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Two New Methods in Stochastic Electrodynamics for Analyzing the Simple Harmonic Oscillator and Possible Extension to Hydrogen

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Abstract: The position probability density function is calculated for a classical electric dipole harmonic oscillator bathed in zero-point plus Planckian electromagnetic fields, as considered in the physical theory of stochastic electrodynamics (SED). The calculations are carried out via two new methods. They start from a general probability density expression involving the formal integration over all probabilistic values of the Fourier coefficients describing the stochastic radiation fields. The first approach explicitly carries out all these integrations; the second approach shows that this general probability density expression satisfies a partial differential equation that is readily solved. After carrying out these two fairly long analyses and contrasting them, some examples are provided for extending this approach to quantities other than position, such as the joint probability density distribution for positions at different times, and for position and momentum. This article concludes by discussing the application of this general probability density expression to a system of great interest in SED, namely, the classical model of hydrogen.

Keywords: stochastic electrodynamics; classical physical dynamics; hydrogen; harmonic oscillator; nonlinear dynamics

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1. Introduction

This paper involves the physics of stochastic electrodynamics (SED) and the exploration of a new approach for analyzing probabilities associated with charged particle motion due to the interaction with stochastic electromagnetic radiation. SED involves the movement of classical charged point particles while interacting with a specific form of fluctuating classical electromagnetic radiation. Despite SED being completely classical, agreement has been shown between SED and quantum mechanics (QM) and even quantum electrodynamics (QED), under an interesting range of conditions. The "classical physics" aspects of SED consist of electromagnetic radiation that obeys Maxwell's classical, microscopic electromagnetic equations, while the classical charged particles obey the relativistic Lorentz–Dirac classical equation of motion [1,2]. The physical predictions of SED that agree with QM and QED hold for classical systems with a linear differential equation of motion. A very good demonstration of this point is Ref. [3] for the simple harmonic oscillator (SHO). Even the complicated fully retarded van der Waals forces between atoms modelled by electric dipole oscillators fulfill this agreement, as do Casimir forces between continuum materials; this agreement holds in both cases for all temperature conditions. Interestingly enough, at one point many who have studied and explored SED thought that SED might form the basis for QM and QED. However, complications have since been found to persuade most researchers that this is not the case; for more details, see Refs. [3–8].

The disagreement between SED and QM for all arbitrary "atomic systems" has some bearing for motivating the investigation in this paper. An interesting point to explain why all "atomic systems" covered by QM do not also hold for SED was first made and analyzed by Boyer [9] and subsequently followed up in a different way by the present author [10]. The point was this: the binding force for all atomic systems in nature is due to the Coulombic force. Hence, not just any binding force inserted into the SED description, other than the Coulombic-based one, should be expected to share agreement with real atomic systems in nature; moreover, SED should not be expected to agree with nonphysical "atomic systems" containing arbitrary binding forces in QM. The difficulty here is that such Coulombic-based systems are inherently nonlinear for the equation of motion for the classical electrons interacting with the classical nucleus; hence, such systems are far more complicated to analyze in SED than when the differential equation of motion is linear. More about this point is discussed in the concluding section of Section 4, but this forms much of the motivation in this study for examining a new way of calculating probabilities in SED.

Key aspects of SED concern classical charged particles interacting with classical electromagnetic radiation at some temperature T, with the recognition that at $T = 0$, the radiation is nonzero, with particular properties that enable a statistical equilibrium between charged particles and radiation. Some of the specific properties of classical electromagnetic zero-point (ZP) radiation at $T = 0$ include that the radiation frequency distribution must be Lorentz invariant [11,12] and the fundamental definition of $T = 0$ must be obeyed by ZP radiation [13–15]. In SED, the $T = 0$ stochastic radiation is referred to as ZP radiation. The stochastic radiation for $T \geq 0$ is referred here as "ZP plus Planckian" (ZPP) radiation.

This study involves exploring the use of the following expression:

$$P_{3\mathrm{x}}(\mathbf{x}) = \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dA_N \cdots \int_{-\infty}^{\infty} dB_1 \cdots \int_{-\infty}^{\infty} dB_N \times P_{\mathrm{F,A\text{-}B}}(A_1, \cdots, A_N, B_1, \cdots, B_N)\delta^3\left[\mathbf{x} - \mathbf{x}_{A_1,\cdots,B_N}(t)\right] \ , \qquad (1)$$

for the probability density of finding a point charged particle at position $\mathbf{x}$, in the steady state condition. The restriction to "steady state" can be removed, but that results in a time dependence $P_{3\mathrm{x}}(\mathbf{x}, t)$, which is not treated here. Also, beside the above expression for $P_{3\mathrm{x}}(\mathbf{x})$, such as those for the energy and momentum, are discussed in Section 2.2.

The "3x" notation in Equation (1) indicates that the function $P_{3\mathrm{x}}(\mathbf{x})$ refers to the position vector point in 3D space, while the "F,A-B" notation refers to the probability density function for the Fourier coefficients of the electromagnetic radiation described next. Specifically, $A_1, \cdots, A_N, B_1, \cdots, B_N$ in Equation (1) represent the coefficients in the Fourier expression for the electric and magnetic fields that the particle is "immersed" within. For more details about these Fourier coefficients, see Refs. [4,16]; however, these coefficients are also explained in Section 2.1 when expressions for the radiation electromagnetic fields are introduced. At the end of the calculations, $N \to \infty$ is imposed. Finally, $\mathbf{x}_{A_1,\cdots,B_N\cdots}(t)$ in Equation (1) is the steady state trajectory of the particle and is a function of $A_1, \cdots, A_N, B_1, \cdots, B_N$, while $\delta^3\left[\mathbf{x} - \mathbf{x}_{A_1,\cdots,B_N}(t)\right]$ is the Dirac delta function in 3D.

All "A" and "B" coefficients are real quantities and are integrated from $-\infty$ to $+\infty$. Their values control the steady state solution for the particle's trajectory of $\mathbf{x}_{A_1,\cdots,B_N}(t)$. The probability density of these fields, $P_{\mathrm{F,A\text{-}B}}(A_1, \cdots, A_N, \cdots, B_N)$ dictates their contribution to how $\delta^3\left[\mathbf{x} - \mathbf{x}_{A_1,\cdots,B_N}(t)\right]$ "selects" the contribution to the final particle probability density $P_{3\mathrm{x}}(\mathbf{x})$.

Two solution methods are examined in this paper. In Section 2.1, a slight variation to Equation (1) is used to fully evaluate the analytic probability density $P_{1\mathrm{x}}(x)$ for the position of a one-dimensional (1D) SHO. Here, each Fourier coefficient is explicitly integrated over. In Section 2.2, a few other examples using expressions similar to Equation (1) are also discussed for the 1D and 3D SHOs, including the particle's joint probability densities of $P_{\mathrm{x,x}}(x_1, t_1; x_2, t_2)$ and $P_{3\mathrm{x},3\mathrm{p}}(\mathbf{x}, \mathbf{p})$, where $\mathbf{p}$ is the particle's momentum, as well as the probability density for the kinetic plus potential energy of the oscillator.

The remainder of this paper has two more Sections. In Section 3, a second method, different from the direct integration method in Section 2.1, is described for deducing the analytic expression for $P_{1\mathrm{x}}(\mathrm{x})$. This second method uses a partial differential equation (PDE) approach. Finally, Section 4 provides some concluding remarks, including a discussion of generalizing this study to more complicated systems, in particular the classical hydrogen atom.

2. Calculating Analytic SHO Probability Density Functions from Initial General Expression

2.1. Direct Integration Method for Analytic Expression of SHO Probability Density, $P_{1x}(x)$

The use of probability density expressions like Equation (1) has been explored in Refs. [17,18], but mainly for the stochastic electric field values of the $T \geq 0$ radiation fields. In this paper, the following is analyzed: the stochastic properties of a classical charged point electron, bound by an SHO potential and bathed in stochastic classical electromagnetic ZPP radiation.

