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Dirac Theory in Noncommutative Phase Spaces

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Abstract: Based on the position and momentum of noncommutative relations with a noncanonical map, we study the Dirac equation and analyze its parity and time reversal symmetries in a noncommutative phase space. Noncommutative parameters can be endowed with the Planck length and cosmological constant such that the noncommutative effect can be interpreted as an effective gauge potential or a $(0, 2)$-type curvature associated with the Plank constant and cosmological constant. This provides a natural coupling between dynamics and spacetime geometry. We find that a free Dirac particle carries an intrinsic velocity and force induced by the noncommutative algebra. These properties provide a novel insight into the Zitterbewegung oscillation and the physical scenario of dark energy. Using perturbation theory, we derive the perturbed and nonrelativistic solutions of the Dirac equation. The asymmetric vacuum gaps of particles and antiparticles reveal the particle–antiparticle symmetry breaking in the noncommutative phase space, which provides a clue to understanding the physical mechanisms of particle–antiparticle asymmetry and quantum decoherence through quantum spacetime fluctuation.

Keywords: Dirac equation; noncommutative phase space; particle–antiparticle asymmetry; quantum decoherence

1. Introduction

When the physical horizons are extended from the macroscopic scale to microscopic and cosmological scales, many novel phenomena emerged at different scales, which induce novel concepts and mathematical structures. Quantum entanglement and quantum fluctuation are rooted in wave–particle duality and noncommutative algebra [1]. Spacetime geometry can be interpreted as gravity in cosmology [2]. However, these discoveries perplex us as to why spacetime plays different roles at different scales, which inspire many attempts to construct a unified framework for all physical scales [3,4].

The cosmological observations of dark matter and dark energy provide us with clues to understanding the incompatibility of spacetime on microscopic and cosmological scales [5–8]. The discoveries of dark matter and dark energy stimulate more radical ideas, such as spacetime quantization, noncommutative geometry, quantum cosmology, and quantum gravity [9–13]. Several schemes have been proposed to quantize spacetime with noncommutative algebra [9–12], such as the Moyal product techniques [9,10,14–18] and a noncanonical map [14,17,19–27]. In particular, the two-dimensional (2D) quantum Hall system in canonical quantum mechanics can be mapped to 2D free electronic systems in a noncommutative phase space [22,23,28–32]. Many interesting phenomena have been found in noncommutative phase spaces, such as magnetic monopoles, Berry phase [33,34], the Aharonov–Bohm effect [35–39], and generalized and extended uncertainty relations [40–44].

Interestingly, in a 3D noncommutative phase space, we found an intrinsic quantum force induced by the noncommutative algebra that breaks the translation and rotation symmetries [45]. To understand this intrinsic force or quantum fluctuation, we proposed a parameterization scheme of noncommutative parameters associated with the Planck constant, Planck length, and cosmological constant [45]. Thus, the noncommutative effects

can be interpreted as an effective magnetic vector potential (or effective "gauge" potential) associated with the Planck constant and cosmological constant [45].

Recently, we extended noncommutative relations to those in the 4D case and we generalized the Schrödinger equation to the Klein–Gordon equation in a noncommutative phase space [46]. Moreover, spin and intrinsic dipole moments emerge in the noncommutative phase space, which are related to the double quantization [47–52], the deformed special relativity [53,54], the Snyder model [55], and quantum dissipation [56,57]. However, more importantly, there are still some puzzles in noncommutative quantum mechanics. What physical mechanisms are hidden in the noncommutative phase space? What are the fundamental properties of the Dirac particles in the noncommutative phase space?

In this paper, we follow our previous formulation in a 4D noncommutative phase space with the Klein–Gordon equation [45,46] to study the Dirac equation in the noncommutative phase space. In Section 2, we briefly review the Heisenberg representation of the 4D noncommutative relations based on a noncanonical map.

In Section 3, we derive the Dirac equation and continuity equation in the noncommutative phase space including the canonical (Hamiltonian) and Lorentz covariant forms of the Dirac equation. We find that the noncommutative effects can be interpreted as an analog with the effective gauge potential. We discuss the spin and helicity of Dirac particles in the noncommutative phase space. We find that the helicity depends on the effective gauge potential. Then, we give the probability current densities and the continuity equation in the noncommutative phase space. Moreover, we discuss the parity and time reversal symmetries of the Dirac equation in the noncommutative phase space.

In Section 4, using the perturbation theory, we obtain the perturbation solution and its corresponding probability and current densities. In the nonrelativistic approximation, we give the Pauli equation in the noncommutative phase space.

In Section 5, the parameterization scheme related to the Planck length and cosmological constant allows us to give the effective gauge field in terms of the Plank constant and cosmological constant. We find an intrinsic velocity and force induced by the noncommutative effect, which can be understood as a Zitterbewegung oscillation. Moreover, we derive the equation of motion of a free Dirac particle and give an effective Lorentz-type force induced by the noncommutative effect. The perturbation solution gives us a hint to understanding the particle–antiparticle asymmetry and quantum decoherence with an intrinsic curved spacetime.

Finally, in Section 6, we present the conclusions and future outlook. In the Appendices, we provide the basic commutative relations, the perturbed matrix elements, and derivation notes.

2. Noncommutative Relations and Their Heisenberg Representation

2.1. From Canonical Commutative Relations to Noncommutative Relations

Let the 4-vector operators of positions, x^μ, and momenta, $\widehat{p}_\mu$, live in the Heisenberg canonical phase space:

$$x^\mu \quad := \quad (ct, \mathbf{x}) \equiv (ct, x, y, z), \tag{1}$$

$$\widehat{p}_\mu \quad := \quad (\widehat{p}_0, \widehat{\mathbf{p}}) \equiv \left(\widehat{E}/c, \widehat{p}_x, \widehat{p}_y, \widehat{p}_z \right), \tag{2}$$

where t is time, the Greek indices μ take the values, $0, 1, 2$ and 3 labeling t (time), x, y, and z (space coordinates), and c denotes the speed of light. The 4-vector operators of position and momentum obey the following canonical commutative relations:

$$[x^\mu, x^\nu] = 0, \tag{3}$$

$$[\widehat{p}_\mu, \widehat{p}_\nu] = 0, \tag{4}$$

$$[x^\mu, \widehat{p}_\nu] = i\hbar \delta^\mu{}_\nu, \tag{5}$$

where $\hbar$ is the reduced Planck constant and $\delta^\mu{}_\nu$ is the Kronecker delta.

Suppose that the 4-vector operators of positions X^μ and momenta $\widehat{P}_\mu$ live in a noncommutative phase space:

$$\widehat{X}^\mu \quad := \quad \left(ct, \widehat{\mathbf{X}}\right) \equiv \left(ct, \widehat{X}, \widehat{Y}, \widehat{Z}\right), \tag{6}$$

$$\widehat{P}_\mu \quad := \quad \left(\widehat{E}/c, \widehat{\mathbf{P}}\right) \equiv \left(\widehat{E}/c, \widehat{P}_x, \widehat{P}_y, \widehat{P}_z\right). \tag{7}$$

They obey the following noncommutative relations (see Appendix B)

$$\left[\widehat{X}^\mu, \widehat{X}^\nu\right] \quad := \quad i\Theta^{\mu\nu}, \tag{8}$$

$$\left[\widehat{P}_\mu, \widehat{P}_\nu\right] \quad := \quad i\Phi_{\mu\nu}, \tag{9}$$

$$\left[\widehat{X}^\mu, \widehat{P}_\nu\right] \quad := \quad i\Omega^\mu{}_\nu. \tag{10}$$

The constant matrices on the right-hand side of Equations (8)–(10) describe the noncommutative features. In Section 5.1, we use a set of physical parameters to characterize the noncommutative strength.

