

# Bounds on Energies and Dissipation Rates in Forced Dynamos

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**Abstract:** This paper is concerned with limits on kinetic and magnetic energies and dissipation rates in forced flows that lead to dynamo action and a finite amplitude magnetic field. Rigorous results are presented giving upper and lower limits on the values of these quantities, in a simple cubic geometry with periodic boundary conditions, using standard inequalities. In addition to the general case, results in the special case of the Archontis dynamo are presented, in which fields and flows are closely similar in much of the domain.

**Keywords:** magnetohydrodynamics; dynamo theory; rigorous bounds

## 1. Introduction

This paper sets out some rigorous bounds on statistically steady dynamos driven by steady forcing. Such dynamos have been widely investigated by a number of authors; for a review with references see [1]. Of particular interest is the so-called Archontis dynamo [2,3] in which the magnetic field is almost aligned with the velocity field over much of the domain. In the latter case the velocity field and magnetic field can be almost steady with steady forcing while in other cases the fields and flows are disordered and aperiodic in time, so that averages over both space and time have to be assumed constant to make progress. This permits the construction of both upper and lower bounds on the time-averaged magnetic energy or dissipation, and the ratios of these quantities to their kinetic equivalents. A number of these results have been given elsewhere (see references in Section 3), but others are apparently novel. Some general results are given, and also more restrictive bounds that obtain (approximately) when the fields and flows are of Archontis type. The analysis shows results for the simple case where the magnetic Prandtl is unity and also where it is small, as found in laboratory fluids.

The plan of this paper is as follows: After an introduction in Sections 1 and 2, the mathematical problem is presented in Section 3.

## 2. Methods

The analysis is performed using a number of known inequalities (Poincaré, Cauchy–Schwarz, Hölder, and others) that have a long history in the literature; while the bounds that are derived are only approximate in the sense that results for real flows will not be close to the extreme limits derived, they have the merit of being universal and are useful as a consistency check on any numerical computations. There have been a number of rigorous results derived for magnetic field growth due to prescribed flow; see, e.g., [4]. Here, flows that are driven by a steady space-periodic forcing are examined. Results are presented both for the flows and fields themselves, and for the Elsasser variables, defined as the sum and difference of the velocity and magnetic fields.

## 3. Governing Equations

We consider forced incompressible flow  $\mathbf{u}^*$  and dynamo generated magnetic field  $\mathbf{b}^*$  in a periodic box of side  $L$ . The forcing  $\mathbf{f}^*(\mathbf{x}^*)$  is also taken to be incompressible. As well



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as the velocity forcing we have the Lorenz force due to the magnetic field. The momentum Equation for  $\mathbf{u}^*$  and the induction equation for  $\mathbf{b}^*$  can be written in the dimensionless forms:

$$\frac{\partial \mathbf{u}}{\partial t} + R\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + R\mathbf{b} \cdot \nabla \mathbf{b} + \nabla^2 \mathbf{u} + \mathbf{f}, \tag{1}$$

$$\frac{\partial \mathbf{b}}{\partial t} = R\nabla \times (\mathbf{u} \times \mathbf{b}) + P_m^{-1} \nabla^2 \mathbf{b}, \tag{2}$$

where  $\mathbf{x}^* = \mathbf{x}/d$ ,  $\mathbf{u}^* = Rv\mathbf{u}/d$ ,  $\mathbf{b}^*$  (measured in Alfvén velocity units)  $= Rv\mathbf{b}/d$ ,  $\mathbf{f}^* = v^2 R d^{-3} \mathbf{f}$  and  $P_m = v/\eta$  (the magnetic Prandtl number), where  $v$  is the kinematic viscosity,  $d = L/2\pi$  where  $L$  is the length scale of the system, and  $\eta$  the magnetic diffusivity. The parameter  $R$  is defined such that  $\langle |\mathbf{f}|^2 \rangle_S = 1$ , where  $\langle \cdot \rangle_S$  is defined as an average over the periodic box. Thus,  $R$  plays the role of a Reynolds number or inverse viscosity. For simplicity, the assumption is made that the forcing  $\mathbf{f}$  is independent of time, and it is further supposed that all quantities are periodic over a box of side  $L$ . It is also supposed that  $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{f} = \nabla \cdot \mathbf{b} = 0$ , and that  $\langle \mathbf{b} \rangle_S = \langle \mathbf{u} \rangle_S = \langle \mathbf{f} \rangle_S = 0$ . (It may be checked that these averages do not change during the evolution of the system.) We are interested in a system in a statistically steady state, and define  $\langle \cdot \rangle$  to be the time average of  $\langle \cdot \rangle_S$  over time.