To start, let us describe the radiation fields by considering a large region of space, where "large" means as compared to the size of the space that any charged particles, "bound" by a classical potential, occupy via traversing within the confines of this classical potential. Let us consider a rectangular parallelepiped region that the radiation fields are confined within, where the rectangular parallelepiped has dimensions in the space of L_x, L_y, and L_z, along the x, y, and z axes. Although other shapes can be considered (see, e.g., Ref. [19]), the rectangular parallelepiped offers mathematical simplicity, especially since at the end of the calculations, L_x, L_y, and L_z are typically taken in the limit of infinite in size.

In what follows, the ZP or ZPP radiation fields are represented as an infinite sum of plane waves, with periodic boundary conditions (bcs) imposed. This imposition enables the use of Fourier series to describe the fields. For a large region of space, this periodicity does not effect the physical analysis, but it does simplify the subsequent mathematical analysis. Hence, the following sum of plane waves is used for the "free" electric $\mathbf{E}(\mathbf{x},t)$ and magnetic $\mathbf{B}(\mathbf{x},t)$ radiation fields in this large parallelepiped volume [4] (see p. 76, Equations (3.65) and (3.66) in Ref. [4]):

$$\mathbf{E}_{\text{rad}}(\mathbf{x},t) = \frac{1}{\left(L_x L_y L_z\right)^{1/2}} \sum_{n_x,n_y,n_z=-\infty}^{\infty} \sum_{\lambda=1,2} \hat{\varepsilon}_{\mathbf{k_n},\lambda} \left[\begin{array}{c} A_{\mathbf{k_n},\lambda} \cos(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) \\ + B_{\mathbf{k_n},\lambda} \sin(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) \end{array} \right] , \quad (2)$$

$$\mathbf{B}_{\text{rad}}(\mathbf{x},t) = \frac{1}{\left(L_x L_y L_z\right)^{1/2}} \sum_{n_x,n_y,n_z=-\infty}^{\infty} \sum_{\lambda=1,2} \left(\hat{\mathbf{k}}_{\mathbf{n}} \times \hat{\varepsilon}_{\mathbf{k_n},\lambda}\right) \left[\begin{array}{c} A_{\mathbf{k_n},\lambda} \cos(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) \\ + B_{\mathbf{k_n},\lambda} \sin(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) \end{array} \right] . \quad (3)$$

Here,

$$\mathbf{k_n} = \frac{2\pi n_x}{L_x}\hat{\mathbf{x}} + \frac{2\pi n_y}{L_y}\hat{\mathbf{y}} + \frac{2\pi n_z}{L_z}\hat{\mathbf{z}} , \quad (4)$$

and n_x, n_y, and n_z are integers. Moreover, $\omega_{\mathbf{n}} = c|\mathbf{k_n}|$, $\mathbf{k_n}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda} = \mathbf{k_n}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda'} = 0$, and $\hat{\varepsilon}_{\mathbf{k_n},\lambda}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda'} = 0$ for $\lambda \neq \lambda'$, where λ and λ' indicate the linear polarization direction, and c denotes the speed of light. Here, λ and λ' are essentially indices that take on only two values, so each might be represented by the values 1 or 2. Moreover, $\hat{\mathbf{k}}_{\mathbf{n}} = \mathbf{k_n}/|\mathbf{k_n}|$; similarly, all other vectors with a hat are meant to be unit vectors, and all quantities in bold are vectors.

It should be noted that one can show, using the free space Maxwell's equations, that Equations (2) and (3) satisfy the wave equations of $\nabla^2\mathbf{E}(\mathbf{x},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{x},t)$ and $\nabla^2\mathbf{B}(\mathbf{x},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}(\mathbf{x},t)$. Moreover, the presence of $\hat{\varepsilon}_{\mathbf{k_n},\lambda}$ and $\left(\hat{\mathbf{k}}_{\mathbf{n}} \times \hat{\varepsilon}_{\mathbf{k_n},\lambda}\right)$ in Equations (2) and (3), plus the cited relationships of $\mathbf{k_n}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda} = \mathbf{k_n}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda'} = 0$, and $\hat{\varepsilon}_{\mathbf{k_n},\lambda}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda'} = 0$ for $\lambda \neq \lambda'$, enable all four free space Maxwell's equations to be satisfied.

In SED, the stochastic nature of the radiation fields in Equations (2) and (3) arises from the probability distribution of the Fourier coefficients $A_1, \cdots, A_N, B_1, \cdots, B_N$ over a large ensemble of equally sized space regions, which in the case considered here has dimensions L_x, L_y, L_z. The fields within the ensemble of space regions are characterized by the temperature $T \geq 0$; consequently, the Fourier coefficients will have a probabilistic distribution over the ensemble. For each member of the ensemble, the Fourier coefficients are fixed; only when examining each cavity in the ensemble will the Fourier coefficients be different and follow a probabilistic distribution in values.

Although this description is followed in SED, this basic behavior goes back to Planck in the first half of Planck's major treatise [20] and later Einstein and Hopf [21,22] (see English translations in Refs. [23,24], respectively).The main difference with this much older studies and SED is that in SED the assumption is not made that the radiation fields fall to zero at $T = 0$.

Here, a classical charged point particle, with charge, q, is considered oscillating in one dimension, namely, the $\hat{\mathbf{x}}$ direction, constrained along the $\hat{\mathbf{x}}$ direction by a SHO potential, $\frac{1}{2}m\omega_0^2 x^2$, giving rise to a binding force, $-m\omega_0^2 x(t)$, along the $\hat{\mathbf{x}}$ direction. If one imagines a sphere of uniform charge density and net charge $-q$ that the $+q$ point charge oscillates within, then the SHO force acting on the $+q$ charge can be pictured as originating in this way. This neutral system will look like an electric dipole oscillator at distances far from the oscillator system.

As for the oscillating $+q$ charge, one can describe its motion using the nonrelativistic approximation of the Lorentz–Dirac equation:

$$m\ddot{x}(t) = -m\omega_0^2 x(t) + m\Gamma \dddot{x}(t) + qE_{\mathrm{rad},x}(\mathbf{x}=\mathbf{0},t) \ , \tag{5}$$

where $\Gamma = \frac{2}{3}\frac{q^2}{mc^3}$, and $m\Gamma\dddot{x}$ is the nonrelativistic expression for the radiation reaction for a charged point particle of mass m and charge q. $E_{\mathrm{rad},x}$ is the net electric field in the $\hat{\mathbf{x}}$ direction due to the sum of the radiation fields, assuming them to be either ZP or ZPP. The reason for $E_{\mathrm{rad},x}(\mathbf{x}=\mathbf{0},t)$ is that the dipole approximation is being made when evaluating the electric field component of the Lorentz force. The magnetic field component of the Lorentz force is assumed to be much smaller in magnitude than $qE_{\mathrm{rad},x}(\mathbf{x}=\mathbf{0},t)$ and is ignored here.

Finally, a common approximation to the weak "force" of $m\Gamma\dddot{x}(t)$, due to the small magnitude of Γ, is first that $m\ddot{x} \approx -m\omega_0^2 x$, or $\ddot{x} \approx -\omega_0^2 x$, so that

$$\dddot{x} = \frac{d}{dt}\ddot{x} \approx \frac{d}{dt}\left(-\omega_0^2 x\right) = -\omega_0^2 \dot{x} \ . \tag{6}$$

After dividing through by m, Equation (5) becomes:

$$\ddot{x}(t) = -\omega_0^2 x(t) - \Gamma\omega_0^2 \dot{x}(t) + \frac{q}{m}E_{\mathrm{rad},x}(\mathbf{x}=\mathbf{0},t) \ . \tag{7}$$

Using Equation (7) and the properties of the ZP and ZPP radiation fields, Boyer showed in Ref. [3] that a detailed agreement exists between SED versus QM and QED, for the stochastic properties of this SHO system, for $T \geq 0$.

