirac adjoint of the charge conjugate is

$$\overline{\psi}^{\mathcal{C}} = \left(\psi^{\mathcal{C}}\right)^+\,\gamma^0 = \psi^{\mathrm{T}}(-\mathrm{i})\left(\gamma^2\right)^+\,\gamma^0 = \psi^{\mathrm{T}}(-\mathrm{i})\,(-\gamma^2)\,\gamma^0 = \psi^{\mathrm{T}}\,\mathrm{i}\,\gamma^2\,\gamma^0\,. \tag{A3}$$

This leads to a verification of the sign flip of the mass terms in the gravitationally coupled Lagrangian for antimatter, given in Equation (28) (see, also, Equations (29) and (30)),

$$\overline{\psi}^{\mathcal{C}}\,\psi^{\mathcal{C}} = (\psi^{\mathrm{T}}\mathrm{i}\gamma^2)\,\gamma^0\,(\mathrm{i}\,\gamma^2\,\psi^*) = -(\mathrm{i})^2\psi^{\mathrm{T}}\,(\gamma^2)^2\,\gamma^0\psi^* = -\psi^{\mathrm{T}}\,\gamma^0\,\psi^* = -\overline{\psi}\,\psi\,. \tag{A4}$$

Two useful identities *(i)* $\gamma^0\,C^+\,\gamma^0 = C$ and *(ii)* $C^{-1} = -C$ have been used in Section 3. These will be derived in the following. The explicit form of the γ^2 matrix in the Dirac representation implies that $\left(\gamma^2\right)^+ = -\gamma^2$. Based on this relation, we can easily show that

$$C^+ = \left(\mathrm{i}\,\gamma^2\,\gamma^0\right)^+ = -\mathrm{i}\,\gamma^0\,\left(\gamma^2\right)^+ = \mathrm{i}\,\gamma^0\,\gamma^2 = -\mathrm{i}\,\gamma^2\,\gamma^0 = -C\,. \tag{A5}$$

The first identity $\gamma^0\,C^+\,\gamma^0 = C$ can now be shown as follows,

$$\gamma^0\,C^+\,\gamma^0 = \gamma^0\,\left[-\mathrm{i}\,\gamma^2\,\gamma^0\right]\,\gamma^0 = -\mathrm{i}\,\gamma^0\,\gamma^2 = \mathrm{i}\,\gamma^2\,\gamma^0 = C\,. \tag{A6}$$

Furthermore, one has

$$C\,C^+ = C\,(-C) = \mathrm{i}\,\gamma^2\,\gamma^0\,\mathrm{i}\,\gamma^0\,\gamma^2 = -\left(\gamma^2\right)^2 = -\,(-\mathbb{1}_{4\times 4}) = \mathbb{1}_{4\times 4}\,, \tag{A7}$$

so that

$$C^{-1} = C^+ = -C\,, \tag{A8}$$

which proves, in particular, that $C^{-1} = -C$.

Appendix B. General Considerations

A few illustrative remarks are in order. These concern the following questions: (i) To which extent do gravitational and electrostatic interactions differ for relativistic particles? This question is relevant because, in the nonrelativistic limit, in a central field, both interactions are described by potentials of the same functional form ("$1/R$ potentials"). (ii) In addition, we should clarify why the integrals (29) and (30) represent the dominant terms in the evaluation of the Dirac particle energies, in the nonrelativistic limit.

After some rather deliberate and extensive considerations, one can show [12] that, up to corrections which combine momentum operators and potentials, the general Hamiltonian for a Dirac particle in a combined electric and gravitational field is

$$H_D = \vec{\alpha} \cdot \vec{p} + \beta\{m(1 + \phi_G)\} + e\phi_C \,, \tag{A9}$$

where ϕ_G is the gravitational, and ϕ_C is the electrostatic potential. In addition, $\vec{\alpha}$ is the vector of Dirac α matrices, $\vec{p}$ is the momentum operator, and $\beta = \gamma^0$ is the Dirac β matrix. After a Foldy–Wouthuysen transformation [26], one sees that the gravitational interaction respects the particle-antiparticle symmetry, while the Coulomb potential does not, commensurate with the opposite sign of the charge for antiparticles. Question (i) as posed above can thus be answered with reference to the fact that, in leading approximation, the gravitational potential enters the Dirac equation as a scalar potential, modifying the mass term, while the electrostatic potential can be added to the free Dirac Hamiltonian $vec\alpha \cdot \vec{p} + \beta m$ by covariant coupling [1].

The second question posed above is now easy to answer: Namely, in the nonrelativistic limit, one has

$$\vec{\alpha} \cdot \vec{p} \to 0 \,, \tag{A10}$$

and furthermore, the gravitational and electrostatic potentials can be assumed to be weak against the mass term, at least for non-extreme Coulomb fields [65]. Under these assumptions, one has $H_D \to \beta m$, and the matrix element $\langle\psi|H_D|\psi\rangle$ assumes the form $\int \mathrm{d}^3 r \, \psi^+(\vec{r}) \, \gamma^0 \, m\psi(\vec{r})$ (see Equation (29)).

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