### 4. Results

#### 4.1. No Magnetic Field

Equations (1) and (2) have solutions with  $\mathbf{b} \equiv 0$ . In that case we can find bounds on the kinetic energy and viscous dissipation in terms of properties of  $\mathbf{f}$ . Defining  $\mathcal{E}^2 \equiv \langle |\mathbf{u}| \rangle^2$ ,  $\mathcal{D}^2 \equiv \langle |\nabla \mathbf{u}| \rangle^2$  we have the following results, noting that  $\mathcal{E}^2 \leq \mathcal{D}^2$  in the chosen geometry.

##### 4.1.1. Upper Bound

Multiplying Equation (1) by  $\mathbf{u}$  and averaging and use of the Cauchy and Poincaré inequalities gives

$$\mathcal{E}^2 \leq \mathcal{D}^2 = \langle \mathbf{u} \cdot \mathbf{f} \rangle \leq \mathcal{D} \langle |\nabla \mathbf{g}|^2 \rangle^{\frac{1}{2}}, \tag{3}$$

where  $\mathbf{g}$  is defined by  $\mathbf{f} = -\nabla^2 \mathbf{g}$ ,  $\langle \mathbf{g} \rangle_S = 0$ , so that one has the upper bound,

$$\mathcal{E} \leq \mathcal{D} \leq \Gamma, \text{ where } \Gamma^2 = \langle |\nabla \mathbf{g}|^2 \rangle \leq 1. \tag{4}$$

The last of these inequalities is shown, recalling that  $\langle |\mathbf{f}|^2 \rangle = 1$ , by noting that  $\Gamma^2 = \langle \mathbf{f} \cdot \mathbf{g} \rangle \leq \langle |\mathbf{g}|^2 \rangle^{\frac{1}{2}} \leq \Gamma$ .

##### 4.1.2. Lower Bound

Multiplying Equation (1) by  $\mathbf{g}$  and averaging, recalling that  $\mathbf{g}$  is independent of time, one obtains:

$$-R \langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{g}) \rangle - \langle \mathbf{u} \cdot \nabla^2 \mathbf{g} \rangle = -R \langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{g}) \rangle + \mathcal{D}^2 = \langle \mathbf{f} \cdot \mathbf{g} \rangle = \Gamma^2. \tag{5}$$

One can bound the terms on the left hand side using the above inequalities and the Hölder inequality to get (defining  $\max_{i,j,x} (|\partial g_i / \partial x_j|) = A$ , where  $A$  is a constant):

$$(RA + 1)\mathcal{D}^2 \geq RA\mathcal{E}^2 + \mathcal{D}^2 \geq \Gamma^2. \tag{6}$$

Results similar to both these upper and lower bounds have been obtained earlier, e.g., [5,6].

### 4.2. Dynamo Inequalities

We now introduce the induction equation and the Lorentz force term and try to generalise the above results. These are now framed as inequalities on the magnetic energy and magnetic dissipation; we define  $\mathcal{E}_m^2 \equiv \langle |\mathbf{b}|^2 \rangle$ ,  $\mathcal{D}_m^2 \equiv \langle |\nabla \mathbf{b}|^2 \rangle$ .