2.2. Heisenberg Representation of Noncommutative Relations

The position–momentum noncommutative relations (8)–(10) provide the operator algebra beyond the canonical (or Heisenberg) algebra in canonical quantum mechanics. In general, there are two ways to implement these algebras, either by directly playing the noncommutative algebra or constructing a map to transform the noncommutative algebra to the Heisenberg algebra. The first approach is generally not straightforward to explicitly reveal the noncommutative effects beyond the canonical algebra. The another approach is based on the idea of the Seiberg–Witten map, to construct a noncanonical map to make the noncommutative algebra reduce the Heisenberg algebra [19,20,25–27]. Here, based on this idea, we propose a noncanonical map to transform the noncommutative algebra into Heisenberg algebra [33,34,36].

Let us define the position and momentum operators, $\widehat{X}^\mu, \widehat{P}_\mu \in \widehat{O}_{\text{nc}}$ in the noncommutative phase space and define $\widehat{x}^\mu, \widehat{p}_\mu \in \widehat{O}_{\text{H}}$, in the Heisenberg phase space. We construct a map from the noncommutative phase to the Heisenberg phase space [46] as follows:

$$\widehat{X}^\mu \quad = \quad \widehat{x}^\mu + \lambda^{\mu\nu}\widehat{p}_\nu \tag{11}$$

$$\widehat{P}_\mu \quad = \quad \widehat{p}_\mu + \pi_{\mu\nu}\widehat{x}^\nu, \tag{12}$$

where

$$[\lambda^{\mu\nu}] = \begin{pmatrix} 0 & -\theta/\hbar & -\theta/\hbar & -\theta/\hbar \\ 0 & 0 & -\theta/(2\hbar) & -\theta/(2\hbar) \\ 0 & \theta/(2\hbar) & 0 & -\theta/(2\hbar) \\ 0 & \theta/(2\hbar) & \theta/(2\hbar) & 0 \end{pmatrix} \tag{13}$$

and

$$[\pi_{\mu\nu}] = \begin{pmatrix} 0 & \eta/\hbar & \eta/\hbar & \eta/\hbar \\ 0 & 0 & \eta/(2\hbar) & \eta/(2\hbar) \\ 0 & -\eta/(2\hbar) & 0 & \eta/(2\hbar) \\ 0 & -\eta/(2\hbar) & -\eta/(2\hbar) & 0 \end{pmatrix}, \tag{14}$$

where θ and η are noncommutative parameters. Note that the position operators $\widehat{x}^\mu$ have the same form as x^μ in the Heisenberg representation. In what follows, the hat on top of x^μ is omitted without loss of generality. The noncommutative parameters θ and η are endowed with physical meanings in Section 5.1. Using the noncanonical map (11) and (12), we obtain

the matrices of the noncommutative relations of the operators in Equations (8)–(10) [46] (see Appendix B):

$$[\Theta^{\mu\nu}] = \begin{pmatrix} 0 & \theta & \theta & \theta \\ -\theta & 0 & \theta & \theta \\ -\theta & -\theta & 0 & \theta \\ -\theta & -\theta & -\theta & 0 \end{pmatrix}, \tag{15}$$

$$[\Phi_{\mu\nu}] = \begin{pmatrix} 0 & \eta & \eta & \eta \\ -\eta & 0 & \eta & \eta \\ -\eta & -\eta & 0 & \eta \\ -\eta & -\eta & -\eta & 0 \end{pmatrix}, \tag{16}$$

and

$$[\Omega^{\mu}{}_{\nu}] = \begin{pmatrix} \hbar + 3\theta\eta/\hbar & \theta\eta/\hbar & 0 & -\theta\eta/\hbar \\ \eta\theta/\hbar & \hbar + \theta\eta/(2\hbar) & \theta\eta/(4\hbar) & -\theta\eta/(4\hbar) \\ 0 & \theta\eta(4\hbar) & \hbar + \theta\eta/(2\hbar) & \theta\eta/(4\hbar) \\ -\eta\theta/\hbar & -\theta\eta/(4\hbar) & \theta\eta/(4\hbar) & \hbar + \theta\eta/(2\hbar) \end{pmatrix}. \tag{17}$$

Using the map (11) and (12), any operator noncommutative relations in noncommutative phase space can be transformed to calculate their Heisenberg commutation relations. However, this map is neither unitary nor canonical.

2.3. Effective Gauge Potential in Heisenberg Representation

As an analog of an electrodynamic field with the minimum coupling, the momenta (12) in the noncommutative phase space is as

$$\widehat{P}_\mu = \widehat{p}_\mu - A_\mu, \tag{18}$$

where

$$A_\mu := -\pi_{\mu\nu} x^\nu \tag{19}$$

is the effective "gauge" potential and

$$\widehat{p}_\mu = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) \equiv i\hbar \partial_\mu \tag{20}$$

is the 4-vector momentum in the Heisenberg canonical representation. Consequently, the noncommutative effect of momentum is equivalent to the momentum in the Heisenberg commutative phase space coupled with an effective gauge potential induced by the noncommutative algebra. Note that $\widehat{P}^\mu = g^{\mu\nu}\widehat{P}_\nu$, where $g^{\mu\nu}$ is the Minkowski metric with a Lorentz signature of $(+ - - -)$.

3. Dirac Equation
3.1. Canonical form of Dirac Equation

Let us assume that the energy–momentum relation in the noncommutative phase space still holds and the 4-momentum is given by $\widehat{P}_\mu = \widehat{p}_\mu - A_\mu$ in the noncommutative phase space, the Schrödinger-like form of the Dirac equation is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H} \Psi, \tag{21}$$

where $\Psi \equiv \Psi(x^\mu)$ is a four-component wave function. In what follows, we omit the variable x^μ without loss of generality for convenience unless specifically noticed. The Hamiltonian $\widehat{H}$ is given by

$$\widehat{H} = c\alpha \cdot (\widehat{\mathbf{p}} - \mathbf{A}) + \beta m_0 c^2 + cA_0, \tag{22}$$

where m_0 denotes the mass of the Dirac particle and

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{23}$$

with σ_i the Pauli matrices and I a 2×2 unit matrix. The α_i and β matrices obey Clifford algebra:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \tag{24}$$
$$\{\alpha_i, \beta\} = 0, \tag{25}$$
$$\alpha_i^2 = \beta^2 = I, \tag{26}$$

where $\{C, D\} \equiv CD + DC$ is the anti-commutator. The effective gauge potential is given by

$$A_\mu = (A_0, \mathbf{A}), \tag{27}$$

with

$$A_0 = -\frac{\eta}{\hbar}(x + y + z), \tag{28}$$
$$\mathbf{A} = \frac{\eta}{2\hbar}[(y+z)\mathbf{i} - (x-z)\mathbf{j} - (x+y)\mathbf{k}]. \tag{29}$$

One can see that the effective gauge potential is generated by the noncommutative algebra of momenta in the noncommutative phase space. However, this effective gauge potential does not involve the gauge transformation in the present paper. This point to be discussed in future studies.

It should be remarked that (i) the operators, α, β, A_μ, and $\widehat{p}_\mu$ are Hermitian, namely, $\alpha^\dagger = \alpha, \beta^\dagger = \beta, A_\mu^\dagger = A_\mu$, and $\widehat{p}_\mu^\dagger = \widehat{p}_\mu$. Consequently, the Hamiltonian (22) is also Hermitian. (ii) Strictly speaking, in the noncommutative phase space, the wave function is expressed as $\Psi(\widehat{X}^\mu) = \Psi(x^\mu + \lambda^{\mu\nu}\widehat{p}_\nu)$. Using the Taylor expansion, one has

$$\Psi(x^\mu + \lambda^{\mu\nu}\widehat{p}_\nu) \approx \Psi(x^\mu) + \theta_\mu^h S^{\mu\nu}\widehat{p}_\nu\Psi(x^\mu) + \mathcal{O}(\theta), \tag{30}$$

where $\theta_\mu^h = \frac{\sqrt{\theta}}{2\hbar}(1,1,1,1)$ and

$$S^{\mu\nu} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{31}$$

Thus, as the first-order approximation, the wave function is given by $\Psi(\widehat{X}^\mu) \approx \Psi(x^\mu)$. A detailed discussion on the higher-order approximations is reserved for the future paper.