Rewriting Equation (2) in the dipole approximation,

$$\begin{aligned} E_{\mathrm{rad},x}(\mathbf{x}=\mathbf{0},t) &= \frac{1}{\left(L_x L_y L_z\right)^{1/2}} \sum_p \left(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p\right)\left[A_p\cos(\omega_p t) - B_p \sin(\omega_p t)\right] \\ &= \frac{1}{\left(L_x L_y L_z\right)^{1/2}} \sum_p \left(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p\right) \mathrm{Re}\left[\left(A_p + iB_p\right)e^{i\omega_p t}\right] \ , \end{aligned} \tag{8}$$

where the sum over the index p means the full sum over n_x, n_y, n_z and λ in Equation (2), and the second line in Equation (8) arises due to the A and B coefficients being real.

The steady state particular solution to Equation (7) can be shown to be

$$x_{\mathrm{ss}}(t) = \frac{(q/m)}{\sqrt{L_x L_y L_z}} \sum_p \left(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p\right) \mathrm{Re}\left[\frac{e^{i\omega_p t}\left(A_p + iB_p\right)}{\left(-\omega_p^2 + \omega_0^2 + i\Gamma\omega_p\omega_0^2\right)}\right] \ , \tag{9}$$

as can be checked by substituting Equation (9) into Equation (5) and by looking at times large enough that the homogeneous solution dies out.

Let us relabel $x_{ss}(t)$ as $x_{A_1,\cdots,B_N}(t)$, to emphasize the dependence on the Fourier coefficients, as in Equation (1). Moreover, to help make better use of Equation (9), let us make the following definitions:

$$x_{A,p}(t) \equiv \frac{(q/m)}{\sqrt{L_x L_y L_z}}(\hat{\mathbf{x}} \cdot \hat{\varepsilon}_p) A_p \,\mathrm{Re}\left[\frac{e^{i\omega_p t}}{-\omega_p^2 + \omega_0^2 + i\Gamma\omega_p\omega_0^2}\right] \tag{10}$$

and

$$\begin{aligned} x_{B,p}(t) &\equiv \frac{(q/m)}{\sqrt{L_x L_y L_z}}(\hat{\mathbf{x}} \cdot \hat{\varepsilon}_p) B_p \,\mathrm{Re}\left[\frac{ie^{i\omega_p t}}{-\omega_p^2 + \omega_0^2 + i\Gamma\omega_p\omega_0^2}\right] \\ &= -\frac{(q/m)}{\sqrt{L_x L_y L_z}}(\hat{\mathbf{x}} \cdot \hat{\varepsilon}_p) B_p \,\mathrm{Im}\left[\frac{e^{i\omega_p t}}{-\omega_p^2 + \omega_0^2 + i\Gamma\omega_p\omega_0^2}\right] \quad , \end{aligned} \tag{11}$$

Simplifying further, let us define $x'_{A,p}(t)$ and $x'_{B,p}(t)$ via:

$$x_{A,p}(t) \equiv A_p x'_{A,p}(t) \tag{12}$$

and

$$x_{B,p}(t) \equiv B_p x'_{B,p}(t) \quad . \tag{13}$$

In SED, the Fourier coefficients $A_1, \cdots, A_N, B_1, \cdots, B_N$ are assumed to be independent random variables with Gaussian probability density distribution,

$$P_{\mathrm{F}}(A_p) = \frac{1}{\sqrt{2\pi\left(\sigma_p^A\right)^2}} \exp\left(-\frac{A_p^2}{2(\sigma_p)^2}\right) \quad . \tag{14}$$

The same distribution holds for the Fourier coefficient B_p. The label "F" is added to specify that this probability density function P_{F} refers to the Fourier coefficients. Moreover, σ_p depends on $\mathbf{k_n}$ as in Equation (4), as well as the temperature T. This dependence has been studied considerably in SED. In particular, the functional form of σ_p at $T = 0$ is a cornerstone for SED and is expressed by $[\sigma(\omega_{\mathbf{n}}, T = 0)]^2 = 2\pi\hbar\omega_{\mathbf{n}}$where $\hbar$ is the Planck's reduced constant. It is this functional form at $T = 0$ that was referred to in Section 1 and that is deduced first via the imposition of Lorentz invariance by Marshall [11] and Boyer [12], and much later by the author by imposing the thermodynamic definition of $T = 0$ [13–15].

In general, for $T \geq 0$,

$$\sigma_p \to [\sigma(\omega_{\mathbf{n}}, T)]^2 = 2\pi\hbar\omega_{\mathbf{n}} + \frac{4\pi\hbar\omega_{\mathbf{n}}}{\exp\left(\frac{\hbar\omega_{\mathbf{n}}}{k_{\mathrm{B}}T}\right) - 1} = 2\pi\hbar\omega_{\mathbf{n}} \coth\left(\frac{\hbar\omega_{\mathbf{n}}}{2k_{\mathrm{B}}T}\right) \quad , \tag{15}$$

where $\omega_{\mathbf{n}} = c|\mathbf{k_n}|$.

Replacing $\delta^3\left[\mathbf{x} - \mathbf{x}_{A_1,\cdots,B_N}(t)\right]$ in Equation (1) with the 1D Dirac delta function of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} ds e^{is\left(x - x_{A_1,\cdots,B_N}(t)\right)} \quad ,$$

and realizing that $P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N)$ equals $P_{\text{F}}(A_1)\cdots P_{\text{F}}(A_N)P_{\text{F}}(B_1)\cdots P_{\text{F}}(B_N)$ due to the random variable independence of these Fourier coefficients, then the 1D position probability density function for this 1D SHO is

$$\begin{aligned}
&P_{1x}(x) \\
&= \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dA_N \cdots \int_{-\infty}^{\infty} dB_1 \cdots \int_{-\infty}^{\infty} dB_N P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N)\delta[x - x_{A_1,\cdots,B_N}(t)] \\
&= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dB_N \frac{1}{2\pi} e^{is\left(x - x_{A_1,\cdots,B_N}(t)\right)} \\
&\qquad \times \frac{1}{\sqrt{2\pi(\sigma_1)^2}} \exp\left(-\frac{A_1^2}{2\sigma_1^2}\right) \cdots \frac{1}{\sqrt{2\pi(\sigma_N)^2}} \exp\left(-\frac{B_N^2}{2\sigma_N^2}\right) \\
&= \frac{1}{2\pi} \frac{1}{2\pi\sigma_1^2} \cdots \frac{1}{2\pi\sigma_N^2} \int_{-\infty}^{\infty} ds e^{isx} \\
&\qquad \times \int_{-\infty}^{\infty} dA_1 \exp\left(-\frac{A_1^2}{2\sigma_1^2} - isA_1 x'_{A,1}\right) \cdots \int_{-\infty}^{\infty} dB_N \exp\left(-\frac{B_N^2}{2\sigma_N^2} - isB_N x'_{B,N}\right) .
\end{aligned} \tag{16}$$

The evaluation of Equation (16) can be conducted by completing squares. Specifically:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dA_p \exp\left(-\frac{A_p^2}{2\sigma_p^2} - isA_p x'_{A,p}\right) \\
&= \int_{-\infty}^{\infty} dA_p \exp\left[-\frac{\left(A_p^2 + 2isA_p x'_{A,p}\sigma_p^2\right)}{2\sigma_p^2}\right] \\
&= \int_{-\infty}^{\infty} dA_p \exp\left\{-\frac{\left[A_p^2 + 2isA_p x'_{A,p}\sigma_p^2 - s^2\left(x'_{A,p}\right)^2\sigma_p^4\right]}{2\sigma_p^2} - \frac{s^2\left(x'_{A,p}\right)^2\sigma_p^4}{2\sigma_p^2}\right\} \\
&= \exp\left[-\frac{s^2\left(x'_{A,p}\right)^2\sigma_p^2}{2}\right] \int_{-\infty}^{\infty} dA_p \exp\left\{-\frac{\left(A_p + isx'_{A,p}\sigma_p^2\right)^2}{2\sigma_p^2}\right\} \\
&= \exp\left[-\frac{s^2\left(x'_{A,p}\right)^2\sigma_p^2}{2}\right] \sqrt{2\pi\sigma_p^2} .
\end{aligned} \tag{17}$$