#### 4.2.1. Upper Bound

Multiplying Equation (1) by  $\mathbf{u}$  and Equation (2) by  $\mathbf{b}$ , averaging and adding, one finds that the cubic integrals cancel and that

$$(P_m^{-1} \mathcal{D}_m^2 + \mathcal{D}^2) = \langle \mathbf{u} \cdot \mathbf{f} \rangle \leq \Gamma \mathcal{D}, \quad \text{so that} \quad (7)$$

$$\mathcal{D}_m^2 \leq P_m (\Gamma \mathcal{D} - \mathcal{D}^2) \leq \frac{P_m \Gamma^2}{4}. \quad (8)$$

#### 4.2.2. Lower Bound

More surprising perhaps is the fact that the magnetic dissipation is bounded below, for given values of  $R$  and  $\mathcal{D}$ . Multiplying Equation (1) by  $\mathbf{g}$ , using Equation (8) and averaging, one finds:

$$-R \langle (\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{g}) - \mathbf{b} \cdot (\mathbf{b} \cdot \nabla \mathbf{g})) \rangle + (P_m^{-1} \mathcal{D}_m^2 + \mathcal{D}^2) = \Gamma^2. \quad (9)$$

Then by analogous methods to the non-magnetic case one finally obtains:

$$RA(\mathcal{D}_m^2 + \mathcal{D}^2) + (P_m^{-1} \mathcal{D}_m^2 + \mathcal{D}^2) \geq RA(\mathcal{E}_m^2 + \mathcal{E}^2) + (P_m^{-1} \mathcal{D}_m^2 + \mathcal{D}^2) \geq \Gamma^2, \quad \text{or} \quad (10)$$

$$\mathcal{D}_m^2 \geq \frac{\Gamma^2 - RA\mathcal{E}^2 - \mathcal{D}^2}{RA + P_m^{-1}} \geq \frac{\Gamma^2 - (RA + 1)\mathcal{D}^2}{RA + P_m^{-1}}.$$

These results are related to those shown by Tilgner [7] in the context of the G.O. Roberts dynamo.

#### 4.2.3. Dissipation Ratio

One can write these results alternately in terms of the ratio of magnetic to kinetic dissipation. Let  $\mathcal{D}_m^2 = k^2 \mathcal{D}^2$ ; then (here  $\mathcal{G} = 1/\mathcal{D}$ ):

$$\frac{\Gamma^2 \mathcal{G}^2 - (1 + RA)}{RA + P_m^{-1}} \leq k^2 \leq P_m (\Gamma \mathcal{G} - 1). \quad (11)$$

The largest possible value of  $k_{\max}^2$  of  $k^2$  is given by the intersection of the boundary curves of the inequalities in Equation (11). Eliminating  $X = \Gamma \mathcal{G}$  at the intersection, one finds:

$$k_{\max}^2 = \frac{1}{2} \left[ P_m (Z - 1) + \sqrt{P_m^2 (1 - Z)^2 + 4P_m (2Z + 1)} \right], \quad \text{where } Z = P_m RA. \quad (12)$$

$k_{\max}^2$  appears to increase monotonically with  $R$ . For many liquid metals  $P_m \ll 1$ ; in that case, if  $k_{\max}^2 = O(1)$ , one must have  $R = O(P_m^{-2})$ .

### 4.3. The Archontis Dynamo

Now these results can be applied to the Archontis dynamo to get what are probably rather conservative estimates on the relation between  $R$  and  $P_m$  necessary to sustain such a dynamo.

The original forcing function used by Archontis takes the form,

$$\mathbf{f} = \sqrt{\frac{2}{3}} (\sin z, \sin x, \sin y). \quad (13)$$

Then one can see that  $\mathbf{g} = \mathbf{f}$ ,  $A = \sqrt{2/3}$ ,  $\Gamma = 1$ . For the Archontis dynamo,  $k^2 \approx 1$  and so the boundary of the region in  $(R, P_m)$  space, where this is possible, is given by putting both the expressions in Equation (11) equal to unity, giving the approximate inequality for such a dynamo:

$$R \geq \sqrt{\frac{1}{6} P_m^{-2}}. \tag{14}$$

Since, as previously noted,  $P_m \ll 1$  in many physical systems, it can be seen that Archontis type dynamos can occur only for large values of  $R$  for such systems.

