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Phase Transition in the Galam's Majority-Rule Model with Information-Mediated Independence

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Abstract: We study the Galam's majority-rule model in the presence of an independent behavior that can be driven intrinsically or can be mediated by information regarding the collective opinion of the whole population. We first apply the mean-field approach where we obtained an explicit time-dependent solution for the order parameter of the model. We complement our results with Monte Carlo simulations where our findings indicate that independent opinion leads to order–disorder continuous nonequilibrium phase transitions. Finite-size scaling analysis show that the model belongs to the mean-field Ising model universality class. Moreover, results from an approach with the Kramers–Moyal coefficients provide insights about the social volatility.

Keywords: social dynamics; Galam model; collective phenomena in social systems; nonequilibrium phase transitions; order-disorder Monte Carlo simulation

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1. Introduction

Opinion dynamics is one of the most attractive topics in sociophysics. This recent research area uses tools and concepts of statistical physics to describe some aspects of social and political behavior [1–4]. From the theoretical point of view, opinion models are interesting to physicists because they can present order–disorder transitions, hysteresis, scaling, and universality, among other typical features of physical systems, which have attracted much of attention [5–14]. Concerning sociologists, these methods are useful to improve forecasting by means of controlled toy models that can be run multiple times and help fine-tune field studies as well [15]. In addition to the interesting properties of opinion dynamics models, per se, such dynamics have also been applied in various fields such as finance and business [16], and epidemic dynamics with the presence of conflicting opinions [17–23], among others [3].

Among the most studied models, one can highlight the voter model [24,25], the Sznajd model [26], the Deffuant model [27], the kinetic exchange opinion models [28], and the majority rule model [1,29–31]. All the mentioned models are built based on distinct microscopic rules that control the dynamics of interactions among agents. The Sznajd model considers a two-state (up/down spins) outflow dynamics, where a group of agents sharing a common opinion influence the group's neighbors to follow the group's opinion. The model presents a phase transition between the positive and the negative consensus: initial densities of spins up smaller than 1/2 lead eventually to all spins down, and densities greater than 1/2 to all spins up, i.e., consensus absorbing states where the system cannot escape [26]. On the other hand, the Deffuant model considers the opinions as continuous variables, and the interactions depend on the "distance" among pairs of opinions, which defines the concept of bounded confidence. Depending on the value of such bounded confidence, the population can evolve to consensus (all equal opinions) or to polarization (population divided into two distinct opinions). No phase transition is observed [27]. The majority rule model considers groups of g agents that interact through a quite simple rule: all agents in the group follow the local majority. In case of even values of g, a probability k defines which opinion will win the debate inside the group. The results of the model, regarding consensus and phase transitions, are similar to those observed in the Sznajd model [29]. Some applications of the majority rule model are mentioned in the following. Finally, the kinetic exchange opinion models are based on dynamics of wealth exchange. Interactions are pairwise and consider continuous opinions originally, or discrete three-state opinions ($+1$, -1 or 0 states) [28]. Both formulations lead the population to undergo order–disorder phase transitions, similar to what occurs in spin models. Absorbing states, where all agents are in the neutral state (all opinions 0 in the population), are observed. One can find that such absorbing states are distinct from the ones observed in the previous models, where all agents share opinion $+1$ or -1. Such kinds of consensus states are observed in kinetic exchange opinion models only in particular situations [28].

We are especially interested in the majority rule model, proposed by Serge Galam [1]. In this model, random groups of agents are chosen, and after the interaction of such agents, all of them assume the initial majority opinion. The model was studied elsewhere [32–39], and it was applied to a series of practical problems, like antivax movement [40], USA [41] and French [42] presidential elections, and terrorism [43], among many others.

Independence in opinion making and the failure of group influence was considered in several opinion dynamics models [44–52]. A recent extension of Galam's model in Ref. [32] considered the impact of independence in social dynamics. In that case, with probability q, an individual acts independently of the majority opinion of their group and chooses at random one of the two possible opinions. The introduction of that condition, quantified by the parameter, paves the way to the occurrence of an order–disorder nonequilibrium phase transition that does not occur in the original majority-rule model [1].

In the present study, we move farther afield than the independence mechanism considered in Ref. [32], and we take into account the overall global opinion of the population when an agent decides to act independently of the group's opinion. With this, one can find a more detailed picture of the process of independence, since agents can now take global opinion into account when they ignore in-group majority. This change manages to incorporate the concept of "impersonal influence" [53] established within political science, the goal of which is to quantify the influence of the anonymous mass of individuals outside one's small world composed of family, (close) friends, and acquaintances. That impersonal influence encompasses polls, reader's comments on news on digital media, and the individual's general perception by consulting social networks that can have an effect on their decision-making process.

The strength of the information-mediated independence can be controlled by a new parameter, g, that gauges the impact of the global population opinion, which is the macrostate, on the individuals. This impact can be of a contrarian nature, for negative values of g, where agents tend to take the opposite opinion to the population or it can be positive and reinforce the predominant opinion, thus helping the building of consensus.

In this paper, we move along the lines of canonical considerations over complex systems for which microscopic and macroscopic features influence one another. We develop an analytical framework in order to understand the results from numerical simulations. All results suggest the occurrence of order–disorder transitions, and the estimates of the critical exponents indicate that the model is in the mean-field Ising model universality class.

2. Model and Methods

Herein, we analyze a majority-rule model with independence; however, differently to Ref. [32] we assume a density-dependent probability, f_t, where t denotes the time, for changing the current opinion independently of the interacting group.