3.2. Lorentz-Covariant form of Dirac Equation

Using the 4-vector gauge potential A_μ, the Dirac equation can be rewritten as a tensorial form:

$$\left[\gamma^\mu(\widehat{p}_\mu - A_\mu) - m_0 c\right]\Psi = 0, \tag{32}$$

where $\gamma^\mu = (\gamma^0, \gamma^i)$ are the Dirac matrices with $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$. These operators obey Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{33}$$
$$\gamma^{\mu 2} = g^{\mu\nu}, \tag{34}$$

and

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{35}$$

is the Dirac representation, and

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{36}$$

is the Weyl representation.

One can verify that Dirac equation (32) is Lorentz-covariant. Interestingly, the effective gauge potential coupled with the Dirac equation does not contain a coupled constant (charge). In other words, the coupled constant is dimensionless.

3.3. Spin and Helicity

Let us consider the spin of a Dirac particle, which is defined by

$$\mathbf{S} := \frac{\hbar}{2}\Sigma, \tag{37}$$

where

$$\Sigma := \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}. \tag{38}$$

As with the canonical case, the helicity is generalized to

$$h_{\mathbf{S}} := \mathbf{S} \cdot \frac{\mathbf{p} - \mathbf{A}}{|\mathbf{p} - \mathbf{A}|} = \frac{\hbar}{2}\Sigma \frac{\mathbf{p} - \mathbf{A}}{|\mathbf{p} - \mathbf{A}|}, \tag{39}$$

in a noncommutative phase space. The helicity depends not only on momentum but also on the effective gauge potential. One can verify

$$\left[h_{\mathbf{S}}, \widehat{H} \right] = 0, \tag{40}$$

which implies that the helicity is still conserved in the noncommutative phase space.

3.4. Probability Current and Continuity Equation

To obtain the probability current and the continuity equation, by following similar steps in the Dirac equation coupled with an electromagnetic field and calculating the product of the complex conjugated wave function Ψ^* on the left-hand side of the Dirac equation (21) and then subsequently subtracting its complex conjugate, one obtains the following continuity equation [58]:

$$\partial_\mu J^\mu = 0, \tag{41}$$

where $J^\mu = (c\rho, \mathbf{j})$ is the 4-vector probability current density. This tensorial form of the current continuity equation is Lorentz-invariant. The probability density is given by $\rho = \Psi^\dagger \Psi$, and the expectation values of the current densities are given, respectively, by

$$\langle J^\mu \rangle = c\Psi^\dagger \gamma^\mu \Psi, \tag{42}$$

where

$$\langle \mathbf{j} \rangle = c\Psi^\dagger \alpha \Psi. \tag{43}$$

3.5. Symmetry

Let us explore the fundamental symmetries of the noncommutative relations and the Dirac equation in a noncommutative phase space. In a 4D noncommutative phase space, let us define the parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) transformations as follows:

$$\mathcal{P} \quad : \quad \widehat{X}_\mu \to -\widehat{X}_\mu, \quad \widehat{P}_\mu \to -\widehat{P}_\mu, \tag{44}$$

$$\mathcal{T} \quad : \quad t \to -t, \quad i \to i. \tag{45}$$

$$\mathcal{PT} \quad : \quad \widehat{X}_\mu \to -\widehat{X}_\mu, \quad \widehat{P}_\mu \to -\widehat{P}_\mu, \tag{46}$$

$$t \to -t, \quad i \to i, \tag{47}$$

where *i* is the imaginary unit.

One can verify that the noncommutative relation and the Dirac equation cannot remain invariants under either a parity or time reversal transformation. However, in the composites of the parity and time reversal transformations, the fundamental noncommutative relations of the position and momentum operators are invariants [46].

Note that the effective gauge potential under the parity and time reversal transformations in Equations (44)–(47) is $A_\mu \to -A_\mu$, and then $\mathcal{PT}(\widehat{p}_\mu - A_\mu)\mathcal{TP} = -(\widehat{p}_\mu - A_\mu)$. The wave function under the $\mathcal{PT}$ transformation is given by $\mathcal{PT} : \Psi(\widehat{X}^\mu) \to \Psi(-\widehat{X}^\mu) \approx \Psi(-x_\mu)$. Since $i \to i$ under the $\mathcal{PT}$ transformation, this implies that $\mathcal{PT}\gamma^\mu\mathcal{TP} = \gamma^\mu$. Thus, we have

$$\mathcal{PT}\left[\gamma^\mu(\widehat{p}_\mu - A_\mu) - m_0c\right]\mathcal{PT} = -\gamma^\mu(\widehat{p}_\mu - A_\mu) - m_0c \tag{48}$$

and infer that Dirac equation (32) under the $\mathcal{PT}$ transformations in Equations (44)–(47) becomes

$$\left[\gamma^\mu(\widehat{p}_\mu - A_\mu) + m_0c\right]\Psi(-x_\mu) = 0. \tag{49}$$

In other words, the quantum states of the particle and antiparticle are inverted, which does not involve a charge conjugation transformation.

It should be remarked that the sign of the imaginary unit *i* does not change under the definition of the time reversal transformations in Equations (44)–(47), which is different from those in the Schrödinger equation in the canonical phase space, where $i \to -i$. This property provides some hints for understanding some unsolved puzzles, such as the particle–antiparticle asymmetry and quantum decoherence discussed in Section 5.

4. Perturbation Solution of Dirac Equation

4.1. Eigen Energies and Wave Functions

To explore the basic properties of the Dirac equation, we solve the Dirac equation using perturbation theory. Note that the noncommutative parameters, θ and η, are constants with length and momentum square dimensions, respectively. Let us assume that the noncommutative effects are very small, that means the noncommutative parameters $\sqrt{\theta}, \sqrt{\eta} \ll 1$. The Hamiltonian (22) can be separated into two terms:

$$\widehat{H} = \widehat{H}_0 + \widehat{H}', \tag{50}$$

where

$$\widehat{H}_0 \quad = \quad c\alpha \cdot \widehat{\mathbf{p}} + \beta m_0 c^2, \tag{51}$$

$$\widehat{H}' \quad = \quad -c\alpha \cdot \mathbf{A} + cA_0, \tag{52}$$

where $\widehat{H}_0$ is the conventional part in the canonical Dirac equation, and $\widehat{H}'$ is the perturbed part. The Hamiltonian $\widehat{H}_0$ can be expressed in a matrix form:

$$\widehat{H}_0 = \begin{pmatrix} m_0c^2 & c\sigma \cdot \widehat{\mathbf{p}} \\ c\sigma \cdot \widehat{\mathbf{p}} & -m_0c^2 \end{pmatrix}. \tag{53}$$

Let us assume that the ansatz is given by $\Psi(x) = \psi(x)e^{-iEt/\hbar}$, where

$$\psi(x) = \begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}. \tag{54}$$

Substituting the wave vectors of Equation (54) into the Schrödinger-like equation, $i\hbar\partial_t\Psi = \hat{H}_0\Psi$, one has

$$\begin{pmatrix} m_0c^2 & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -m_0c^2 \end{pmatrix} \begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix} = E\begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix}. \tag{55}$$