The same holds true for the B terms.
Hence, from Equations (16) and (17):

$$\begin{aligned}
P_{1x}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{isx} \left\{ \exp\left[-\frac{s^2\left(x'_{A,1}\right)^2\sigma_1^2}{2}\right] \cdots \exp\left[-\frac{s^2\left(x'_{A,N}\right)^2\sigma_N^2}{2}\right] \right\} \\
&\qquad \times \left\{ \exp\left[-\frac{s^2\left(x'_{B,1}\right)^2\sigma_1^2}{2}\right] \cdots \exp\left[-\frac{s^2\left(x'_{B,N}\right)^2\sigma_N^2}{2}\right] \right\} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \exp\left[isx - \frac{s^2}{2} \sum_p \left[\left(x'_{A,p}\sigma_p\right)^2 + \left(x'_{B,p}\sigma_p\right)^2\right]\right] .
\end{aligned} \tag{18}$$

To better identify the significance of this expression, let us calculate

$$\left\langle \left[x_{A_1,\cdots,B_N}(t)\right]^2\right\rangle = \left\langle [x_{ss}(t)]^2\right\rangle = \left\langle \left\{ \frac{(q/m)}{\sqrt{L_xL_yL_z}} \sum_p (\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)\,\mathrm{Re}\left[\frac{e^{i\omega_p t}(A_p+iB_p)}{\left(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2\right)}\right]\right\}^2\right\rangle \ , \tag{19}$$

where the angle brackets mean that the ensemble average is to be taken. Let us impose that

$$\langle A_{\mathbf{k_n},\lambda}\rangle = \langle B_{\mathbf{k_n},\lambda}\rangle = 0 \ , \tag{20}$$

from Equation (14), which also holds for the $B_{\mathbf{k_n},\lambda}$ variables, and also impose the assumption of independent random variables,

$$\left\langle A_{\mathbf{k_n},\lambda}B_{\mathbf{k_{n'}},\lambda'}\right\rangle = 0 \ , \tag{21}$$

and:

$$\left\langle A_{\mathbf{k_n},\lambda}A_{\mathbf{k_{n'}},\lambda'}\right\rangle = \left\langle B_{\mathbf{k_n},\lambda}B_{\mathbf{k_{n'}},\lambda'}\right\rangle = 0 \ , \ \text{if } \mathbf{n}\neq\mathbf{n}' \text{ or } \lambda\neq\lambda' \ , \tag{22}$$

while

$$\langle A_{\mathbf{k_n},\lambda}A_{\mathbf{k_n},\lambda}\rangle = \langle B_{\mathbf{k_n},\lambda}B_{\mathbf{k_n},\lambda}\rangle = [\sigma(\omega_{\mathbf{n}},T)]^2 \ , \tag{23}$$

are assumed to be functions of the frequency of the radiation and of the temperature T, which were previously labelled as σ_p^2 .

Continuing with the evaluation of Equation (19), let us first note that:

$$x_{ss}(t) = \frac{(q/m)}{\sqrt{L_xL_yL_z}}\sum_p(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)\left\{\begin{array}{c} A_p\,\mathrm{Re}\left[\frac{e^{i\omega_p t}}{(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2)}\right] \\ -B_p\,\mathrm{Im}\left[\frac{e^{i\omega_p t}}{(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2)}\right]\end{array}\right\} \ . \tag{24}$$

Taking the statistical properties into account of Equations (21)–(23), we obtain:

$$\begin{aligned}\left\langle [x_{ss}(t)]^2\right\rangle &= \frac{(q/m)^2}{L_xL_yL_z}\sum_p(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)^2\sigma_p^2\left(\begin{array}{c}\left\{\mathrm{Re}\left[\frac{e^{i\omega_p t}}{(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2)}\right]\right\}^2 \\ +\left\{\mathrm{Im}\left[\frac{e^{i\omega_p t}}{(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2)}\right]\right\}^2\end{array}\right) \\ &= \frac{(q/m)^2}{L_xL_yL_z}\sum_p(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)^2\left|\frac{1}{\left(-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2\right)}\right|^2\sigma_p^2 \ .\end{aligned} \tag{25}$$

To simplify later expressions, let

$$\sigma_x^2 \equiv \left\langle [x_{ss}(t)]^2\right\rangle \ . \tag{26}$$

Now one just needs to relate the terms $\sum_p \sigma_p^2\left[\left(x'_{A_p}\right)^2+\left(x'_{B_p}\right)^2\right]$ in Equation (18) to $\left\langle [x_{ss}(t)]^2\right\rangle$. As shown below, they are exactly equal. From Equations (10)–(13):

$$\begin{aligned}&\sum_p\sigma_p^2\left[\left(x'_{A,p}\right)^2+\left(x'_{B,p}\right)^2\right] \\ &= \sum_p\sigma_p^2\frac{(q/m)^2}{L_xL_yL_z}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)^2\left(\left\{\mathrm{Re}\left[\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right]\right\}^2+\left\{\mathrm{Im}\left[\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right]\right\}^2\right) \\ &= \sum_p\frac{(q/m)^2}{L_xL_yL_z}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)^2\frac{1}{\left|-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2\right|^2}\sigma_p^2 = \left\langle [x_{ss}(t)]^2\right\rangle \ .\end{aligned} \tag{27}$$

Equation (18) then becomes, after completing the square of s and then integrating over s:

$$\begin{aligned} P_{1x}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} ds \exp\left[isx - \frac{s^2}{2}\left\langle [x_{ss}(t)]^2\right\rangle\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} ds \exp\left[-\frac{\sigma_x^2}{2}\left(s^2 - \frac{2isx}{\sigma_x^2} - \frac{x^2}{(\sigma_x^2)^2}\right) - \frac{x^2}{2\sigma_x^2}\right] \\ &= \frac{1}{2\pi}\sqrt{\frac{2\pi}{\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \ . \end{aligned} \tag{28}$$

This Gaussian result has been deduced in SED previously by researchers in SED, but not as far as this author knows, starting via the general probability expression in Equation (1). Although the above is a much longer derivation than deductions published earlier, it is still illuminating, as discussed in Sections 2.2, 3 and 4 below.

Moreover, while the expression (28) is connected with QED [3], one can relate it to the relevant expression in QM as calculated from Schrödinger's equation when taking into account the probability density at temperature T, and summing over all probability density functions $|\psi_n(x)|^2$, each times $\frac{1}{Z}\exp(-E_n/kT)$, where $Z = \sum_n \exp(-E_n/kT)$. To obtain this agreement, and essentially dropping the QED effects, in SED one would make what some call the continuum and resonant approximations, where first the sum over $\mathbf{n}$ is approximated as a 3D integral, and then later the charge q is assumed to be small.

More specifically, the continuum approximation consists of the following approximations, from Equation (4):

$$dk_x = 2\pi\frac{dn_x}{L_x} \ , \ dk_y = 2\pi\frac{dn_y}{L_y} \ , \ dk_z = 2\pi\frac{dn_z}{L_z} \ , \tag{29}$$

so that

$$\sum_{\mathbf{n}} \cdots = \sum_{n_1,n_2,n_3} \cdots \approx \int dn_x \int dn_y \int dn_z \cdots \approx \frac{L_x L_y L_z}{(2\pi)^3}\int dk_x \int dk_y \int dk_z \ldots \ . \tag{30}$$

Consequently,

$$\begin{aligned} \left\langle [x_{ss}(t)]^2 \right\rangle &= \sum_p \sigma_p^2 \frac{(q/m)^2}{L_x L_y L_z}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_p)^2 \frac{1}{\left|-\omega_p^2 + \omega_0^2 + i\Gamma\omega_p\omega_0^2\right|^2} \\ &= \frac{(q/m)^2}{L_x L_y L_z}\sum_{n_x,n_y,n_z=-\infty}^{\infty} \sigma_{\mathbf{k_n}}^2 \sum_{\lambda=1,2}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_{\mathbf{k_n},\lambda})^2 \frac{1}{\left|-\omega_{\mathbf{k_n}}^2 + \omega_0^2 + i\Gamma\omega_{\mathbf{k_n}}\omega_0^2\right|^2} \\ &\approx \frac{(q/m)^2}{(2\pi)^3}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \sigma_{\mathbf{k}}^2 \sum_{\lambda=1,2}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_{\mathbf{k},\lambda})^2 \frac{1}{\left|-\omega_{\mathbf{k}}^2 + \omega_0^2 + i\Gamma\omega_{\mathbf{k}}\omega_0^2\right|^2} \ . \end{aligned} \tag{31}$$