#### 4.4. Use of Elsasser Variables

One can instead use the Elsasser variables  $\mathbf{z}_{\pm} = \mathbf{u} \pm \mathbf{b}$ . The equations can be written

$$\frac{\partial \mathbf{z}_{\pm}}{\partial t} + R \mathbf{z}_{\mp} \cdot \nabla \mathbf{z}_{\pm} = -\nabla P + \frac{1}{2}(1 + P_m^{-1}) \nabla^2 \mathbf{z}_{\pm} + \frac{1}{2}(1 - P_m^{-1}) \nabla^2 \mathbf{z}_{\mp} + \mathbf{f}. \tag{15}$$

Then multiplying each equation by  $\mathbf{z}_{\pm}$ , one finds, in the statistically steady state:

$$\langle \mathbf{z}_{\pm} \cdot \mathbf{f} \rangle = \frac{1}{2}(1 + P_m^{-1}) \langle |\nabla \mathbf{z}_{\pm}|^2 \rangle + \frac{1}{2}(1 - P_m^{-1}) \langle \nabla \mathbf{z}_{\pm} \cdot \nabla \mathbf{z}_{\mp} \rangle. \tag{16}$$

Now defining  $\mathcal{D}_{\pm}^2 = \langle |\nabla \mathbf{z}_{\pm}|^2 \rangle$  and using similar inequalities to those in previous Sections, one finds:

$$\Gamma \mathcal{D}_{\pm} + \frac{1}{2}|1 - P_m^{-1}| \mathcal{D}_{\pm} \mathcal{D}_{\mp} \geq \frac{1}{2}(1 + P_m^{-1}) \mathcal{D}_{\pm}^2, \tag{17}$$

or, dividing through,

$$\Gamma \geq \frac{1}{2}(1 + P_m^{-1}) \mathcal{D}_{\pm} - \frac{1}{2}|1 - P_m^{-1}| \mathcal{D}_{\mp}. \tag{18}$$

As a check consider the non-magnetic case for which  $\mathbf{z}_+ = \mathbf{z}_-$ . For  $P_m < 1$  we reproduce an inequality similar to Equation (4). When  $P_m > 1$  one obtains a weaker result. One can find relations corresponding to Equation (10) by multiplying Equation (15) by  $\mathbf{g}$  and averaging, leading to

$$VR \mathcal{D}_{\pm} \mathcal{D}_{\mp} + \frac{1}{2}(1 + P_m^{-1}) \Gamma \mathcal{D}_{\pm} + \frac{1}{2}|1 - P_m^{-1}| \Gamma \mathcal{D}_{\mp} \geq \Gamma^2. \tag{19}$$

where  $\max_{\mathbf{x}}(|\mathbf{g}|) = V$ . To understand what these inequalities imply consider the special case  $P_m = 1$ , and the Archontis forcing, where  $V = \sqrt{2}$ ,  $\Gamma = 1$ . Then Equations (18) and (19) become:

$$\mathcal{D}_{\pm} \leq 1, \quad \sqrt{2} R \mathcal{D}_{\pm} \mathcal{D}_{\mp} + \mathcal{D}_{\pm} \geq 1. \tag{20}$$