2.1. Model

Let us consider a population of N individuals, $i = 1, ..., N$, with opinions A or B, with respect to a given issue, that map into a stochastic variable, o_i, such that $o_i(A, t) = +1$ and $o_i(B, t) = -1$. Macroscopically, we compute the density of agents with opinion A,

$$\eta_A(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{o_i(t),+1}, \quad (1)$$

with $\delta_{a,b}$ the Kronecker delta, and the density of agents with opinion B,

$$\eta_B(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{o_i(t),-1} = 1 - \eta_A(t). \quad (2)$$

The mean opinion, from which we establish the macroscopic state of the system, reads:

$$m(t) \equiv \frac{1}{N} \sum_{i=1}^{N} o_i(t) = \eta_A(t) - \eta_B(t). \quad (3)$$

The dynamics of each individual is governed at each time step, t, by the following set of rules:

- an individual with opinion A can change to opinion B through two mechanisms:
  - with probability q, the individuals act independently of their group. In that case, the individuals change their opinion with probability $f^{AB}(t) = f(1 - g\, m(t))$;
  - otherwise, if the individuals do not act on their own, then there is a probability $1 - q$ that they change their opinion according to a local majority-rule, $A + 2B \to 3B$.
- on the other hand, an individual with opinion B can flip to opinion A through two mechanisms:
  - with probability q, the individuals decide to whether act independently of their group or not. In that case, the agent will change their opinion with probability $f^{BA}(t) = f(1 + g\, m(t))$;
  - otherwise, if the the individuals do not act on their own, then there is a probability $1 - q$ that those individuals change their opinion according to a local majority-rule, $B + 2A \to 3A$.

The rules listed above are translated into the transition matrix,

$$W(t) \equiv \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} = \begin{bmatrix} q\, f^{AB}(t) & 1 - q \\ 1 - q & q\, f^{BA}(t) \end{bmatrix}. \quad (4)$$

Note that the definitions $f(t)^{AB} = f(1 - g\, m(t))$ and $f(t)^{BA} = f(1 + g\, m(t))$ imply that

$$\eta_A(t) > \eta_B(t) \Rightarrow m(t) > 0 \Rightarrow f(t)^{BA} > f(t)^{AB},$$

as expected.

Let us have a closer look at the parameters involved in the model: the parameter q is related to the backbone of our approach establishing the relative weight of the local peer-pressure, $p = 1 - q$, leading to a decision-making process wherein the individual either submits to the local majority (a conformist behavior) or the decision-making dynamics is carried out on its own. The probability $f^{XY}(t)$—related to the latter case—is naturally shaped by the assessment of the state of affairs provided by the global state, $m(t)$, so that a standard propensity to change opinion through reflection, f, is either boosted or mitigated. Epistemologically, the shaping of the probability is equivalent to the process of risk taking versus risk aversion, described within prospect theory [54]. Herein, we assume a linearized form $f^{XY}(m(t)) = f + v\, m(t) + \mathcal{O}(m(t)^2)$, so that, depending on the sign of v, one has either a follower or a contrarian impact. If $g = 0$, then $f^{XY}(t) = f$ and one recovers the results of Ref. [32].

2.2. Simulation Details

Our Monte Carlo simulations are structured within an agent-based framework, as individuals constitute the underlying object of study in social theories [55]. In the algorithm here, we consider a computational array of size N to store the opinion of each agent. In each time t, a Monte Carlo step (MCS) that represents a complete iteration through all agents is applied. During each interaction, the simulation chooses a group of 3 agents at random, considering their current opinion and applying specific rules. These rules are summarized in Table 1 and define how an agent's opinion may change based on various conditions and probabilities. After each MCS, we implement a simultaneous-parallel updating. This means that the updated opinions are applied to all agents at the same time, ensuring that the changes in opinions are synchronized across the entire population.

Table 1. Agent-based rules of the model.

Three-Agent Interaction	Transition Probability
Each agent with opinion A can flip to opinion B through two mechanisms:	
1. $A \to B$	$p^{(1)}_{A\to B} = q\, f^{AB}(t)$
2. $A + 2B \to 3B$	$p^{(2)}_{A\to B} = (1-q)\,\eta_B(t)^2$
Each agent with opinion B can flip to opinion A through two mechanisms:	
1. $B \to A$	$p^{(3)}_{B\to A} = q\, f^{BA}(t)$
2. $B + 2A \to 3A$	$p^{(4)}_{B\to A} = (1-q)\,\eta_A(t)^2$

3. Results and Discussion

3.1. Analytical Results

Using the mean-field approach, one can obtain a set of ordinary differential equations that describes the time evolution of the competing opinions in a population. To derive the rate of change of opinions A and B at time t, one needs to consider that each opinion is influenced by the intrinsic independent behavior (controlled by the parameter f), information-driven independence (modulated by the parameter g), and local interactions. Thus, based on the rules summarized in Table 1, one obtains the following mean-field equations:

$$\frac{d\eta_A(t)}{dt} = q\, f^{BA}(t)\,\eta_B(t) + (1-q)\,\eta_A(t)^2\,\eta_B(t) - q\, f^{AB}(t)\,\eta_A(t) - (1-q)\,\eta_A(t)\,\eta_B(t)^2, \tag{5}$$

$$\frac{d\eta_B(t)}{dt} = q\, f^{AB}(t)\,\eta_A(t) + (1-q)\,\eta_B(t)^2\,\eta_A(t) - q\, f^{BA}(t)\,\eta_B(t) - (1-q)\,\eta_B(t)\,\eta_A(t)^2, \tag{6}$$

$$f^{AB}(t) = f(1 - g\, m(t)), \tag{7}$$

$$f^{BA}(t) = f(1 + g\, m(t)). \tag{8}$$

From Equations (1)–(3), namely, that

$$\eta_A(t) = \frac{1}{2}(1+m(t)), \quad \eta_B(t) = \frac{1}{2}(1-m(t)), \quad \eta_A(t)\,\eta_B(t) = \frac{1}{4}(1-m(t))^2, \tag{9}$$

the set of differential equations yields the ordinary differential equation for the global state, $m(t)$:

$$\frac{dm(t)}{dt} = -2\,q\,f\,(1-g)m(t) + (1-q)\,m(t)\,\frac{1-m(t)^2}{2}. \tag{10}$$