One can show that

$$\sum_{\lambda=1,2}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_{\mathbf{k},\lambda})^2 = \sum_{\lambda=1,2}(\hat{\varepsilon}_{\mathbf{k},\lambda,x})^2 = \left[1 - \left(\frac{k_x^2}{\mathbf{k}}\right)^2\right] \ . \tag{32}$$

Now, using spherical coordinates in Equation (31) and noting that $\sigma_{\mathbf{k}}$ actually depends only on the magnitude of $\mathbf{k}$, or $|\mathbf{k}| = k$, and likewise, $\omega_{\mathbf{k}} = \omega_k = ck$, one finds:

$$\left\langle [x_{ss}(t)]^2 \right\rangle \approx \frac{(q/m)^2}{(2\pi)^3} \int_0^\infty dk \int_0^\pi d\theta k \int_0^{2\pi} d\phi k (\sin\theta) \sigma_k^2 \frac{\left[1 - \left(\frac{k_x^2}{k}\right)^2\right]}{\left[\left(-\omega_k^2 + \omega_0^2\right)^2 + \left(\Gamma \omega_k \omega_0^2\right)^2\right]} \quad . \tag{33}$$

Integrating over θ and ϕ and noting by symmetry that

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \left[1 - \left(\frac{k_x^2}{k}\right)^2\right] = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3} \quad , \tag{34}$$

then:

$$\left\langle [x_{ss}(t)]^2 \right\rangle \approx \frac{(q/m)^2}{(2\pi)^3} \frac{8\pi}{3} \frac{1}{c^3} \int_0^\infty d\omega \frac{\omega^2 \sigma^2(\omega, T)}{\left[\left(-\omega^2 + \omega_0^2\right)^2 + \left(\Gamma \omega \omega_0^2\right)^2\right]} \quad . \tag{35}$$

For small values of q in $\Gamma = \frac{2}{3}\frac{q^2}{mc^3}$ (for an electron, Γ is very small, about 6.27×10^{-24} s), $\left[\left(-\omega^2 + \omega_0^2\right)^2 + \left(\Gamma \omega \omega_0^2\right)^2\right]^{-1}$ becomes strongly peaked when $\omega \approx \omega_0$. The above integral can then be well approximated by

$$\int_0^\infty d\omega \frac{\omega_0^2 \sigma^2(\omega_0, T)}{(\omega - \omega_0)^2 (2\omega_0)^2 + \left(\Gamma \omega_0^3\right)^2} \approx \int_{-\infty}^\infty d\omega \frac{\omega_0^2 \sigma^2(\omega_0, T)}{(\omega - \omega_0)^2 (2\omega_0)^2 + \left(\Gamma \omega_0^3\right)^2} \quad . \tag{36}$$

Making use of the integral,

$$\int_{-\infty}^\infty d\omega \frac{1}{\omega^2 A^2 + B^2} = \frac{\pi}{AB} \ . \tag{37}$$

Equation (35) becomes:

$$\begin{aligned} \left\langle [x_{ss}(t)]^2 \right\rangle &\approx \frac{(q/m)^2}{(2\pi)^3} \frac{8\pi}{3} \frac{1}{c^3} \omega_0^2 \sigma^2(\omega_0, T) \frac{\pi}{(2\omega_0)\left(\frac{2}{3}\frac{q^2}{mc^3}\omega_0^3\right)} \\ &= \frac{\hbar}{2m\omega_0} \coth\left(\frac{\hbar\omega_0}{2kT}\right) \quad . \end{aligned} \tag{38}$$

This is more recognizable for QM, and even more so for $T \to 0$, becoming

$$\left\langle [x_{ss}(t)]^2 \right\rangle_{T=0} = \frac{\hbar}{2m\omega_0} \quad . \tag{39}$$

2.2. *Examples of Other Analytic SHO Probability Density Functions That Can Similarly Be Deduced*

The method of Section 2.1 can be used to obtain many other types of probability density functions. Using

$$m\ddot{\mathbf{x}}(t) = -m\omega_0^2 \mathbf{x}(t) + m\Gamma \dddot{\mathbf{x}}(t) + q\mathbf{E}_{rad}(\mathbf{x} = \mathbf{0}, t) \quad , \tag{40}$$

one can certainly generalize the previous 1D SHO to a 3D SHO with the probability density function of $P_{3x}(\mathbf{x})$ in Equation (1).

Moreover, the position and momentum joint probability density function for this 3D SHO can be expressed by

$$P_{3x,3p}(\mathbf{x},\mathbf{p}) = \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dB_N P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N)\delta^3[\mathbf{x}-\mathbf{z}_{A_1\ldots B_N}(t)]\delta^3\left[\mathbf{p}-\mathbf{p}_{A_1\ldots B_N}(t)\right] \ , \quad (41)$$

which can be used to find an analytic expression for $P_{3x,3p}(\mathbf{x},\mathbf{p})$ in a similar manner as carried out in Section 2.1 for the 1D SHO and $P_{1x}(x)$.

In addition, one can express the probability density for the nonrelativistic energy of a 3D SHO via

$$P_{\text{E}}(\mathcal{E}) = \int dA_1 \cdots \int dB_N P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N)\delta^3\{\mathcal{E}-[\text{KE}(t)+\text{PE}(t)]\} \ , \quad (42)$$

where

$$\text{KE}(t)+\text{PE}(t) = \frac{m}{2}[\dot{\mathbf{x}}_{\text{ss}}(t)]^2 + \frac{m\omega_0^2}{2}[\mathbf{x}_{\text{ss}}(t)]^2 \ . \quad (43)$$

As another example, using the same method, one can calculate the following 1D SHO joint probability position density distribution $P_{1x,1x}(x_1,t_1;x_2,t_2)$. The final result reads:

$$\begin{aligned}
&P_{1x,1x}(x_1,t_1;x_2,t_2) \\
&= \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dB_N P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N)\delta[x_1 - x_{A_1\ldots B_N}(t_1)]\delta[x_2 - x_{A_1\ldots B_N}(t_2)] \\
&= \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dB_N P_{\text{F,A-B}}(A_1,\cdots,A_N,B_1,\cdots,B_N) \\
&\quad \times \frac{1}{2\pi}\int_{-\infty}^{\infty} ds_1 e^{is_1\left(x_1 - x_{A_1,\cdots,B_N}(t_1)\right)} \frac{1}{2\pi}\int_{-\infty}^{\infty} ds_2 e^{is_2\left(x_2 - x_{A_1,\cdots,B_N}(t_2)\right)} \\
&= \frac{\exp\left(-\frac{\left[x_1^2\langle x_{\text{ss}}^2(t)\rangle + x_2^2\langle x_{\text{ss}}^2(t)\rangle - 2x_1x_2\langle x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)\rangle\right]}{2\left[\langle x_{\text{ss}}^2(t)\rangle^2 - \langle x(t_1)x(t_2)\rangle^2\right]}\right)}{2\pi\sqrt{\langle x_{\text{ss}}^2(t)\rangle^2 - \langle x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)\rangle^2}} \ .
\end{aligned} \quad (44)$$

Here, the ensemble average of the square of the steady state solution $x_{\text{ss}}(t)$, or $\langle x_{\text{ss}}^2(t)\rangle$, is independent of t, as shown in Equation (31), and as given in the more familiar continuum and resonant approximation in Equation (38), as $\frac{\hbar}{2m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$. However, the ensemble average of the product $x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)$, or $\langle x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)\rangle$, depends on the time difference $t_1 - t_2$, and is given by

$$\langle x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)\rangle = \frac{q^2}{m^2(L_xL_yL_z)}\sum_{\mathbf{n},\lambda}(\hat{\mathbf{x}}\cdot\hat{\varepsilon}_{\mathbf{n},\lambda})^2(\sigma_{\mathbf{n},\lambda})^2 \frac{\cos[\omega_{\mathbf{n}}(t_2-t_1)]}{(-\omega_{\mathbf{n}}^2+\omega_0^2)^2 + (\Gamma\omega_0^2\omega_{\mathbf{n}})^2} \ . \quad (45)$$

In the continuum and resonant approximation, this simplifies to

$$\langle x_{\text{ss}}(t_1)x_{\text{ss}}(t_2)\rangle \approx \frac{\hbar}{2m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\cos[\omega_0(t_2-t_1)] \ . \quad (46)$$

Thus, the general expressions are quite straightforwad to formulate, as in Equations (1), (16), (41), (42) and (44), although carrying out all the integrations to arrive at an analytic expression, as in Equations (28) and (44), can be quite nontrivial.