Using the mathematical identity,

$$(\sigma\cdot\mathbf{C})(\sigma\cdot\mathbf{D}) = \mathbf{C}\cdot\mathbf{D}I + i\sigma\cdot(\mathbf{C}\times\mathbf{D}), \tag{56}$$

to solve the eigen Equation (55), the eigen energies are obtained [58,59]:

$$E_{\pm} = \pm c\sqrt{\mathbf{p}^2 + m_0^2c^2} \equiv \pm E_p = \lambda E_p, \tag{57}$$

where $E_p = c\sqrt{\mathbf{p}^2 + m_0^2c^2}$ and $\lambda = +1$ and -1 denote particle (positive) and antiparticle (negative) eigen energies, respectively. The corresponding eigen vectors are given by

$$\begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix} = \begin{pmatrix} \varphi_a \\ c\sigma\cdot\mathbf{p}/(\lambda E_p + m_0c^2\varphi_a) \end{pmatrix}. \tag{58}$$

Using $(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p}) = \mathbf{p}^2$ and normalizing the eigen vectors (58), one obtains:

$$\|\varphi_a\| = \sqrt{\frac{\lambda E_p + m_0c^2}{2\lambda E_p}}. \tag{59}$$

Thus, the normalized wave vector is obtained as follows:

$$\Psi_{\lambda,\tau}^{(0)} = \sqrt{\frac{\lambda E_p + m_0c^2}{2\lambda E_p}} \begin{pmatrix} \chi^{\tau} \\ \frac{c\sigma\cdot\mathbf{p}}{\lambda E_p + m_0c^2}\chi^{\tau} \end{pmatrix} e^{i(\mathbf{p}\cdot\mathbf{x} - \lambda E_p t)/\hbar}, \tag{60}$$

where "(0)" superscript denotes the zero-order correction, $\tau = +1$ and -1 denotes the spin-up and spin-down states, respectively, and

$$\chi^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{61}$$

Using the perturbation theory, for a given state in the phase space, namely, a pair of 4-vector position and momentum (x_μ, p_μ), the first-order corrections of the eigen energies are given by the diagonal elements of the perturbation Hamiltonian,

$$E_{p,\lambda,\tau}^{(1)} = \left\langle \Psi_{p,\lambda,\tau}^{(0)} | H' | \Psi_{p,\lambda,\tau}^{(0)} \right\rangle \tag{62}$$

and the corresponding first-order corrections of the eigen vectors are expressed as

$$\Psi_{p,\lambda,\tau}^{(1)} = \sum_{\lambda',\tau} \frac{\left\langle \Psi_{p,\lambda',\tau}^{(0)} | H' | \Psi_{p,\lambda,\tau}^{(0)} \right\rangle}{E_{p,\lambda,\tau} - E_{p,\lambda',\tau'}} \Psi_{p,\lambda',\tau}^{(0)}. \tag{63}$$

By substituting the eigen vectors (60) and the perturbation Hamiltonian (52) into Equation (62), the first-order corrections of the eigen energies are given

$$E_{p,\lambda,\tau}^{(1)} = \frac{\lambda E_p + m_0 c^2}{2\lambda E_p}\left[cA_0\left(1 + \frac{c^2\mathbf{p}^2}{(\lambda E_p + m_0 c^2)^2}\right) - \frac{2c^2\mathbf{p}\cdot\mathbf{A}}{\lambda E_p + m_0 c^2}\right]. \tag{64}$$

Thus, the first-order approximation eigen energies are expressed as

$$E_{p,\lambda,\tau} = \lambda E_p + E_{p,\lambda,\tau}^{(1)}. \tag{65}$$

To calculate the first-order corrections of the eigen vectors, we first calculate the off-diagonal elements of the perturbation Hamiltonian. We have

$$H_{p,\lambda,\tau;p,\lambda,-\tau}^{(1)} = H_{p,\lambda,-\tau;p,\lambda,\tau}^{(1)} = 0 \tag{66}$$

and

$$H_{p',\mp,\pm;p,\pm,\pm}^{(1)} = \frac{c}{pE_p}\left[m_0 c^2\mathbf{p}\cdot\mathbf{A} \mp i\lambda E_p(\mathbf{p}\times\mathbf{A})_z\right], \tag{67}$$

$$H_{p',\mp,\pm;p,\pm,\pm}^{(1)} = \mp\frac{i\lambda c}{p}\left[i(\mathbf{p}\times\mathbf{A})_y - (\mathbf{p}\times\mathbf{A})_x\right]. \tag{68}$$

The detailed forms of the off-diagonal elements of the perturbed Hamiltonian are given in Appendix C. Thus, by putting the off-diagonal matrix elements (67), (68), (A17), (A18), (A20) and (A21) into Equation (63), we obtain the first-order approximation wave vectors of the Dirac equation:

$$\Psi_{p,\lambda,\tau}^{(1)} = \Psi_{p,\lambda,\tau}^{(0)} + c_{p,\lambda,\tau}\Psi_{p,-\lambda,\tau}^{(0)}, \tag{69}$$

where

$$c_{p,\lambda,\tau} \equiv \frac{H_{p,\lambda,\tau;p,-\lambda,\tau}^{(1)}}{E_{p,\lambda,\tau} - E_{p,-\lambda,\tau}}. \tag{70}$$

4.2. Probability and Current Densities

By using the first-order approximation wave vectors, the probability and current densities (69) are expressed as

$$\rho_{p,\lambda,\tau}^{(1)} = \Psi_{p,\lambda,\tau}^{(1)\dagger}\Psi_{p,\lambda,\tau}^{(1)}, \tag{71}$$

$$\mathbf{j}_{p,\lambda,\tau}^{(1)} = c\Psi_{p,\lambda,\tau}^{(1)\dagger}\boldsymbol{\alpha}\Psi_{p,\lambda,\tau}^{(1)}, \tag{72}$$

where $\Psi_{p,\lambda,\tau}^{(1)}$ is given in Equation (69). The probability density is positive definite. This is different from those of the Klein–Gordon equation, where the probability density is not positive definite.

4.3. Nonrelativistic Approximation

In the nonrelativistic approximation and using the canonical form of the Dirac equation [58,59],

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\alpha\cdot(\widehat{\mathbf{p}} - \mathbf{A}) + \beta m_0 c^2 + cA_0\right]\Psi, \tag{73}$$

Let

$$\Psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}; \quad \psi_u = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \quad \psi_d = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{74}$$

Note that using

$$\alpha\cdot(\widehat{\mathbf{p}} - \mathbf{A})\Psi = \begin{pmatrix} \mathbf{0} & \sigma\cdot(\widehat{\mathbf{p}} - \mathbf{A}) \\ \sigma\cdot(\widehat{\mathbf{p}} - \mathbf{A}) & \mathbf{0} \end{pmatrix}\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}, \tag{75}$$

the Dirac equation (73) can be rewritten as

$$i\hbar\frac{\partial\psi_u}{\partial t} = c\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_d + (\beta m_0 c^2 + cA_0)\psi_u, \tag{76}$$

$$i\hbar\frac{\partial\psi_d}{\partial t} = c\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_u + (\beta m_0 c^2 + cA_0)\psi_{d'}. \tag{77}$$

In the nonrelativistic approximation, $|A_0| \ll m_0 c$ and $|\mathbf{A}| \ll m_0 c$, so that one has

$$\left(i\hbar\frac{\partial}{\partial t} + cA_0\right)\psi_{u,d}^{\pm} = m_0 c^2\left(\pm 1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right)\psi_{u,d'}^{\pm} \tag{78}$$

where v is the velocity of the Dirac particle. Inserting Equation (78) into Equations (76) and (77), one has

$$m_0 c^2\left(1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right)\psi_u^+ = c\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_d^+, \tag{79}$$

$$m_0 c^2\left(-1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right)\psi_d^- = c\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_u^-, \tag{80}$$

which can be rewritten as

$$\psi_u^+ = \frac{1}{m_0 c}\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_d^+ + \mathcal{O}\left(\frac{v^2}{c^2}\right), \tag{81}$$

$$\psi_d^- = -\frac{1}{m_0 c}\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})\psi_u^- + \mathcal{O}\left(\frac{v^2}{c^2}\right). \tag{82}$$