In other words, starting from a given initial condition, $m(0) = m_0$, the macroscopic state evolves and eventually reaches a stationary state, $dm/dt = 0$; that state is lowerbounded by the maximal state of disagreement than $m = 0$, whereas when the population presents unanimity, $|m| = 1$. Thus, we expect that for certain conditions dictated by the parameters of the problem, the system can evade the final stationary state of disagreement and end up in a situation for which $|m| \neq 0$, i.e., a majority of individuals favoring opinion $A(B)$. Physically, $m(t)$ is thus defined as an order parameter. That turns out clearer when we consider that the population adjust is macrostate m aiming at minimizing its so-called Hamiltonian function, $\mathcal{H}$.

That is best understood when one recasts the previous equation into

$$\frac{dm(t)}{dt} = -\frac{\partial \mathcal{H}}{\partial m} = r\, m(t) + u\, m(t)^3, \tag{11}$$

where

$$r = \frac{1}{2}\{q[4f(1-g)+1]-1\}, \qquad u = \frac{1}{2}(q-1). \tag{12}$$

Therefore, the analytical form of $\mathcal{H}$,

$$\mathcal{H}(m) = -\frac{1}{2} r\, m^2 - \frac{1}{4} u\, m^4, \tag{13}$$

dictates not only the dynamics of the parameter m, but its stable outcome. First, since $q \leq 1$ and $u < 0$, the stability of the process is assured, as the fourth-order term is positive. In the limit $q \to 1$, the agents act independently from the local group, and totally rely on their assessment of the position of the whole population, and one has: $\lim_{q\to 1} u = 0^-$ and $\lim_{q\to 1} r > 0$, which leads to nontrivial minima of $\mathcal{H}$ at $m_c \neq 0$. By $u < 0$, the emergence of those $m \neq 0$ minima are related to the change of convexity of $\mathcal{H}$ at $m = 0$ from $\frac{d^2\mathcal{H}}{dm^2}|_{m=0} > 0$ to a concave profile $\frac{d^2\mathcal{H}}{dm^2}|_{m=0} < 0$. The fulfillment of the concave condition implies

$$|m| = m_c = \sqrt{-\frac{r}{u}} = \sqrt{1 - \frac{4fq(1-g)}{1-q}}, \qquad |m| \leq 1, \tag{14}$$

the graphical representation of which can be seen in Figure 1. Using relations (12) one finds:

$$m \sim (q_c - q)^{\beta}, \tag{15}$$

where $\beta = 1/2$ and

$$q_c = \frac{1}{1 + 4f(1-g)} \tag{16}$$

defines the critical peer-pressure relative weight, $p_c \equiv 1 - q_c$. Let us note that the instances with $g < 0$ imply a smaller value of q_c and, then a larger p_c in what we assume as a freethinker-prone behavior; on the other hand, when $g > 0$ we regard it as a conformist-prone case.

Equation (15) with $\beta = 1/2$ suggests a phase transition in the same universality class of the mean-field Ising model. We discuss this point in more detail in Section 3.3, where we exhibit the results of Monte Carlo simulations of the model.

Equation (16) corresponds to the limit $t \to \infty$ of the solution to Equation (17):

$$m(t) = \left[\exp(-2rt)\left(\frac{1}{m_0^2} + \frac{u}{r}\right) - \frac{u}{r}\right]^{-1/2} = m_c\left[\exp(-2rt)\left(\frac{m_c^2}{m_0^2} - 1\right) + 1\right]^{-1/2}, \tag{17}$$

where m_0 is the initial condition of the system.

One can further explore the dynamical behavior of the system, especially when the parameters are set at their critical values and one lets the system evolve. In that case, two situations deserve a particular attention: (i) when the initial state corresponds to an unanimity, $m_0 = 1$, the factor given by $\exp(-2rt)\left(m_c^2/m_0^2 - 1\right)$ in Equation (17) can be seen as perturbation, (ii) whereas for the same factor, this factor dominates Equation (17) when the initial condition is that of full disagreement ($m_0 \to 0$). These two possibilities result in two quite different behaviors of $m(t)$ in the short-term.