3. A PDE Approach for Deducing the SHO Probability Density Function $P_{1x}(x)$

Rather than directly integrating over all the A_p and B_p radiation Fourier coefficients in Equation (16) to obtain the analytic expression for $P_{1x}(x)$ in Equation (28), here it is shown that Equation (16) satisfies a PDE that enables $P_{1x}(x)$ in Equation (28) to be deduced. In some ways, this approach is less complicated than the direct integration method of Section 2.1 and might provide insight for more complicated systems than the SHO, such as the classical hydrogen case.

Without integrating over all $A_{1,} \cdots, B_N$ variables as in Section 2.1, nor by only showing that Equation (28) solves Equation (47) below, here, let us directly take the 1D version of Equation (1) and illustrate how this expression satisfies

$$\frac{\partial}{\partial(\sigma_x^2)} P_{1x}(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} P_{1x}(x) \ , \tag{47}$$

where σ_x^2 is given by Equations (25) and (26), or in the continuum resonant approximations by Equation (38). More specifically, as is shown below, in order for Equation (47) to hold with $P_{1x}(x)$ given by the 1D version of Equation (1) (i.e., the top line of Equation (16)), then σ_x^2 must be given by Equation (25). Moreover, upon imposing that $P_{1x}(x) \to 0$ as $|x| \to \infty$, with symmetry about $x = 0$, and that the function monotonically decreases as $|x|$ increases, one obtains the solution Equation (28) for Equation (47).

To show that $P_{1x}(x)$ in the 1D version of Equation (1) satisfies Equation (47), let us start with

$$\begin{aligned}
&\frac{\partial}{\partial(\sigma_x^2)} P_{1x}(x) \\
&= \frac{\partial}{\partial(\sigma_x^2)} \int_{-\infty}^{\infty} dA_1, \cdots, \int_{-\infty}^{\infty} dA_N \int_{-\infty}^{\infty} dB_1, \cdots, \int_{-\infty}^{\infty} dB_N \\
&\qquad P_F(A_1), \cdots, P_F(A_N), P_F(B_1), \cdots, P_F(B_N)\delta\left[x - x_{A_1,\cdots,B_N}(t)\right] \\
&= \int_{-\infty}^{\infty} dA_1, \cdots, \int_{-\infty}^{\infty} dA_N \int_{-\infty}^{\infty} dB_1, \cdots, \int_{-\infty}^{\infty} dB_N \delta\left[x - x_{A_1,\cdots,B_N}(t)\right] \\
&\qquad \times \frac{\partial}{\partial(\sigma_x^2)}\left[P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)\right] \ .
\end{aligned} \tag{48}$$

Clearly,

$$\begin{aligned}
&\frac{\partial}{\partial(\sigma_x^2)}\left[P_F(A_1)\cdots P_F(B_N)\right] \\
&= \left[\frac{\partial}{\partial(\sigma_x^2)} P_F(A_1)\right] P_F(A_2)\cdots P_F(A_N)P_F(B_1)P_F(B_2)\cdots P_F(B_N) \\
&\quad + P_F(A_1)\left[\frac{\partial}{\partial(\sigma_x^2)} P_F(A_2)\right]\cdots P_F(A_N)P_F(B_1)P_F(B_2)\cdots P_F(B_N) \\
&\quad + \cdots + P_F(A_1)P_F(A_2)\cdots P_F(A_N)P_F(B_1)P_F(B_2)\cdots \left[\frac{\partial}{\partial(\sigma_x^2)} P_F(B_N)\right] \ .
\end{aligned} \tag{49}$$

Moreover, since one can readily show that

$$\frac{\partial}{\partial\left(\sigma_p^2\right)}\left\{\frac{1}{\sqrt{2\pi\sigma_p^2}}\exp\left[-\frac{(A_p)^2}{2\sigma_p^2}\right]\right\} = \frac{1}{2}\frac{\partial^2}{\partial A_p^2}\left\{\frac{1}{\sqrt{2\pi\sigma_p^2}}\exp\left[-\frac{(A_p)^2}{2\sigma_p^2}\right]\right\} \ , \tag{50}$$

then,

$$\frac{\partial}{\partial(\sigma_x^2)} P_{\mathrm{F}}(A_p) = \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \frac{\partial}{\partial\left(\sigma_p^2\right)} P_{\mathrm{F}}(A_p) = \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \frac{1}{2} \frac{\partial^2}{\partial A_p^2} \left\{ \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left[-\frac{(A_p)^2}{2\sigma_p^2} \right] \right\} , \tag{51}$$

and likewise for $P_{\mathrm{F}}(B_p)$.

From Equations (48)–(51):

$$\frac{\partial}{\partial\sigma_x^2} P_{1x}(x) = \int_{-\infty}^{\infty} dA_1 \cdots \int_{-\infty}^{\infty} dB_N \left\{ \begin{array}{c} \frac{\partial(\sigma_1^2)}{\partial(\sigma_x^2)} \frac{1}{2} \frac{d^2 P_{\mathrm{F}}(A_1)}{dA_1^2} P_{\mathrm{F}}(A_2) \cdots P_{\mathrm{F}}(B_N) \\ + P_{\mathrm{F}}(A_1) \frac{\partial(\sigma_2^2)}{\partial(\sigma_x^2)} \frac{1}{2} \frac{d^2 P_{\mathrm{F}}(A_2)}{dA_2^2} P_{\mathrm{F}}(A_3) \cdots P_{\mathrm{F}}(B_N) \\ + \cdots + [P_{\mathrm{F}}(A_1) \cdots P_{\mathrm{F}}(B_{N-1})] \frac{\partial(\sigma_N^2)}{\partial(\sigma_x^2)} \frac{1}{2} \frac{d^2 P_{\mathrm{F}}(B_N)}{dB_N^2} \end{array} \right\} \delta[x - x_{A_1,\cdots,B_N}(t)] . \tag{52}$$

Each of the $2N$ terms in the sum within the curly brackets above can be integrated by parts twice. Considering the pth A_p term,

$$\begin{aligned} &\frac{1}{2} \int_{-\infty}^{\infty} dA_1 P_{\mathrm{F}}(A_1) \cdots \int_{-\infty}^{\infty} dA_{p-1} P_{\mathrm{F}}(A_{p-1}) \left\{ \int_{-\infty}^{\infty} dA_p \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \frac{d^2 P_{\mathrm{F}}(A_p)}{dA_p^2} \right\} \\ &\times \int_{-\infty}^{\infty} dA_{p+1} P_{\mathrm{F}}(A_{p+1}) \cdots \int_{-\infty}^{\infty} dA_N P_{\mathrm{F}}(A_N) \int_{-\infty}^{\infty} dB_1 P_{\mathrm{F}}(B_1) \cdots \int_{-\infty}^{\infty} dB_N P_{\mathrm{F}}(B_N) \delta[x - x_{A_1,\cdots,B_N}(t)] , \end{aligned} \tag{53}$$

and integrating by parts for this pth term yields the following. Note that neither σ_x^2 from Equation (25) nor σ_p^2 from Equation (15) depend on A_p or B_p, so $\partial\left(\sigma_p^2\right)/\partial(\sigma_x^2)$ is not involved with this integration by parts, for either A_p or B_p.

