

Article

# Entangled Probability Distributions for Center-of-Mass Tomography

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**Abstract:** We review the formalism of center-of-mass tomograms, which allows us to describe quantum states in terms of probability distribution functions. We introduce the concept of separable and entangled probability distributions for center-of-mass tomography. We obtain the time evolution of center-of-mass tomograms of entangled states of the inverted oscillator.

**Keywords:** entangled probability distributions; entanglement; symplectic tomography; center-of-mass tomography

## 1. Introduction

Conventional probability theory [1] provides the basis to study the properties of quantum systems. The formulation of quantum mechanics is based on the Schrödinger equation [2]. The states of quantum systems are commonly described by the complex wave function or the density operator [2–6]. An alternative way to represent the states is given by quasi-probability distributions like the Wigner function [7,8], the Husimi  $Q$ -function [9,10], and the Glauber–Sudarshan  $P$ -function [11,12]. There exist other quasi-probability functions as well [13–16]. Quasi-probability distributions are not fair probabilities: they are functions on the phase space whose arguments, position, and momentum are not simultaneously measurable due to the uncertainty relations [17–19].

The problem of constructing the probability representation of quantum states has been discussed for many decades, for example, with measuring the quantum system states expressed in terms of the Wigner function. It was found [20,21] that the Wigner function can be expressed in terms of the probability distribution of position, measured using the optical tomography method [22]. Also, one can find arguments based on the statistical interpretation of quantum physics, which are closely related to the probability description of quantum states presented in the study [23]. An analogous connection with the statistical approach to quantum mechanics is available in a textbook by Leslie E. Ballentine; see [24] and references therein.

In addition, the quasi-probability functions can take negative or even complex values [8]. The probability representation of quantum mechanics [25,26] provides the description of the states of quantum systems in terms of nonnegative probability distributions called tomograms, both for discrete variables [27] and continuous variables [25,26]. The tomograms are related to the density operator or quasi-probability distributions by means of invertible integral transforms. For example, the Wigner function and tomogram are connected by the Radon transform [26]. All invertible maps of density operators and observables onto functions are described by the star-product formalism [28]. Special cases of such maps include symplectic tomography [25,26] and center-of-mass tomography [29,30]. Examples of center-of-mass tomograms [29,31] and symplectic tomograms for states of a harmonic oscillator like Fock states and coherent or Schrödinger cat states are presented in Refs. [26,32]. The tomographic methods and their applications to analyze quantum systems were considered in Refs. [33–37]. Some other aspects of quantum systems in the context of



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interactions with light and external fields [38–41] and the entanglement formation [42,43] were discussed.

The tomographic picture allows one to describe the states of both quantum and classical systems by tomograms [25,26]; it finds application in cosmology [44,45]. The difference between quantum systems and classical systems consists in the possibility to be in an entangled state. This feature leads to the concept of entangled probability distributions [46], which has not been studied in classical probability theory. Classical probability theory is described, for example, in Refs. [47,48]. Some new aspects of entanglement phenomena are discussed in recent papers [49–54]. In this paper, we introduce the notion of entangled probability distributions for center-of-mass tomography and study them in view of the probability representation of quantum mechanics. We consider examples of center-of-mass tomograms of entangled states of harmonic and inverted oscillators. We also determine the time evolution of these states using the method of integrals of motion developed in Ref. [32]. The idea of the method is that, for systems like harmonic oscillators or inverted oscillators, the position and momentum operators in the Heisenberg representation are expressed in terms of integrals of motion and are linear in the position and momentum operators. This allows one to obtain the evolution of tomograms by corresponding time-dependent transforms of the parameters of initial tomograms. Also, we consider examples of cluster tomograms [31]—a generalization of center-of-mass tomography.

In the current study, we construct new conditional probability distribution functions and entangled probability distribution functions, which describe quantum states of quantum systems, and to study their properties. Earlier, these probability distributions were not considered, since classical and quantum systems provide different randomness phenomena such as, for example, the existence of the position-and-momentum uncertainty relations for quantum systems [17–19].

This paper is organized as follows. In Section 2, we review the probability representation of quantum states, paying special attention to center-of-mass tomography. In Section 3, we consider examples of center-of-mass tomograms for entangled states and their connection to symplectic tomograms. In Section 4, we obtain the dynamics of tomograms for harmonic and inverted oscillators. Section 5 is devoted to cluster tomography. Summary and prospects are given in Section 6.