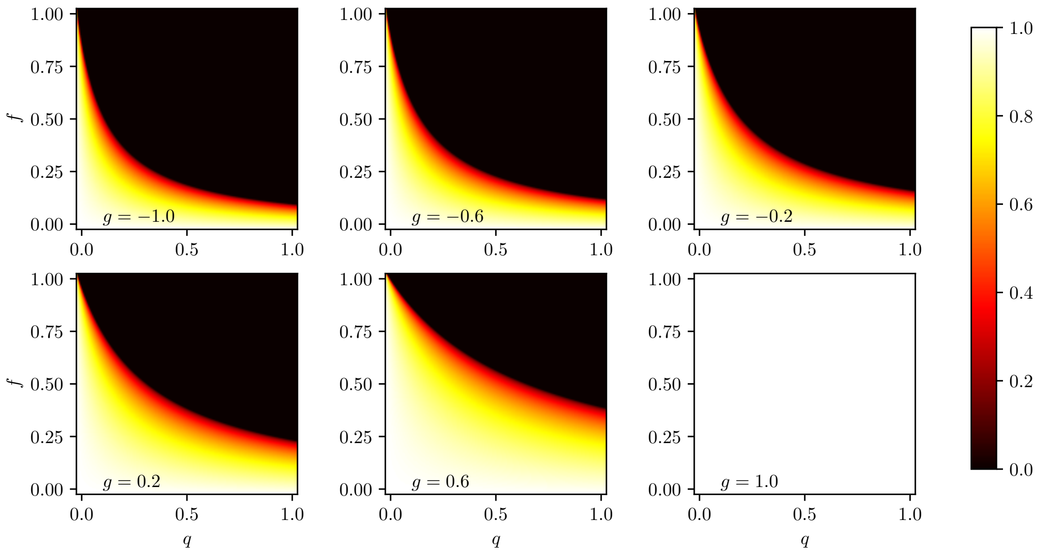

Figure 1. Graphical representations of Equation (14) for the stationary state solution, $|m|$, in the plane f versus q for typical values of g. As negative values of g imply a contrarian effect and positive a follower, the ordered region increases with an increase of g. Note that for $g = 1.0$, $|m| = 1$ for all values of f and q, as predicted by Equation (14).

3.2. Probabilistic Approach

The previous deterministic approach can be further seasoned when fluctuations are taken into account. Recalling that, for a population of N individuals the macroscopic state m, changes by $\mu = \pm 2/N$ every time the individuals switch their opinion with each opinion fraction varying by $1/N$, one finds:

$$\eta_A(t+1) - \eta_A(t) = \frac{1}{N} p^{\dagger}(t) - \frac{1}{N} p(t), \tag{18}$$

as soon as the focus is on the time evolution of the fraction of individuals with opinion A at time $t+1$, with

$$p^{\dagger}(m,t) = w_1\, \eta_A + w_2\, \eta_A\, \eta_B^2, \tag{19}$$

corresponding to the probability that the number of people with opinion A increases by one individual, and

$$p(m,t) = w_3\, \eta_B\, \eta_A^2 + w_4\, \eta_B, \tag{20}$$

giving the probability that the number of people with opinion A diminishes by one individual. The quantities (19) and (20) are identified as the operators of, respectively, creation and destruction in the probability space [56].

Taking into consideration that $p^{\dagger}$ and p correspond to an increment and a reduction of the macroscopic state by $\mu = 2/N$, respectively, one can establish the following master equation for the evolution of m for a time step, $\epsilon = 1/N$:

$$\eta(m, t+\epsilon) = p^{\dagger}(m-\mu, t)\, \eta(m-\mu, t) + p(m+\mu, t)\, \eta(m+\mu, t) + \bar{p}(m,t)\eta(m,t), \tag{21}$$

with $\bar{p} \equiv 1 - p^{\dagger} - p$ quantifying the maintenance of the macroscopic state. Formally, Equation (21) fits within the (normalized) one-step class of stochastic processes and, thus,

$$\eta(m,t) = \exp[\mathbf{L}_{KM}(m,t)]\eta(m_0, 0) \qquad \text{with} \qquad \eta(m,0) = \delta(m - m_0), \tag{22}$$

where, bearing in mind that a normalized quantity is computed and not just $N_A - N_B$, the Kramers–Moyal operator reads:

$$\mathbf{L}_{KM}(m,t) = \sum_{n=1}^{\infty} \frac{(-\mu)^{-n}}{n!} \frac{\partial^n}{\partial m^n} \left[p_{m,t} + (-1)^n \, p^{\dagger}_{m,t} \right]. \tag{23}$$

In considering $\mu \to 0$ so that the variance of m_t is kept fixed and equal to $\sigma^2_m(t)$, we neglect the terms of order $n > 2$, and the formal solution obtains the form of a Fokker–Planck equation:

$$\frac{\partial \eta(m,t)}{\partial t} = -\mu^{-1} \frac{\partial}{\partial m} [D_1(m,t)\, \eta(m,t)] + \frac{\mu^{-2}}{2} \frac{\partial^2}{\partial m^2} [D_2(m,t)\, \eta(m,t)]. \tag{24}$$

Therefrom we identify the Kramers–Moyal coefficients,

$$D_1(m,t) \propto p(m,t) - \, p^{\dagger}(m,t), \tag{25}$$

that defines the shape of the effective potential wherein the macroscopic dynamics of the order parameter evolves in time; on the other hand,

$$D_2(m,t) \propto p(m,t) + \, p^{\dagger}(m,t) \tag{26}$$

characterizes the magnitude of the fluctuations, which in the present social system we associate with the concept of social volatility [57].

Plugging the previous relations (19)/(20) for the probability creation/annihilation operators into Equations (25) and (26), one finally obtains:

$$D_1(m,t) \propto r\, m(t) + u\, m(t)^3, \tag{27}$$

as given by the effective-Hamiltonian, Equation (11) Landau approach. The second-order coefficient,

$$D_2(m,t) \propto 1 + q\,(4f - 1) - [1 + q(4\,f\,g - 1)]\, m(t)^2, \tag{28}$$

indicates a macroscopic feature of this model that is worth noting: the magnitude of the fluctuations—i.e., the social volatility—exhibited by the system depends on its state in such a form that as m increases and approaches $m = 1$ (unanimity), the volatility decreases. On the contrary, when the group shows strong disagreement, $m \approx 0$, the fluctuations approach the sate of maximal volatility. Such a behavior contrasts with what is measured in quantitative finance where the realized volatility is directly proportional to price variations [58]. If one considers that, within a physical context, D_2 is related to the (local) temperature of a physical system, one can assert that our model is able to capture the cooling down and the heating up of a social system as the system approaches or departs from consensus.