Hence:

$$\begin{aligned} &\int_{-\infty}^{\infty} dA_p \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \frac{d^2 P_{\mathrm{F}}(A_p)}{dA_p^2} \delta[x - x_{A_1,\cdots,B_N}(t)] \\ &= \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \left(\begin{array}{c} \int_{-\infty}^{\infty} dA_p \frac{d}{dA_p} \left\{ \frac{dP_{\mathrm{F}}(A_p)}{dA_p} \delta[x - x_{A_1,\cdots,B_N}(t)] \right\} \\ - \int_{-\infty}^{\infty} dA_p \left\{ \frac{dP_{\mathrm{F}}(A_p)}{dA_p} \frac{d}{dA_p} \delta[x - x_{A_1,\cdots,B_N}(t)] \right\} \end{array} \right) \\ &= \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)} \left[\begin{array}{c} \left\{ \frac{dP_{\mathrm{F}}(A_p)}{dA_p} \delta[x - x_{A_1,\cdots,B_N}(t)] \right\} \Big|_{A_p \to -\infty}^{A_p \to \infty} \\ - \int_{-\infty}^{\infty} dA_p \left(\begin{array}{c} \frac{d}{dA_p} \left\{ P_{\mathrm{F}}(A_p) \frac{d}{dA_p} \delta[x - x_{A_1,\cdots,B_N}(t)] \right\} \\ - P_{\mathrm{F}}(A_p) \frac{d^2}{dA_p^2} \delta[x - x_{A_1,\cdots,B_N}(t)] \end{array} \right) \end{array} \right] . \end{aligned} \tag{54}$$

From Equation (14), $P_{\rm F}(A_p)$ and $\frac{d}{dA_p}P_{\rm F}(A_p)$ both go to zero exponentially as $A_p \to \pm\infty$, so the first term in the last line above is zero. Continuing:

$$\begin{aligned}
&\frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\int_{-\infty}^{\infty} dA_p \frac{d^2 P_{\rm F}(A_p)}{dA_p^2}\delta\left[x - x_{A_1,\cdots,B_N}(t)\right] \\
&= \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\left[\begin{array}{l} -P_{\rm F}(A_p)\frac{d}{dA_p}\delta\left[x - x_{A_1,\cdots,B_N}(t)\right]\Big|_{A_p\to-\infty}^{A_p\to\infty} \\ +\int_{-\infty}^{\infty} dA_p P_{\rm F}(A_p)\frac{d^2}{dA_p^2}\delta\left[x - x_{A_1,\cdots,B_N}(t)\right]\end{array}\right] \\
&= \frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\int_{-\infty}^{\infty} dA_p P_{\rm F}(A_p)\frac{d^2}{dA_p^2}\delta\left[x - x_{A_1,\cdots,B_N}(t)\right].
\end{aligned} \tag{55}$$

Thus:

$$\begin{aligned}
&\frac{\partial}{\partial\sigma_x^2}P_{1x}(x) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dA_1\cdots\int_{-\infty}^{\infty} dB_N P_{\rm F}(A_1)\cdots P_{\rm F}(A_N)P_{\rm F}(B_1)\cdots P_{\rm F}(B_N) \\
&\quad\times\sum_{p=1}^{N}\frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\left(\frac{d^2}{dA_p^2}+\frac{d^2}{dB_p^2}\right)\delta\left[x - x_{A_1,\cdots,B_N}(t)\right] \ .
\end{aligned} \tag{56}$$

Since

$$\frac{d^2\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{dA_p^2} = \frac{d}{dA_p}\left\{\left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right)\frac{\partial\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{\partial x_{A_1\ldots B_N}}\right\}, \tag{57}$$

then with $x_{A_1,\cdots,B_N}(t) = x_{\rm ss}(t)$ in Equation (9),

$$\left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right) = \frac{(q/m)}{\sqrt{L_x L_y L_z}}\varepsilon_{p,x}\,{\rm Re}\left(\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right), \tag{58}$$

and

$$\left(\frac{\partial x_{A_1\ldots B_N}}{\partial B_p}\right) = -\frac{(q/m)}{\sqrt{L_x L_y L_z}}\varepsilon_{p,x}\,{\rm Im}\left(\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right). \tag{59}$$

Both $\left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right)$ and $\left(\frac{\partial x_{A_1\ldots B_N}}{\partial B_p}\right)$ are independent of A_p and B_p for all p. This follows from the linearity of $x_{A_1\ldots B_N}$ with the $A's$ and $B's$, which in turn is due to the special case of the SHO obeying the linear ordinary dufferential equation (ODE) in Equation (5).

Hence, from Equation (57):

$$\frac{d^2\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{dA_p^2} = \left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right)^2\frac{\partial^2\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{\partial x_{A_1\ldots B_N}^2}, \tag{60}$$

and, likewise,

$$\frac{d^2\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{dB_p^2} = \left(\frac{\partial x_{A_1\ldots B_N}}{\partial B_p}\right)^2\frac{\partial^2\delta\left[x - x_{A_1\ldots B_N}(t)\right]}{\partial x_{A_1\ldots B_N}^2}. \tag{61}$$

Consequently, from Equations (56), (60) and (61):

$$\frac{\partial}{\partial\sigma_x^2}P_{1x}(x)=\frac{1}{2}\int_{-\infty}^{\infty}dA_1\cdots\int_{-\infty}^{\infty}dB_N P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)$$
$$\times\sum_{p=1}^{N}\frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\left[\left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right)^2+\left(\frac{\partial x_{A_1\ldots B_N}}{\partial B_p}\right)^2\right]\frac{\partial^2\delta\left[x-x_{A_1\ldots B_N}(t)\right]}{\partial x_{A_1\ldots B_N}^2}\,. \quad (62)$$

Using Equations (58) and (59),

$$\left[\left(\frac{\partial x_{A_1\ldots B_N}}{\partial A_p}\right)^2+\left(\frac{\partial x_{A_1\ldots B_N}}{\partial B_p}\right)^2\right]$$
$$=\left[\frac{(q/m)}{\sqrt{L_xL_yL_z}}\varepsilon_{p,x}\,\mathrm{Re}\left(\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right)\right]^2+\left[-\frac{(q/m)}{\sqrt{L_xL_yL_z}}\varepsilon_{p,x}\,\mathrm{Im}\left(\frac{e^{i\omega_p t}}{-\omega_p^2+\omega_0^2+i\Gamma\omega_p\omega_0^2}\right)\right]^2$$
$$=\frac{(q/m)^2(\varepsilon_{p,x})^2}{L_xL_yL_z}\left[\frac{1}{\left(-\omega_p^2+\omega_0^2\right)^2+(\Gamma\omega_p\omega_0^2)^2}\right]\,. \quad (63)$$

Hence:

$$\frac{\partial}{\partial\sigma_x^2}P_{1x}(x)$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}dA_1\cdots\int_{-\infty}^{\infty}dB_N P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)$$
$$\times\left\{\sum_{p=1}^{N}\frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\frac{(q/m)^2(\varepsilon_{p,x})^2}{L_xL_yL_z}\left[\frac{1}{\left(-\omega_p^2+\omega_0^2\right)^2+(\Gamma\omega_p\omega_0^2)^2}\right]\right\}\frac{\partial^2\delta[x-x_{A_1\ldots B_N}(t)]}{\partial x_{A_1\ldots B_N}^2}\,. \quad (64)$$

From Equation (25),

$$\frac{\partial\sigma_x^2}{\partial\sigma_x^2}=1=\frac{(q/m)^2}{(L_xL_yL_z)}\sum_p(\varepsilon_{p,x})^2\frac{1}{\left(-\omega_{\mathbf{n}}^2+\omega_0^2\right)^2+\left(\Gamma\omega_0^2\omega_{\mathbf{n}}\right)^2}\frac{\partial\left(\sigma_p^2\right)}{\partial(\sigma_x^2)}\,. \quad (65)$$

This is where the functional form of σ_x^2 in Equation (25) enters in to enable Equation (47) to be solved by the 1D version of Equation (1).