For the positive and negative eigen energies, the wave vectors read

$$\Psi_u = \begin{pmatrix} 1 \\ \mathcal{O}\left(\frac{v^2}{c^2}\right) \end{pmatrix}\psi_u \quad \text{and} \quad \Psi_d = \begin{pmatrix} \mathcal{O}\left(\frac{v^2}{c^2}\right) \\ 1 \end{pmatrix}\psi_{d'} \tag{83}$$

respectively. Thus, Dirac equations (76)–(77) can be rewritten as

$$i\hbar\frac{\partial\psi_u}{\partial t} = \left\{\frac{1}{2m_0}[\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})]^2 + m_0 c^2 + cA_0 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right\}\psi_u, \tag{84}$$

$$i\hbar\frac{\partial\psi_d}{\partial t} = \left\{\frac{1}{2m_0}[\sigma\cdot(\widehat{\mathbf{p}}-\mathbf{A})]^2 + m_0 c^2 + cA_0 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right\}\psi_d. \tag{85}$$

Using mathematical identity (56), note that

$$(\widehat{\mathbf{p}}-\mathbf{A})\times(\widehat{\mathbf{p}}-\mathbf{A}) = i\hbar\mathbf{B}, \tag{86}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$ is the effective magnetic field. The Dirac equation can be expressed as

$$i\hbar\frac{\partial\Psi}{\partial t} = \beta\left[(\widehat{\mathbf{p}}-\mathbf{A})^2 + \frac{i\hbar}{2m_0}\Sigma\cdot\mathbf{B} + cA_0 + m_0 c^2\right]\Psi. \tag{87}$$

This can be regarded as the Pauli equation in the noncommutative phase space. Thus, in the nonrelatativistic approximation, the Dirac equation (87) is decoupled to two-component 2×2 differential equations, which can be interpreted as particle and antiparticle.

5. What Physics Happens in Noncommutative Phase Space

5.1. Physics of Parameterization Scheme

In the canonical quantum mechanics, the Planck constant, as a noncommutative parameter, quantizes phase space, which implies the existence of a minimum volume in the phase space. The minimum volume of the phase space yields quantum fluctuations

called uncertainty relations. In the noncommutative phase space, the noncommutative parameters θ and η extend the minimum volume of the phase space. What are the physical meanings of these noncommutative relations of the operators? Cosmological observations give us some hints. These cosmological observations indicate the existence of intrinsic spacetime singularities. In particular, the interaction between a photon and gravity implies the existence of a minimum length of spacetime at the Planck scale. The dark energy in cosmology can be interpreted as an intrinsic minimum curvature of spacetime, which can be equivalent phenomenologically to the cosmological constant. Consequently, we propose a parameterization scheme for the noncommutative parameters associated with a set of physical constants [46]:

$$\theta = \ell_{\text{P}}^2; \quad \eta = \hbar^2 \Lambda, \tag{88}$$

where $\ell_{\text{P}} = \sqrt{\hbar G/c^3}$ is the Planck length with G the gravitational constant, and $\Lambda \simeq 10^{-56} \text{cm}^{-2}$ is the cosmological constant. Based on these parameter settings, the matrices of the noncommutative relations in Equations (15)–(17) can be expressed as:

$$\left[\Theta_{\mu\nu}\right] = \begin{pmatrix} 0 & \ell_{\text{P}}^2 & \ell_{\text{P}}^2 & \ell_{\text{P}}^2 \\ -\ell_{\text{P}}^2 & 0 & \ell_{\text{P}}^2 & \ell_{\text{P}}^2 \\ -\ell_{\text{P}}^2 & -\ell_{\text{P}}^2 & 0 & \ell_{\text{P}}^2 \\ -\ell_{\text{P}}^2 & -\ell_{\text{P}}^2 & -\ell_{\text{P}}^2 & 0 \end{pmatrix}, \tag{89}$$

$$\left[\Phi_{\mu\nu}\right] = \hbar \begin{pmatrix} 0 & \hbar\Lambda & \hbar\Lambda & \hbar\Lambda \\ -\hbar\Lambda & 0 & \hbar\Lambda & \hbar\Lambda \\ -\hbar\Lambda & -\hbar\Lambda & 0 & \hbar\Lambda \\ -\hbar\Lambda & -\hbar\Lambda & -\hbar\Lambda & 0 \end{pmatrix}, \tag{90}$$

and

$$\left[\Omega_{\mu\nu}\right] = \begin{pmatrix} \hbar + 3\hbar\Lambda\ell_{\text{P}}^2 & \hbar\Lambda\ell_{\text{P}}^2 & 0 & -\hbar\Lambda\ell_{\text{P}}^2 \\ \hbar\Lambda\ell_{\text{P}}^2 & \hbar + \hbar\ell_{\text{P}}^2\Lambda/2 & \hbar\ell_{\text{P}}^2\Lambda/4 & -\hbar\ell_{\text{P}}^2\Lambda/4 \\ 0 & \hbar\ell_{rmP}^2\Lambda/4 & \hbar + \hbar\ell_{\text{P}}^2\Lambda/2 & \hbar\ell_{\text{P}}^2\Lambda/4 \\ -\hbar\Lambda\ell_{\text{P}}^2 & -\hbar\ell_{\text{P}}^2\Lambda/4 & \hbar\ell_{\text{P}}^2\Lambda/4 & \hbar + \hbar\ell_{\text{P}}^2\Lambda/2 \end{pmatrix}. \tag{91}$$

In general, the parameterization scheme depends on the problem to solve. The parameterization (88) is based on physics at the Planck and cosmological scales. Actually, one can adopt other schemes for physics at microscopic or macroscopic scales, such as the de Broglie wave length or energy-dependent noncommutative parameters [34].

5.2. Noncommutative Algebra, Curvature, and Cosmological Constant

The noncommutative effects can be interpreted as an effective gauge potential. We can define its corresponding gauge field using $F = dA$, which was given by [46]:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{92}$$

From a mathematical viewpoint, the effective gauge field can be interpreted as a $(0, 2)$-type curvature in the noncommutative phase space, namely, $R_{\mu\nu} \equiv F_{\mu\nu}$. Using the parameterization scheme (88), we obtain

$$R_{\mu\nu} = \begin{pmatrix} 0 & \hbar\Lambda & \hbar\Lambda & \hbar\Lambda \\ -\hbar\Lambda & 0 & \hbar\Lambda & \hbar\Lambda \\ -\hbar\Lambda & -\hbar\Lambda & 0 & \hbar\Lambda \\ -\hbar\Lambda & -\hbar\Lambda & -\hbar\Lambda & 0 \end{pmatrix}. \tag{93}$$

Interestingly, this paramaterization scheme provides the geometric feature of the noncommutative relations, namely,

$$\left[\widehat{P}_\mu, \widehat{P}_\nu\right] = \hbar R_{\mu\nu}. \tag{94}$$

The noncommutative effect can be interpreted as the two physical scenarios, namely, on one hand, an effective gauge field or $(0,2)$-type curvature. This curvature depends on the Planck constant and cosmological constant, which implies that the noncommutative algebra modifies the dynamical behavior through an effective gauge field or spacetime curvature associated with the cosmological constant. On other hand, This feature provides a physical scenario of dark energy or a natural coupling between quantum mechanics and gravity.