Alternatively, the fluctuations given by $D_2(m,t)$ can be understood from another perspective: given that $p^{\dagger}$ and p, respectively, correspond to an increment and a reduction of the macroscopic state by $\mu = 2/N$, one can then interpret D_1 as the imbalance between the likelihood of increment and reduction of $p(m)$, whereas D_2 to be related to the average over increment and reduction, which is nonvanishing. Thisis related to the microscopic change of opinion that each individual can make and which corresponds to a source of the macroscopic fluctuations that end up being expressed by the social volatility. Complementarily, those fluctuations yield an entropy production that can be associated with the total information output due to the microscopic interaction between agents. Therefore, around a consensus, one measures a less volatile state as that state is less entropic and vice versa.

3.3. Monte Carlo Simulation and Finite-Size Scaling

Figure 2a shows a comparison between the analytical solution, as elucidated in Section 3.1, and the Monte Carlo simulation for the Galam model with information-mediated independence. We plot the stationary values of the macrostate obtained from simulations for a population size $N = 10^4$ and from Equation (14) for typical values of g and fixed $f = 0.5$. One can see a good agreement between the results obtained by both methods of calculations. One as well observes the order–disorder phase transitions, which mark a collective change in the population behavior. These transitions indicate a macroscopic change from the so-called ordered state, characterized by the presence of a well-defined majority ($|m| > 0$), to the disordered-state, characterized by the absence of a clear majority ($|m| \sim 0$). When $q = 0$, the result of the original Galam model, i.e., a consensus in the population (all agents sharing opinion A or B) is recovered.

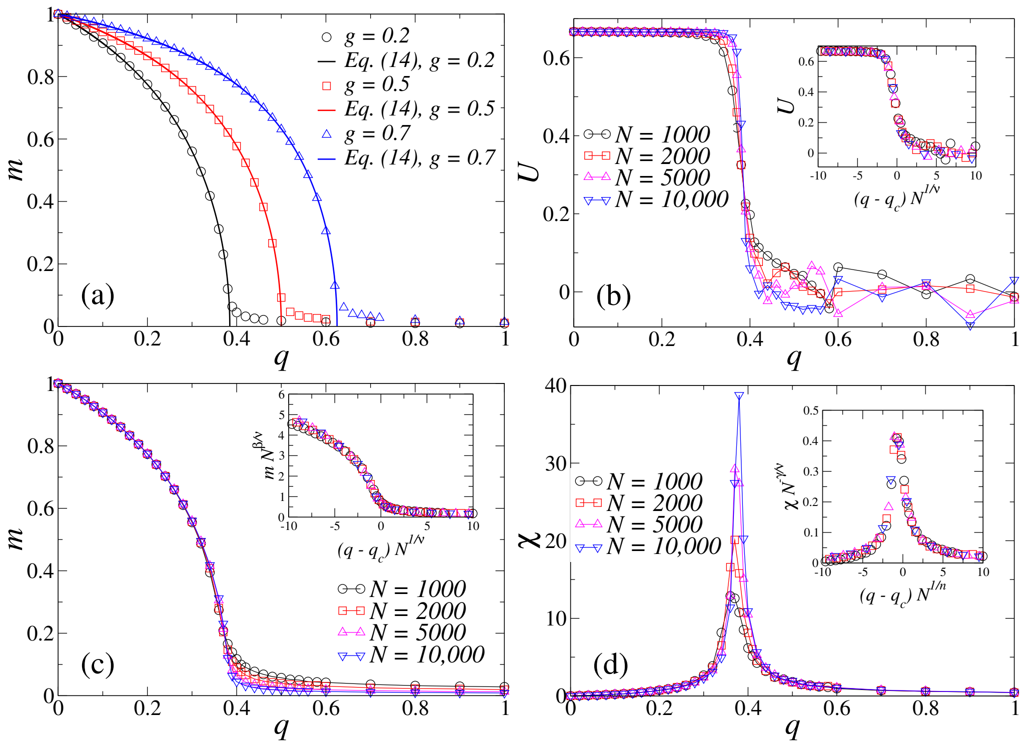

Figure 2. The model Monte Carlo results for $f = 0.5$ averaged over 100 simulations. (**a**) Stationary order parameter, m (collective opinion), as a function of q. The symbols represent the simulations for $N = 10^4$ and typical values of g, and the lines show the analytical results obtained from Equation (14). The results at fixed $g = 0.2$ and distinct population sizes N for, and the corresponding finite-size scaling analysis for (**b**) the Binder cumulant, U (30), (**c**) the order parameter, m, and (**d**) the susceptibility, χ (29) are also shown. The following critical probability, $q_c \approx 0.385$ and the critical exponents, $\beta \approx 0.50$, $\gamma \approx 1.00$, and $\nu \approx 2.00$, are obtained; see text for details.

In order to verify the universality class of the model, we performed numerical simulations for distinct population sizes and applied a so-called scaling analysis. In addition to the order parameter, m we also computed the fluctuations χ of the order parameter (or "susceptibility"), defined as

$$\chi \equiv N\left(\langle m^2 \rangle - \langle m \rangle^2\right) \tag{29}$$

and the Binder cumulant U, defined as [59]

$$U \equiv 1 - \frac{\langle m^4 \rangle}{3\,\langle m^2 \rangle^2}. \tag{30}$$

The brackets here denote the averaging over distinct realizations of the dynamics.