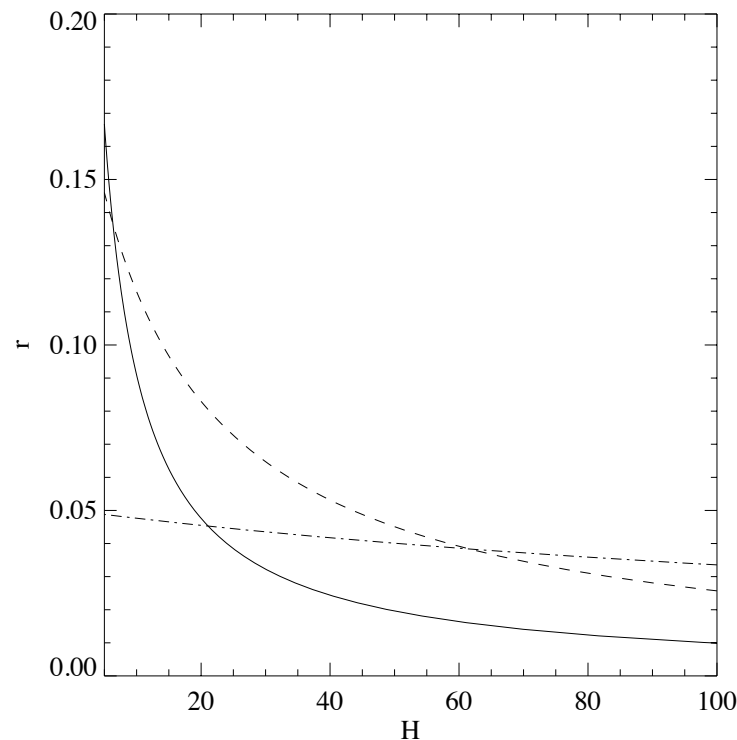
For the Archontis dynamo, we have that  $\mathcal{D}_- \ll \mathcal{D}_+$ , say. It is straightforward to check, using the lower sign in Equation (20), that the smallest value of the ratio  $r = \mathcal{D}_- / \mathcal{D}_+$  compatible with the above inequalities is  $1/(\sqrt{2}R + 1)$ , and so  $R$  has to be large for an Archontis type dynamo, as indeed is found in the experiments; see for example the detailed calculations for large  $R$ , described for  $P_m = 1$  in [8], (with large  $R$  corresponding to small  $\epsilon$  in their notation).

Similar results hold for general  $P_m$ . Consider the interesting case  $P_m < 1$ . Then a similar calculation gives, for  $\mathcal{H} = VR$ ,

$$2\mathcal{H}r \geq 1 + r + \frac{1}{P_m}(1 - 2r - r^2) + \frac{r}{P_m^2}(r - 1). \tag{21}$$

How big should  $R$  - or  $\mathcal{H}$  - be to get close to the Archontis configuration? If  $r \ll 1$  then  $\mathcal{H} \approx (1 + P_m^{-1})/2r \gg 1$ . If  $r \sim P_m \ll 1$  then  $r \approx 1/((P_m(2\mathcal{H} + P_m^{-2})))$  and so  $\mathcal{H}$  has to be very large, of order  $P_m^{-2}$ , to reduce  $r$  significantly below  $P_m$ .

Figure 1 shows the relation between  $\mathcal{H}$  and  $r$  for three values of  $P_m \leq 1$ . It can immediately be seen that as  $P_m$  decreases (i.e.,  $\eta$  increases for fixed  $\nu$ ) it becomes harder to make  $r$  small, i.e., to approach an Archontis state.



**Figure 1.** The minimum value of  $r = \mathcal{D}_- / \mathcal{D}_+$  as a function of  $\mathcal{H} = R\sqrt{2}$  for  $P_m = 1$  (solid line);  $P_m = 0.2$  (dashed line);  $P_m = 0.05$  (dash-dotted line). See text for details.

The values of  $k$  and  $r$  are not straightforwardly related. By use of the Schwartz inequality it can be established that as long as  $k, r < 1$  as envisaged one must have  $(r + 1)(k + 1) \geq 2$ , so that, for instance, if  $r$  is close to zero, then  $k$  must be close to unity but, since  $r$  is bounded below rather than above for fixed  $\mathcal{H}$ , then  $k$  is not constrained.

## 5. Discussion

In this short paper, I explore the rigorous results that constrain the magnetic and kinetic energies and dissipation rates in a forced, fully nonlinear dynamo. It is possible to obtain both lower and upper bounds on the magnetic dissipation for a given forcing. In particular, the parameter  $R$ , which measures the amplitude of the forcing, has to be very large when the magnetic Prandtl number is small. Similar bounds can be found for the Elsasser variables,  $\mathbf{z}_{\pm} = \mathbf{u} \pm \mathbf{z}$ , and these lead to the same conclusion in the case of the Archontis dynamo where  $\mathbf{u}$  and  $\mathbf{b}$  are closely aligned in much of the domain, while some of the bounds are likely to be rather loose, they are at least rigorous, and provide guidance as to where in parameter space to look for exceptional configurations such as those found by Archontis. Work is in progress with D.J.Galloway to examine the accuracy of the bounds through direct numerical simulations. In Appendix A, the author of this paper presents his brief reminiscent about contacts with M. Tribelsky to whose honour this issue is dedicated.