5.3. Intrinsic Velocity, Force, and Zitterbewegung Oscillation

To examine the basic behaviors of the relativistic particle in the noncommutative phase space, let us study a free particle model and its mechanical properties. As with the canonical case, the velocity of a particle is defined by $\widehat{\mathbf{v}} = d\widehat{\mathbf{X}}/dt$ in the noncommutative phase space. Using the Heisenberg equation (or the called as Heisenberg picture), the velocity of a particle is given by

$$\widehat{\mathbf{v}} = \frac{1}{i\hbar}\left[\widehat{\mathbf{X}}, \widehat{H}\right].\tag{95}$$

The Hamiltonian of a free particle in the noncommutative phase space is given by

$$\widehat{H} = c\alpha \cdot \left(\widehat{\mathbf{P}} + \beta m_0 c^2\right),\tag{96}$$

where cA_0 is ignored without loss of generality, and $\widehat{\mathbf{P}} = \widehat{\mathbf{p}} - \mathbf{A}$ in the Heisenberg representation. $\mathbf{A}$ is the 3D effective gauge potential. Using the noncommutative relations (89)–(91), we have

$$\left[\widehat{\mathbf{X}}, \alpha \cdot \widehat{\mathbf{P}}\right] = i\hbar\left(\kappa^\alpha \alpha + \kappa^\beta \widetilde{\alpha}\right),\tag{97}$$

where

$$\kappa^\alpha = \hbar\left(1 + \frac{\ell_{\mathrm{P}}^2 \Lambda}{2}\right), \quad \kappa^\beta = \frac{\hbar \ell_{\mathrm{P}}^2 \Lambda}{4}.\tag{98}$$

and

$$\widetilde{\alpha} = (\alpha_y - \alpha_z)\mathbf{i} + (\alpha_x + \alpha_z)\mathbf{j} + (-\alpha_x + \alpha_y)\mathbf{k}.\tag{99}$$

Thus, the velocity is expressed as

$$\widehat{\mathbf{v}} = c\left(\kappa^\alpha \alpha + \kappa^\beta \widetilde{\alpha}\right).\tag{100}$$

Similarly, the acceleration is defined by

$$\frac{d\widehat{\mathbf{v}}}{dt} = \frac{1}{i\hbar}\left[\widehat{\mathbf{v}}, \widehat{H}\right].\tag{101}$$

Note that

$$\alpha\widehat{H} + \widehat{H}\alpha = 2c\widehat{\mathbf{P}},\tag{102}$$

$$\widetilde{\alpha}\widehat{H} + \widehat{H}\widetilde{\alpha} = 2c\widehat{\widetilde{\mathbf{P}}},\tag{103}$$

where

$$\widehat{\widetilde{\mathbf{P}}} = (\widehat{P}_y - \widehat{P}_z)\mathbf{i} + (\widehat{P}_x + \widehat{P}_z)\mathbf{j} + (-\widehat{P}_x + \widehat{P}_y)\mathbf{k}.\tag{104}$$

Consequently, Equation (101) can be rewritten as

$$\frac{d\widehat{\mathbf{v}}}{dt} = \frac{2c^2}{i\hbar}\left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widetilde{\mathbf{P}}}\right) - \frac{2}{i\hbar}\widehat{H}\widehat{\mathbf{v}}.\tag{105}$$

Solving the differential equation on $\widehat{\mathbf{v}}$, we obtain

$$\widehat{\mathbf{v}}(t) = \left(\widehat{\mathbf{v}}(0) - \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right)}{\widehat{H}} \right) \exp\left(\frac{2i\widehat{H}t}{\hbar} \right) + \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right)}{\widehat{H}}. \tag{106}$$

In substituting Equation (106) into Equation (100) and integrating over t, the position vector reads a function of t:

$$\widehat{\mathbf{X}}(t) = \widehat{\mathbf{X}}(0) + \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right) t}{\widehat{H}} - \frac{i\hbar}{2\widehat{H}} \left(\mathbf{v}(0) - \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right)}{\widehat{H}} \right) \exp\left(\frac{2i\widehat{H}t}{\hbar} \right), \tag{107}$$

where 0 denotes the initial time. Note that the classical (cl) position part is given in Refs. [58,59]:

$$\mathbf{x}^{\mathrm{cl}}(t) = \mathbf{x}^{\mathrm{cl}}(0) + \frac{c^2 \widehat{\mathbf{P}}}{E_p} t. \tag{108}$$

The equation of motion in the noncommutative phase space is rewritten as a sum of the two terms:

$$\widehat{\mathbf{X}}(t) = \widehat{\mathbf{X}}^{\mathrm{cl}}(t) + \widehat{\mathbf{X}}_Z(t), \tag{109}$$

where

$$\widehat{\mathbf{X}}^{\mathrm{cl}}(t) = \widehat{\mathbf{X}}(0) + \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right) t}{\widehat{H}} \tag{110}$$

is the classical part and

$$\widehat{\mathbf{X}}_Z(t) = -\frac{i\hbar}{2\widehat{H}} \left(\mathbf{v}(0) - \frac{c^2 \left(\kappa^\alpha \widehat{\mathbf{P}} + \kappa^\beta \widehat{\widehat{\mathbf{P}}} \right)}{\widehat{H}} \right) \exp\left(\frac{2i\widehat{H}t}{\hbar} \right) \tag{111}$$

represents the Zitterbewegung oscillation.

5.4. Equation of Motion

As with the equation of motion on velocity (95), in the Heisenberg picture, the equation of motion is given by

$$\frac{d\widehat{\mathbf{P}}}{dt} = \frac{1}{i\hbar} \left[\widehat{\mathbf{P}}, \widehat{H} \right]. \tag{112}$$

By calculating the commutation relation (see Appendix D), we obtain

$$\left[\widehat{P}_\mu, c\alpha \cdot \widehat{\mathbf{P}} \right] = i\hbar v^\nu F_{\mu\nu}. \tag{113}$$

where $v_\nu = c\alpha_\nu$ is the velocity, and $F_{\mu\nu}$ is the effective gauge field in Equation (92). Thus, we obtain the equation of motion

$$\frac{d\widehat{P}_\mu}{dt} = v_\nu F_{\mu\nu}, \tag{114}$$

where the right-hand side of Equation (114) is the Lorentz-type force induced by the noncommutative algebra. Interestingly, a free particle carries an intrinsic velocity and Lorentz-type force in the noncommutative phase space.

5.5. Particle–Antiparticle Symmetric Breaking and Quantum Decoherence

Let us recall the Dirac equation of the static states in the canonical phase space:

$$\left(c\alpha \cdot \widehat{\mathbf{p}} + \beta m_0 c^2 \right) \Psi_\lambda = E_\lambda \Psi_\lambda. \tag{115}$$

By solving the eigen Equation (115), the eigen energies E_λ are given in Equation (57). The corresponding eigen vectors are given by

$$\Psi_{\lambda,\tau}^{(0)} = \sqrt{\frac{\lambda E_p + m_0 c^2}{2\lambda E_p}} \left(\begin{array}{c} \chi^\tau \\ \frac{c\sigma \cdot \mathbf{p}}{\lambda E_p + m_0 c^2} \chi^\tau \end{array} \right) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{116}$$

with $\tau = \pm 1$ as defined in Section 4.1 and $\chi^\pm$ are defined in Equation (61). Assuming the vacuum energy is at zero between the E_λ eigen energies, E_+ and E_-, the gap asymmetry of the particle and antiparticle is given by $\Delta E_g = E_+ + E_- = 0$. In other words, the energy gaps of the particle and antiparticle are symmetric for free Dirac particles in canonical quantum mechanics.