As an example, we exhibit in Figure 2 the finite-size scaling (FSS) analysis of the order parameter, the susceptibility, and the Binder cumulant for four lattice sizes, for $f = 0.5$ and $g = 0.2$. We obtain the critical value, q_c, by the crossing of the Binder cumulant curves, as seen in Figure 2b, to be $q_c \approx 0.385$, in quite a good agreement with the analytical result of Equation (16), $q_c \approx 0.3846$. The critical exponents, β, γ, and ν, are found by the best collapse of data. The FSS analysis was based on the standard relations,

$$m(q,N) \sim N^{-\beta/\nu}, \tag{31}$$
$$\chi(q,N) \sim N^{\gamma/\nu}, \tag{32}$$
$$U(q,N) \sim \text{constant}, \tag{33}$$
$$q_c(N) - q_c \sim N^{-1/\nu} \tag{34}$$

Considering Equations (31)–(34), the values of the critical exponents, $\beta \approx 0.5$, $\gamma \approx 1.0$, and $\nu \approx 2.0$, are obtained. The data collapses are exhibited in Figure 2b–d. We have also verified that for other values of f and g, the same exponents are obtained. The results suggest that the model belongs to the mean-field Ising model universality class, and it is also in the same universality class of the Sznajd model and kinetic exchange opinion models in the presence of independence [44,47,50,60].

The above results, namely, Equation (34), bridge with the dynamical analysis as Equation (17) sets up a relaxation timescale, τ, of the macroscopic parameter that is inversely proportional to r. Comparing the exponential factor in Equation (17) with the typical term related to the relaxation, $e^{t/\tau}$, one obtains the relaxation time, $\tau = -1/(2r)$. Then, plugging Equation (16) into the definition of r, one finds: $r = (q - q_c)/(2q_c)$, and, finally,

$$\tau \sim (q_c - q)^{-1}. \tag{35}$$

Explicitly, at criticality, one considers a relaxation timescale of the order parameter, $m(t)$, that diverges with the same scale-invariant functional form as the correlation length. Since the propagator given by the Fokker–Planck Equation (24) rules all relaxation quantities of $m(t)$, the same slowing down near the transition is found for the self-correlation function of $m(t)$, $\langle m(t')\, m(t)\rangle \sim \exp[-|t' - t|/\tau]$.

4. Conclusions

In this paper, we studied an extension of the Galam's majority-rule model. For this purpose, we introduced the mechanism of independence, considering that individuals can act independently of their interaction groups with a given probability q that is complementary to the peer-pressure weight, $p = 1 - q$. In addition, the individuals inspect the global population opinion and such opinion affects their independent probability. When an individual does not act independently of the group, the individual follows the local majority opinion, as in the original Galam model.

We observed that the independence mechanism leads the population to undergo a critical change of behavior at $q = q_c$ in which a minimal consensus $m \neq 0$—where m is the order parameter of the model—optimizes the overall state of the population better than the case of complete disagreement. Within that phase transition context, we derived an expression for order parameter, $m(t)$. From its stationary solution, we obtained the critical behavior, $m \sim (q_c - q)^{\beta}$ with $\beta = 1/2$. We found that as one approached the critical transition, the relaxation of the overall state is ever slower with its typical timescale, $\tau \sim (q_c - q)^{-1}$. The other canonical critical exponents, γ and ν, were obtained through Monte Carlo simulations. From the set of critical exponents, we verified that the model is in the ubiquitous universality class of the mean-field Ising model. This result is expected since, as opinions are mapped into random variables, $o_i = \pm 1$, the phase transition corresponds to a group $\mathbb{Z}_2$ symmetry breaking of which the Ising model is the quintessential case.

Let us mention that, while our model is not defined by a physical Hamiltonian, the identification with the Ising universality class arises from a series of results coming from three methods: mean-field approach, Monte Carlo simulations, and finite-size scaling analysis.

From the microscopic dynamics, we derived the probabilistic evolution of m. The results obtained allowed us to confirm the critical behavior of m from the first Kramers–

Moyal coefficient, and from the second, the nature of the fluctuations that can be coined as social volatility. In respect of the latter, we learned that the magnitude of the volatility depends on the state of the population in a inverse proportion relation way, such that, in this case, herding in opinion tends to induce less agitation in the population. Further insights into this subject matter to be discussed in our future studies.