From Equation (65), the quantity in curly brackets in Equation (64) equals unity, and Equation (64) becomes

$$\frac{\partial}{\partial\sigma_x^2}P_{1x}(x,t)$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}dA_1\cdots\int_{-\infty}^{\infty}dB_N P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)\frac{\partial^2\delta\left[x-x_{A_1\ldots B_N}(t)\right]}{\partial x_{A_1\ldots B_N}^2}$$
$$=\frac{1}{2}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}dA_1\cdots\int_{-\infty}^{\infty}dB_N P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)\delta\left[x-x_{A_1\ldots B_N}(t)\right]$$
$$=\frac{1}{2}\frac{\partial^2}{\partial x^2}P_{1x}(x,t)\,. \quad (66)$$

This completes the proof of Equation (47), carried out without the full integration of $P_F(A_1)\cdots P_F(A_N)P_F(B_1)\cdots P_F(B_N)$, as in Section 2.1. In a sense, certainly integrations were carried out over A_p and B_p via the double integration by parts in Equations (53)–(55).

However, this was a far simpler task than the more complicated operations in Section 2.1 of completing squares for all A_p and B_p variables and then integrating them. Here, the double integration by parts only required that $P_F(A_p) \to 0$ and $\frac{d}{dA_p} P_F(A_p) \to 0$ as $|A_p| \to \infty$ in Equations (54) and (55), and similarly for B_p. This behavior for $P_F(A_p)$ and $\frac{d}{dA_p} P_F(A_p)$ was assured by the Gaussian functional form of $P_F(A_p)$ in Equation (50). Moreover, this Gaussian form was important in the steps of Equations (50) and (51). In Section 2.1, the Gaussian form of $P_F(A_p)$ was important for a different reason, namely, to enable the squares to be carried out when integrating.

4. Concluding Remarks

Two methods were shown in this paper for finding the analytic expression (i.e., Equation (28)) for the probability density of the position of a classical charged point particle in an SHO potential, where the charge is bathed in classical electromagnetic random radiation at a temperature T. Both of these methods used the 1D form of Equation (1) as the starting point. Section 2.1 obtained the analytic expression by explicitly integrating over each of the Fourier coefficients; the calculation was fairly lengthy. In contrast, Section 3 showed that the 1D version of Equation (1) satisfied a PDE, which in turn enabled Equation (28) to be deduced.

Some other relevant points of this study are the following. First, Equation (1) should hold for dynamic systems other than the SHO. Moreover, it should also hold for the SHO in more generality than considered here, namely, not just for the steady state part of the oscillatory motion, but also including the initial transitory motion. Including the probability density of the initial conditions in Equation (1) would enable this to be accomplished. The probability density then changes from $P_{1x}(x)$ to $P_{1x}(x,t)$, as it will now depend on time.

A dynamic system of considerable interest to be considered here is the classical hydrogen, with a $-e$ classical charged point particle as the classical electron, and a much more massive nucleus with charge $+e$ for the proton as the nucleus. In SED, this would again be "bathed" in classical electromagnetic random radiation at temperature T. This system is of interest as it represents a real atomic system, as opposed to the solvable, but hypothetical SHO. Hydrogen is the simplest of atomic systems, so it is a suitable system to be analyzed in detail. Many researchers in SED have tackled this problem, but it still remains an open problem.

Equation (1) should hold for this classical hydrogen system. The quantity $\mathbf{x}_{A_1,\cdots,B_N}(t)$ in $\delta^3[\mathbf{x} - \mathbf{x}_{A_1,\cdots,B_N}(t)]$ would become the trajectory of the classical electron given its initial conditions and how the radiation fields influence the electron's trajectory, just as occurs for the SHO treated here. The equation of motion for the classical electron should be the Lorentz–Dirac equation, where the relativistic version is used, although it would be interesting to obtain and compare the resulting $P_{3x}(\mathbf{x})$ when the nonrelativistic approximation of the Lorentz–Dirac equation is used. It should be noted that the "easiest" and clearest situation to be considered for this problem would be the $T = 0$ case, since we expect that the electron would be bound and would not move off to $|\mathbf{x}| = \infty$ in space. For $T > 0$, there is a nonzero probability that the electron will "ionize" and $|\mathbf{x}(t)| \to \infty$, so this situation is more difficult to analyze with the present scheme.

Despite that Equation (1) should be valid for this classical hydrogen atom, there is a significant difference in using this $P_{3x}(\mathbf{x})$ formulation for the classical hydrogen atom versus the SHO model analyzed here. The "success" of Sections 2.1 and 3 came about because of the following: a nonrelativistic equation of motion was used (Equation (5)), the dipole approximation was made for the radiation's electric field, and the radiation magnetic field effects were ignored. These approximations enabled an analytic result for the motion, $x_{ss}(t) = x_{A_1,\cdots,B_N}(t)$, to be obtained: Equation (9). This expression for $x_{ss}(t)$ is linear in the Fourier coefficients of the radiation's electric field. This linear analytic expression for the particle's motion was used in each of the methods in Sections 2.2 and 3, to arrive at analytic expressions for $P_{1x}(x)$ and σ_x^2: Equations (25), (26) and (28). Without the linear analytic expression of Equation (9), the methods in Sections 2.1 and 3 could not have been carried out in the manners described.

In contrast, an analytic expression is not known for calculating the classical electron's probability distribution for the mentioned classical hydrogen atom, whether one treats the electron's trajectory relativistically, or even with a simpler nonrelativistic approximation. Although the expression for $P_{3x}(\mathbf{x})$ in Equation (1) should be correct for this classical hydrogen atom, other mathematical or numerical methods would need to be developed to carry out similar approaches in Sections 2.1 and 3 of direct integrations or showing the expression satisfies an appropriate PDE, from which $P_{3x}(\mathbf{x})$ can be deduced.

To date, no researcher has found an analytic means within SED for deducing $P_{3x}(\mathbf{x})$ for the classical hydrogen atom. Consequently, the author along with Y. Zou carried out simulation methods in 2003 for deducing $P_{3x}(\mathbf{x})$ for hydrogen, with some degree of success [25]. More recently, extensive simulations have been carried out by Nieuwenhuizen and Liska [26,27] with interesting results, but always with some fraction of the ensemble of hydrogen systems resulting in electrons leaving, or "ionizing," away from the classical nucleus. Despite this problem, all three simulation efforts [25–27] do not result in the classical electron "falling," or spiraling, into the nucleus due to energy radiating off from the classical electron's orbital motion. Thus, these simulations in SED "solve" the old atomic collapse problem of the simple classical atomic model by Rutherford.

Although these simulations are insightful, there are reasons for concern. Simulations in Refs. [25,26] were not relativistic, while Ref. [27] was certainly more relativistic than the others, but still not completely so, and each have various physical approximations. Perhaps of even more concern is that all of these studies deal with a chaotic system, where small errors in electron trajectories cannot of course be avoided numerically, but that build up to large errors quickly. Could these account for the apparent ionizations of some classical electrons in the ensemble of systems investigated? No matter how much the numerical resolutions of the simulations are reduced, this effect cannot go away, as known from chaos theory.

An analytic solution is indeed ideal for overcoming such problems, but may not be possible to obtain, whether nonrelativistically or relativistically. Nevertheless, this goal of exploring Equation (1) for obtaining analytical results was part of the motivation for this study. What is interesting to note is that using Feynman's path integral method in QM [28] was certainly exceptionally successful for a range of systems, but for a long time, starting from about 1948 [29] until Duru's and Kleinert's paper in 1979 [30], the hydrogen atom was not solved via this path integral method. Moreover, it should be mentioned that Equation (1) has some resemblance to a "path integral" formulation, although the Fourier coefficients of the stochastic radiation field are integrated over instead of the possible paths of the particle.

Recapitulating, in this paper, the general expression of Equation (1) was used to obtain an analytic expression of $P_{1x}(x)$ for the 1D electric dipole SHO, one of the first systems analyzed in SED. The calculations were fairly long for each of the two methods discussed, but certainly tractable. Despite Equation (1) being correct for the classical hydrogen atom, evaluating Equation (1) for this system is far more complicated task for reasons discussed.

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Abbreviations

The following abbreviations are used in this paper:

ODE	ordinary differential equation
PDE	partial differential equation
QED	quantum electrodynamics

QM	quantum mechanics
SED	stochastic electrodynamics
SHO	simple harmonic oscillator
1D	one-dimensional
3D	three-dimensional

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