In the noncommutative phase space, the noncommutative effect can be included in the Dirac equation as an effective gauge field. The Dirac equation becomes

$$\left[c\alpha \cdot (\hat{\mathbf{p}} - \mathbf{A}) + \beta m_0 c^2 \right] \Psi_{p,\lambda,\tau} = E_{p,\lambda,\tau} \Psi_{p,\lambda,\tau}. \tag{117}$$

For a given state in the phase space, (x_μ, p_μ), the eigen energies in the first-order approximation can be expressed as

$$E_{p,\lambda,\tau} = \lambda E_p + E_{p,\lambda,\tau}^{(1)}, \tag{118}$$

where $E_{p,\lambda,\tau}^{(1)}$ is given by Equation (64). The energy gap asymmetry in the first-order approximation is given by

$$\begin{aligned} \Delta E_{p,\lambda,\tau} &= E_{p,\lambda,\tau}^{(1)} + E_{p,-\lambda,\tau}^{(1)} \\ &= -\frac{2c^2}{E_p} \mathbf{A} \cdot \mathbf{p}. \end{aligned} \tag{119}$$

In the parameterization scheme (88), the energy gap asymmetry can be rewritten as

$$\Delta E_{p,\lambda,\tau} = \hbar\Lambda \frac{c^2}{E_p} (L_x - L_y + L_z), \tag{120}$$

where $L_i = \varepsilon_{ij}{}^k x^j p_k$ with $\varepsilon_{ij}{}^k$ the Levi-Civita symbol is the 3D angular momentum. In general, $\Delta E_{p,\lambda\tau} \neq 0$. In other words, the energy gap symmetry of the particle and antiparticle is broken by the effective gauge potential in the noncommutative phase space, which is related to the cosmological constant and Planck constant. This implies that curved spacetime or gravity breaks the particle–antiparticle symmetry such that particles are observed in a considerably larger number than antiparticles.

On the other hand, the energy gap shift leads to quantum decoherence. Roughly, let us consider the size of a quantum object to be L. The average of the energy gap shift for a quantum object is, $\Delta\mathcal{E}_{p,\lambda,\tau} \approx \frac{1}{L} \int_0^L \Delta E_{p,\lambda,\tau} dx \propto \hbar\Lambda, L$. Thus, the quantum decoherence time can be given by

$$\tau_d = \frac{\hbar}{\Delta\mathcal{E}_{p,\lambda,\tau}}. \tag{121}$$

In other words, the particle–antiparticle symmetry breaking induces quantum decoherence. The quantum decoherence time depends on the cosmological constant and the object size, which implies that quantum decoherence is induced intrinsically by a gravity or spacetime background [60]. The decoherence time τ_d becomes quite long for objects with relatively small sizes and quite short for those with relatively large sizes. This is consistent with what one expects. Consequently, the noncommutative phase space provides a natural background or scenario for understanding particle–antiparticle asymmetry and quantum decoherence.

Quantum decoherence is a core issue for understanding why quantum coherence is fragile in the macroscopic world [61,62]. It is still a puzzle even when there are many theoretical models for understanding the physical mechanism of quantum decoherence, such as quantum measurement-induced decoherence and environment-induced superselection [60,62]. Here, we present a quantum decoherence model based on noncommutative quantum mechanics.

6. Conclusions and Outlook

The incompatibility of the spacetime background in quantum mechanics and general relativity is still a challenging problem in theoretical physics. In particular, the intrinsic spacetime singularities imply that all physics collapse at the Planck scale. There may be a minimum length in the Planck scale to avoid singularity emergence. The observation of the universe acceleration expansion hints at the existence of a minimum curvature of spacetime associated with the cosmological constant. Understanding these unexpected phenomena has stimulated many attempts to quantize the spacetime background and deform canonical quantum mechanics.

By using a noncanonical map, noncommutative relations are mapped to the Heisenberg canonical commutation relations. We presented the Dirac equation and its corresponding current continuity equation in the noncommutative phase space. We propose a parameterization scheme associated with the Planck length and cosmological constant such that the noncommutative effect can be interpreted as an effective gauge potential or $(0, 2)$-type curvature associated with the cosmological constant. This reveals an intrinsic interlay between the dynamics of gravity and spacetime. Moreover, we found that a free particle carries an intrinsic velocity and acceleration induced by the noncommutative algebra. These novel properties of free Dirac particles provide a deep physical scenario of the Zitterbewegung oscillation and dark energy.

We analyzed the $\mathcal{PT}$ symmetries of the Dirac equation in the noncommutative phase space. We found that the particle and antiparticle states are inverted under $\mathcal{PT}$ transformations if i is invariant. Using the perturbation approach, we obtained the perturbed and nonrelativistic solutions of the Dirac equation. We found that there exists an asymmetric vacuum gap of particles and antiparticles induced by the noncommutative effect. This symmetric breaking reveals deep physical mechanisms for understanding the particle–antiparticle asymmetry and quantum decoherence induced by gravity or quantum spacetime background fluctuation.

These formulations not only open a novel framework of relativistic quantum mechanics in the noncommutative phase space but also inspire some interesting mathematical structures.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. The Basic Commutative Relations

Let us present some basic commutative relations in the noncommutative phase space for convenience,

$$\left[\widehat{X}, \widehat{\mathbf{P}}^2\right] = 2i\left[\kappa^\alpha \widehat{P}_x + \kappa^\beta\left(\widehat{P}_y - \widehat{P}_z\right)\right], \tag{A1}$$

$$\left[\widehat{Y}, \widehat{\mathbf{P}}^2\right] = 2i\left[\kappa^\alpha \widehat{P}_y + \kappa^\beta\left(\widehat{P}_x + \widehat{P}_z\right)\right], \tag{A2}$$

$$\left[\widehat{Z}, \widehat{\mathbf{P}}^2\right] = 2i\left[\kappa^\alpha \widehat{P}_z + \kappa^\beta\left(-\widehat{P}_x + \widehat{P}_y\right)\right], \tag{A3}$$

and

$$\left[\widehat{P}_x, \widehat{\mathbf{P}}^2\right] = 2i\eta\left(\widehat{P}_y + \widehat{P}_z\right), \tag{A4}$$

$$\left[\widehat{P}_y, \widehat{\mathbf{P}}^2\right] = 2i\eta\left(-\widehat{P}_x + \widehat{P}_z\right), \tag{A5}$$

$$\left[\widehat{P}_z, \widehat{\mathbf{P}}^2\right] = -2i\eta\left(\widehat{P}_x + \widehat{P}_y\right). \tag{A6}$$

One can find that in the noncommutative phase space, $\left[\widehat{\mathbf{P}}, \widehat{\mathbf{P}}^2\right] \neq 0$, which implies the translation symmetry breaking. When the noncommutative parameters vanish, the commutative relations (A1)–(A6) reduce to the canonical relations in Heisenberg algebra.

Appendix B. Proof of the Noncanonical Map

Proof. By using noncanonical map (11) and (12), the noncommutative relations between the position operators $\widehat{X}^\mu$ are expressed as

$$\begin{aligned}\left[\widehat{X}^\mu, \widehat{X}^\nu\right] &= [x^\mu + \lambda^{\mu\kappa}\widehat{p}_\kappa, (x^\nu + \lambda^{\nu\sigma}\widehat{p}_\sigma)] \\ &= [x^\mu, \widehat{p}_\sigma]\lambda^{\sigma\nu} + \lambda^{\nu\kappa}[\widehat{p}_\kappa, x^\mu] \\ &= i\hbar(\lambda^{\mu\nu} - \lambda^{\nu\mu}).\end{aligned} \tag{A7}$$

In applying λ-matrix (13) subtraction, the noncommutative relation (8) with the Θ-matrix (15) is obtained.

Similarly, the noncommutative relations between the momentum operators $\widehat{P}^\mu$ are expressed as

$$\begin{aligned}\left[\widehat{P}_\mu, \widehat{P}_\nu\right] &= \left[\widehat{p}_\mu + \pi_{\mu\kappa}x^\kappa, (\widehat{p}^\nu + \pi_{\nu\sigma}x^\sigma)\right] \\ &= \left[\widehat{p}_\mu, x^\sigma\right]\pi_{\sigma\nu} + \pi_{\nu\kappa}\left[x^\kappa, \widehat{p}_\mu\right] \\ &= i\hbar\left(-\pi_{\mu\nu} + \pi_{\nu\mu}\right).\end{aligned} \tag{A8}$$

Through π-matrix (14) subtraction, the noncommutative relation (9) with the Φ-matrix (16) is obtained.