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References

1. Galam, S. Sociophysics: A review of Galam models. *Int. J. Mod. Phys. C* **2008**, *19*, 409–440. [CrossRef]
2. Sen, P.; Chakrabarti, B.K. *Sociophysics: An Introduction*; Oxford University Press: Oxford, UK, 2014.
3. Castellano, C.; Fortunato, S.; Loreto, V. Statistical physics of social dynamics. *Rev. Mod. Phys.* **2009**, *81*, 591–646. [CrossRef]
4. da Luz, M.G.; Anteneodo, C.; Crokidakis, N.; Perc, M. Sociophysics: Social collective behavior from the physics point of view. *Chaos Solit. Fract.* **2023**, *170*, 113379. [CrossRef]
5. Dinkelberg, A.; O'Sullivan, D.J.; Quayle, M.; Maccarron, P. Detecting opinion-based groups and polarization in survey-based attitude networks and estimating question relevance. *Adv. Complex Syst.* **2021**, *24*, 2150006. [CrossRef]
6. Sobkowicz, P. Whither now, opinion modelers? *Front. Phys.* **2020**, *8*, 587009. [CrossRef]
7. Martins, A.C. Discrete opinion dynamics with *M* choices. *Eur. Phys. J. B* **2020**, *93*, 1. [CrossRef]
8. Baumann, F.; Lorenz-Spreen, P.; Sokolov, I.M.; Starnini, M. Modeling echo chambers and polarization dynamics in social networks. *Phys. Rev. Lett.* **2020**, *124*, 048301. [CrossRef] [PubMed]
9. Anagnostopoulos, A.; Becchetti, L.; Cruciani, E.; Pasquale, F.; Rizzo, S. Biased opinion dynamics: When the devil is in the details. *Inf. Sci.* **2022**, *593*, 49–63. [CrossRef]
10. Zubillaga, B.J.; Vilela, A.L.M.; Wang, M.; Du, R.; Dong, G.; Stanley, H.E. Three-state majority-vote model on small-world networks. *Sci. Rep.* **2022**, *12*, 282. [CrossRef] [PubMed]
11. Oestereich, A.L.; Pires, M.A.; Crokidakis, N. Three-state opinion dynamics in modular networks. *Phys. Rev. E* **2019**, *100*, 032312. [CrossRef]
12. Oestereich, A.L.; Pires, M.A.; Queirós, S.D.; Crokidakis, N. Hysteresis and disorder-induced order in continuous kinetic-like opinion dynamics in complex networks. *Chaos Solit. Fract.* **2020**, *137*, 109893. [CrossRef]
13. Vazquez, F.; Saintier, N.; Pinasco, J.P. Role of voting intention in public opinion polarization. *Phys. Rev. E* **2020**, *101*, 012101. [CrossRef] [PubMed]
14. Oestereich, A.L.; Crokidakis, N.; Cajueiro, D.O. Impact of memory and bias in kinetic exchange opinion models on random networks. *Phys. A Stat. Mech. Its Appl.* **2022**, *607*, 128199. [CrossRef]
15. Imai, K. *Quantitative Social Science. An Introduction*; Princetion University Press: Princeton, NJ, USA, 2018.
16. Zha, Q.; Kou, G.; Zhang, H.; Liang, H.; Chen, X.; Li, C.C.; Dong, Y. Opinion dynamics in finance and business: A literature review and research opportunities. *Financ. Innov.* **2020**, *6*, 44. [CrossRef]
17. Fang, F.; Ma, J.; Li, Y. The coevolution of the spread of a disease and competing opinions in multiplex networks. *Chaos Solit. Fract.* **2023**, *170*, 113376. [CrossRef] [PubMed]
18. Pires, M.A.; Oestereich, A.L.; Crokidakis, N. Sudden transitions in coupled opinion and epidemic dynamics with vaccination. *J. Stat. Mech. Theory Exp.* **2018**, *2018*, 053407. [CrossRef]
19. Pires, M.A.; Oestereich, A.L.; Crokidakis, N.; Queirós, S.M.D. Antivax movement and epidemic spreading in the era of social networks: Nonmonotonic effects, bistability, and network segregation. *Phys. Rev. E* **2021**, *104*, 034302. [CrossRef]
20. Oestereich, A.; Pires, M.; Crokidakis, N.; Cajueiro, D. Optimal rewiring in coupled opinion and epidemic dynamics with vaccination. *arXiv* **2023**, arXiv:2305.02488. [CrossRef]