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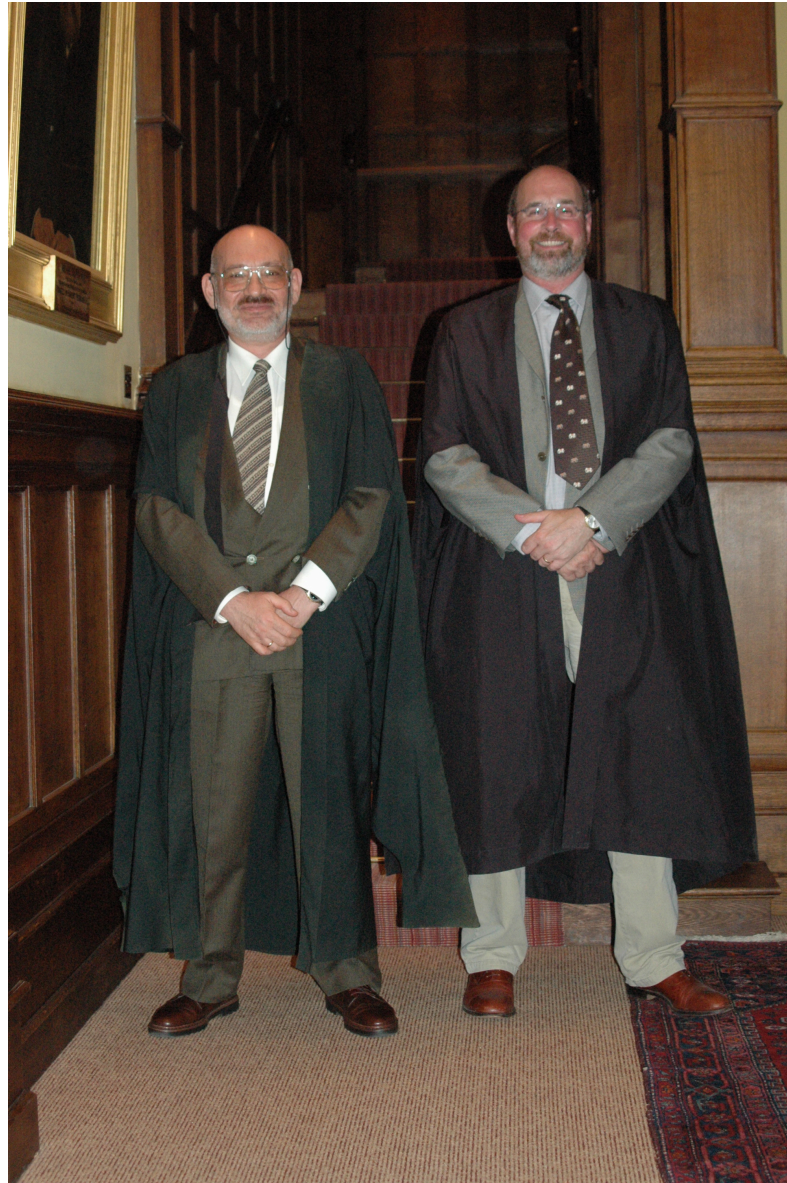
**Data Availability Statement:** Not applicable.

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**Conflicts of Interest:** The author declares no conflict of interest.

## Appendix A

I am delighted to dedicate this paper to my friend and colleague Michael Tribelsky. I have known him since meeting in Bayreuth in 1989 at a conference and we had valuable discussions during his visit to the Isaac Newton Institute, Cambridge, UK in 2005.



**Figure A1.** Michael Tribelsky (left) and Michael Proctor (right): Trinity College, Cambridge, UK in 2005.

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