In the same way, the noncommutative relations between the position and momentum operators are expressed as follows:

$$\begin{aligned}\left[\widehat{X}^\mu, \widehat{P}_\nu\right] &= [x^\mu + \lambda^{\mu\kappa}\widehat{p}_\kappa, \widehat{p}_\nu + \pi_{\nu\sigma}x^\sigma] \\ &= [x^\mu, \widehat{p}_\nu] + \lambda^{\mu\kappa}[\widehat{p}_\kappa, x^\sigma]\pi_{\sigma\nu} \\ &= i\hbar(\delta^\mu{}_\nu - \lambda^{\mu\kappa}\pi_{\kappa\nu}).\end{aligned} \tag{A9}$$

In making the matrix product of λ and π, the noncommutative relation is obtained:

$$\left[\widehat{X}^\mu, \widehat{P}_\nu\right] = i\Omega^\mu{}_\nu, \tag{A10}$$

where

$$[\Omega^\mu{}_\nu] = \begin{pmatrix} \hbar + 3\theta\eta/\hbar & \theta\eta/\hbar & 0 & -\theta\eta/\hbar \\ \eta\theta/\hbar & \hbar + \theta\eta(2\hbar) & \theta\eta/(4\hbar) & -\theta\eta(4\hbar) \\ 0 & \theta\eta/(4\hbar) & \hbar + \theta\eta(2\hbar) & \theta\eta(4\hbar) \\ -\eta\theta/\hbar & -\theta\eta/(4\hbar) & \theta\eta/(4\hbar) & \hbar + \theta\eta/(2\hbar) \end{pmatrix}. \tag{A11}$$

These representations of the position and momentum operators can be regarded as Heisenbereg representations of the noncommutative quantum mechanics. The noncanonical map provides an efficient way to approximately perform the Heisenberg algebra in the noncommutative phase space even though the map is not unitary or canonical. It can be verified that the commutative relations obtained satisfy the Lorentz invariants. $\quad\square$

Appendix C. The Perturbed Matrix Elements

Using the mathematical identity

$$\sigma \cdot (\mathbf{A} \times \mathbf{p}) + \sigma \cdot (\mathbf{p} \times \mathbf{A}) = 0, \tag{A12}$$

one obtains

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \sigma \cdot (\mathbf{p} \times \mathbf{A}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\mathbf{p} \times \mathbf{A})_z, \tag{A13}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \sigma \cdot (\mathbf{p} \times \mathbf{A}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -(\mathbf{p} \times \mathbf{A})_z, \tag{A14}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \sigma \cdot (\mathbf{p} \times \mathbf{A}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\mathbf{p} \times \mathbf{A})_x - i(\mathbf{p} \times \mathbf{A})_y, \tag{A15}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \sigma \cdot (\mathbf{p} \times \mathbf{A}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\mathbf{p} \times \mathbf{A})_x + i(\mathbf{p} \times \mathbf{A})_y. \tag{A16}$$

Using the mathematical identities (A13)–(A16), we obtain the perturbed matrix elements:

$$
\begin{aligned}
H^{(1)}_{p',\lambda',\pm;p,\lambda,\pm} &= M_{p'\lambda';p\lambda}\left[cA_0\left(1 + \frac{c^2\mathbf{p}'\cdot\mathbf{p}}{(m_0c^2+\lambda'E_p')(m_0c^2+\lambda E_p)}\right) - c^2\left(\frac{\mathbf{p}\cdot\mathbf{A}}{m_0c^2+\lambda E_p} + \frac{\mathbf{p}'\cdot\mathbf{A}}{m_0c^2+\lambda'E_p'}\right)\right.\\
&\quad - \left. ic^2\left(\frac{(\mathbf{A}\times\mathbf{p})_z}{m_0c^2+\lambda E_p} \pm \frac{(\mathbf{p}'\times\mathbf{A})_z}{m_0c^2+\lambda'E_p}\right) + \frac{c^3A_0(\mathbf{p}'\times\mathbf{p})_z}{(m_0c^2+\lambda'E_p')(m_0c^2+\lambda E_p)}\right],
\end{aligned}
\tag{A17}
$$

$$
\begin{aligned}
H^{(1)}_{p',\lambda',\pm;p,\lambda,\mp} &= M_{p'\lambda';p\lambda}\left[\mp ic^2\left(\frac{1}{m_0c^2+\lambda E_p} - \frac{1}{m_0c^2+\lambda'E_p}\right)((\mathbf{p}\times\mathbf{A})_y - (\mathbf{p}\times\mathbf{A})_x)\right.\\
&\quad + \left. c^3A_0\frac{\mp i(\mathbf{p}'\times\mathbf{p})_y + (\mathbf{p}'\times\mathbf{p})_x}{(m_0c^2+\lambda'E_p')(m_0c^2+\lambda E_p)}\right],
\end{aligned}
\tag{A18}
$$

where

$$M_{p\lambda';p\lambda} = \sqrt{\frac{(m_0c^2+\lambda'E_p')(m_0c^2+\lambda E_p)}{2\lambda'\lambda E_p'E_P}}. \tag{A19}$$

For the case of $p' = p$, the matrix elements (A17) and (A18) reduce to

$$
\begin{aligned}
H^{(1)}_{p,\lambda',\pm;p,\lambda,\pm} &= N_{p\lambda';p\lambda}\left[cA_0\left(1 + \frac{c^2\mathbf{p}^2}{(m_0c^2+\lambda'E_p)(m_0c^2+\lambda E_p)}\right) - \frac{c^2\mathbf{p}\cdot\mathbf{A}}{m_0c^2+\lambda E_p} - \frac{c^2\mathbf{p}\cdot\mathbf{A}}{m_0c^2+\lambda'E_p}\right.\\
&\quad - \left. ic^2\left(\frac{(\mathbf{p}\times\mathbf{A})_z}{m_0c^2+\lambda E_p} \pm \frac{(\mathbf{p}\times\mathbf{A})_z}{m_0c^2+\lambda'E_p}\right)\right],
\end{aligned}
\tag{A20}
$$

$$H^{(1)}_{p,\lambda',\pm;p,\lambda,\mp} = N_{p\lambda';p\lambda}\left[\mp ic^2\left(\frac{1}{m_0c^2+\lambda E_p} - \frac{1}{m_0c^2+\lambda'E_p}\right)\left(i(\mathbf{p}\times\mathbf{A})_y - (\mathbf{p}\cdot\mathbf{A})_x\right)\right], \tag{A21}$$

where

$$N_{p\lambda';p\lambda} = \frac{1}{2E_p}\sqrt{\frac{(m_0c^2+\lambda'E_p)(m_0c^2+\lambda E_p)}{\lambda'\lambda}}. \tag{A22}$$

Appendix D. Lorentz-Type Force

Using noncanonical map (11) and (12) with its Heisenberg representation (18), we calculate the following commutator:

$$\begin{aligned}
\left[\widehat{P}_\mu, c\alpha \cdot \widehat{\mathbf{P}}\right] &= \left[\widehat{P}_\mu, c\alpha^\nu \cdot \widehat{P}_\nu\right] \\
&= \left[(\widehat{p}_\mu - A_\mu), c\alpha^\nu \cdot (\widehat{p}_\nu - A_\nu)\right] \\
&= -c\alpha^\nu\left[\widehat{p}_\mu, A_\nu\right] - c\alpha^\nu\left[A_\mu, \widehat{p}_\nu\right] \\
&= i\hbar c\alpha^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= i\hbar c\alpha^\nu F_{\mu\nu} \\
&= i\hbar v^\nu F_{\mu\nu}.
\end{aligned} \tag{A23}$$

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