21. Franceschi, J.; Pareschi, L.; Bellodi, E.; Gavanelli, M.; Bresadola, M. Modeling opinion polarization on social media: Application to Covid-19 vaccination hesitancy in Italy. *arXiv* **2023**, arXiv:2302.01028. [CrossRef]
22. Harari, G.S.; Monteiro, L.H.A. An epidemic model with pro and anti-vaccine groups. *Acta Biotheor.* **2022**, *70*, 20. [CrossRef]
23. Wang, J.; Xu, W.; Chen, W.; Yu, F.; He, J. Information sharing can suppress the spread of epidemics: Voluntary vaccination game on two-layer networks. *Phys. A Stat. Mech. Its Appl.* **2021**, *583*, 126281. [CrossRef]
24. Clifford, P.; Sudbury, A. A model for spatial conflict. *Biometrika* **1973**, *60*, 581–588. [CrossRef]
25. Holley, R.A.; Liggett, T.M. Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probab.* **1975**, *3*, 643–663. Available online: https://www.jstor.org/stable/2959329 (accessed on 20 July 2023). [CrossRef]
26. Sznajd-Weron, K.; Sznajd, J.; Weron, T. A review on the Sznajd model—20 years after. *Phys. A Stat. Mech. Its Appl.* **2021**, *565*, 125537. [CrossRef]
27. Deffuant, G.; Neau, D.; Amblard, F.; Weisbuch, G. Mixing beliefs among interacting agents. *Adv. Complex Syst.* **2000**, *3*, 87–98. [CrossRef]
28. Biswas, S.; Chatterjee, A.; Sen, P.; Mukherjee, S.; Chakrabarti, B.K. Social dynamics through kinetic exchange: The BChS model. *Front. Phys.* **2023**, *11*, 1196745. [CrossRef]
29. Galam, S. Application of statistical physics to politics. *Phys. A Stat. Mech. Its Appl.* **1999**, *274*, 132–139. [CrossRef]
30. Galam, S. Minority opinion spreading in random geometry. *Eur. Phys. J. B-Condens. Matter Complex Syst.* **2002**, *25*, 403–406. [CrossRef]
31. Krapivsky, P.L.; Redner, S. Dynamics of Majority Rule in Two-State Interacting Spin Systems. *Phys. Rev. Lett.* **2003**, *90*, 238701. [CrossRef]
32. Crokidakis, N.; de Oliveira, P.M.C. Inflexibility and independence: Phase transitions in the majority-rule model. *Phys. Rev. E* **2015**, *92*, 062122. [CrossRef]
33. Crokidakis, N.; de Oliveira, P.M.C. The first shall be last: Selection-driven minority becomes majority. *Phys. A Stat. Mech. Its Appl.* **2014**, *409*, 48–52. [CrossRef]
34. Abilhoa, W.D.; de Oliveira, P.P.B. Density classification based on agents under majority rule: Connectivity influence on performance. In *Distributed Computing and Artificial Intelligence, Proceedings of the 16th International Conference, Ávila, Spain, 26–28 June 2019;* Herrera, F., Matsui , K., Rodríguez-González, S., Eds.; Springer: Cham, Switzerland, 2020; pp. 163–170. [CrossRef]
35. Galam, S. *Sociophysics. A Physicist's Modeling of Psycho-Political Phenomena*; Springer Science+Business Media, LLC: New York, NY, USA, 2012; Ch. 7. [CrossRef]
36. Gimenez, M.C.; Reinaudi, L.; Paz-García, A.P.; Centres, P.M.; Ramirez-Pastor, A.J. Opinion evolution in the presence of constant propaganda: Homogeneous and localized cases. *Eur. Phys. J. B* **2021**, *94*, 35. [CrossRef]
37. Krapivsky, P.; Redner, S. Divergence and consensus in majority rule. *Phys. Rev. E* **2021**, *103*, L060301. [CrossRef]
38. Muslim, R.; Anugraha, R.; Sholihun, S.; Rosyid, M.F. Phase transition and universality of the three-one spin interaction based on the majority-rule model. *Int. J. Mod. Phys. C* **2021**, *32*, 2150115. [CrossRef]
39. Azhari; Muslim, R. The external field effect on the opinion formation based on the majority rule and the q-voter models on the complete graph. *Int. J. Mod. Phys. C* **2023**, *34*, 2350088. [CrossRef]
40. Pires, M.A.; Crokidakis, N. Dynamics of epidemic spreading with vaccination: Impact of social pressure and engagement. *Phys. A Stat. Mech. Its Appl.* **2017**, *467*, 167–179. [CrossRef]
41. Galam, S. The Trump phenomenon: An explanation from sociophysics. *Int. J. Mod. Phys. B* **2017**, *31*, 1742015. [CrossRef]
42. Galam, S. Unavowed abstention can overturn poll predictions. *Front. Phys.* **2018**, *6*, 24. [CrossRef]
43. Galam, S. Identifying a would-be terrorist: An ineradicable error in the data processing? *Chaos Solit. Fract.* **2023**, *168*, 113119. [CrossRef]
44. Sznajd-Weron, K.; Tabiszewski, M.; Timpanaro, A.M. Phase transition in the Sznajd model with independence. *Europhys. Lett.* **2011**, *96*, 48002. [CrossRef]
45. Nyczka, P.; Sznajd-Weron, K. Anticonformity or independence?—Insights from statistical physics. *J. Stat. Phys.* **2013**, *151*, 174–202. [CrossRef]
46. Nyczka, P.; Sznajd-Weron, K.; Cisło, J. Phase transitions in the q-voter model with two types of stochastic driving. *Phys. Rev. E* **2012**, *86*, 011105. [CrossRef] [PubMed]
47. Muslim, R.; Kholili, M.J.; Nugraha, A.R. Opinion dynamics involving contrarian and independence behaviors based on the Sznajd model with two-two and three-one agent interactions. *Phys. D Nonlinear Phenom.* **2022**, *439*, 133379. [CrossRef]
48. Vieira, A.R.; Crokidakis, N. Phase transitions in the majority-vote model with two types of noises. *Phys. A Stat. Mech. Its Appl.* **2016**, *450*, 30–36. [CrossRef]
49. Muslim, R.; Anugraha, R.; Sholihun, S.; Rosyid, M.F. Phase transition of the Sznajd model with anticonformity for two different agent configurations. *Int. J. Mod. Phys. C* **2020**, *31*, 2050052. [CrossRef]
50. Crokidakis, N. Phase transition in kinetic exchange opinion models with independence. *Phys. Lett. A* **2014**, *378*, 1683–1686. [CrossRef]
51. Abramiuk, A.; Sznajd-Weron, K. Generalized independence in the q-voter model: How do parameters influence the phase transition? *Entropy* **2020**, *22*, 120. [CrossRef]
52. Vieira, A.R.; Anteneodo, C.; Crokidakis, N. Consequences of nonconformist behaviors in a continuous opinion model. *J. Stat. Mech. Theory Exp.* **2016**, *2016*, 023204. [CrossRef]

53. Mutz, D.C. Impersonal influence: Effects of representations of public opinion on political attitudes. *Political Behav.* **1992**, *14*, 89–122. [CrossRef]
54. Kahneman, D.; Tversky, A. Prospect theory: An analysis of decision under risk. *Econometrica* **1979**, *47*, 263–291. [CrossRef]
55. Conte, R.; Gilbert, N.; Bonelli, G.; Cioffi-Revilla, C.; Deffuant, G.; Kertesz, J.; Loreto, V.; Moat, S.; Nadal, J.P.; Sanchez, A.; et al. Manifesto of computational social science. *Eur. Phys. J. Spec. Top.* **2012**, *214*, 325–346. [CrossRef]
56. van Kampen, N.G. *Stochastic Processes in Physics and Chemistry*; North Holland: Amsterdam, The Netherlands, 2007.
57. Behrens, T.E.J.; Hunt, L.T.; Woolrich, M.W.; Rushworth, M.F.S. Associative learning of social value. *Nature* **2008**, *456*, 245–249. [CrossRef]
58. Queirós, S.M.D. On non-Gaussianity and dependence in financial time series: A nonextensive approach. *Quant. Financ.* **2005**, *5*, 475–487. [CrossRef]
59. Binder, K. Finite size scaling analysis of Ising model block distribution functions. *Z. Phys. B.* **1981**, *43*, 119–140. [CrossRef]
60. Calvelli, M.; Crokidakis, N.; Penna, T.J. Phase transitions and universality in the Sznajd model with anticonformity. *Phys. A Stat. Mech. Its Appl.* **2019**, *513*, 518–523. [CrossRef]