## 2. Entangled Probability Distributions

In the probability representation of a quantum state, such as a one-dimensional oscillator, the state is determined by the tomogram, which is the conditional probability distribution,  $w(X|\mu, \nu)$ , of one random variable. The wave function,  $\psi(X)$ , introduced in quantum mechanics can be mapped onto the probability distribution  $w(X|\mu, \nu)$  of random variable  $X$ , which is the position measured in an ensemble of reference frames in the oscillator’s phase space, where  $\mu$  and  $\nu$  determine the axes in the phase space [55] (in what follows, we consider Planck’s constant  $\hbar = 1$ ),

$$w(X|\mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(Y) \exp\left(\frac{i\mu}{2\nu} Y^2 - \frac{i}{\nu} XY\right) dY \right|^2. \tag{1}$$

This function is the conditional probability distribution of random position  $X$ , measured in the set of reference frames in the phase space. In these frames, the axes of the position and momentum were first rescaled, in view of the transformations  $q' = sq$  and  $p' = s^{-1}p$ , and then they were rotated as  $q'' = \cos \Theta q' + \sin \Theta p'$ . It turns out that the conditional probability distribution function  $w(X|\mu, \nu)$  determines the density matrix,  $\rho(x, x') = \psi(x)\psi^*(x')$  [3], of the pure state with the wave function  $\psi(x)$ . The conditional probability distribution function contains the same information on the oscillator state as the wave function does. The conditional probability distribution is an infinite set of probability distributions of one random variable  $X$  used in conventional probability theory [51,52].

We consider a quantum system with two degrees of freedom. In center-of-mass tomography, the state of a quantum system is described by the center-of-mass tomogram.

The center-of-mass tomogram of a quantum state with the density operator,  $\hat{\rho}$ , is defined as follows [29]:

$$w(X|\mu_1, \nu_1, \mu_2, \nu_2) = \text{Tr}(\hat{\rho} \delta(X - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2)), \tag{2}$$

where  $\hat{q}_j$  and  $\hat{p}_j$  are the position and momentum operators for each degree of freedom. The name “center-of-mass tomogram” for function (2) reflects a formal coincidence of the linear form  $X = \mu_1 q_1 + \nu_1 p_1 + \mu_2 q_2 + \nu_2 p_2$  in the argument of Dirac  $\delta$ -function with the center-of-mass expression. It is so if one considers parameters  $\mu_1, \nu_1, \mu_2,$  and  $\nu_2$  as masses of four particles and parameters  $q_1, p_1, q_2,$  and  $p_2$  as positions of the particles. Then, the parameter  $X$  coincides with the center-of-mass system, if the sum of the four masses is equal to 1.

The center-of-mass tomogram is a nonnegative probability distribution of the random variable  $X$  associated with the center-of-mass position of the system in the phase space in rotated and scaled reference frames, which are determined by parameters  $\mu_1, \nu_1, \mu_2,$  and  $\nu_2$ . Note that the tomogram effectively operates with four variables, due to the homogeneity property of the Dirac delta-function. The density operator of a state can be reconstructed from the center-of-mass tomogram; it reads

$$\hat{\rho} = \frac{1}{4\pi^2} \int w(X|\mu_1, \nu_1, \mu_2, \nu_2) \exp(i(X - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2)) \times dX d\mu_1 d\nu_1 d\mu_2 d\nu_2. \tag{3}$$

The center-of-mass tomogram can be treated as the conditional probability distribution [31], where  $\mu_1, \nu_1, \mu_2,$  and  $\nu_2$  are parameters describing the condition of measuring  $X$ . The treatment follows from the “no-signalling” property [56],

$$\int dX w(X|\mu_1, \nu_1, \mu_2, \nu_2) = 1, \tag{4}$$

which holds for parameters  $\mu_1, \nu_1, \mu_2,$  and  $\nu_2$ . In the case of pure states, the center-of-mass tomogram is given in Ref. [29]; it is

$$w(X|\mu_1, \mu_2, \nu_1, \nu_2) = \int dY_1 dY_2 \frac{\delta(X - Y_1 - Y_2)}{4\pi^2 |\nu_1 \nu_2|} \times \left| \int dq_1 dq_2 \psi(q_1, q_2) \exp\left(\frac{i\mu_1}{2\nu_1} q_1^2 + \frac{i\mu_2}{2\nu_2} q_2^2 - \frac{iY_1}{\nu_1} q_1 - \frac{iY_2}{\nu_2} q_2\right) \right|^2. \tag{5}$$

There exist other probability distributions that can be identified with quantum states; for instance, the symplectic tomogram. The symplectic tomogram is the nonnegative probability distribution of random variables  $X_1$  and  $X_2$  associated with the position of the subsystem in the phase space in rotated and scaled reference frames determined by parameters  $\mu_1, \nu_1, \mu_2,$  and  $\nu_2$ ; it reads

$$w^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \text{Tr}(\hat{\rho} \delta(X_1 - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1) \delta(X_2 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2)). \tag{6}$$

The inverse transform is

$$\hat{\rho} = \frac{1}{4\pi^2} \int w^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) \exp(i(X_1 + X_2 - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2)) \times dX_1 dX_2 d\mu_1 d\nu_1 d\mu_2 d\nu_2. \tag{7}$$

The center-of-mass tomogram and symplectic tomogram are related as follows:

$$w^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{1}{4\pi^2} \int w(X|k_1 \mu_1, k_2 \mu_2, k_1 \nu_1, k_2 \nu_2) e^{i(X - k_1 X_1 - k_2 X_2)} dk_1 dk_2 dX. \tag{8}$$

The state of the first subsystem can be found in terms of the center-of-mass tomogram of the whole system, in view of the formula [31]

$$w_1(X_1|\mu_1, \nu_1) = \text{Tr}(\hat{\rho}_1 \delta(X_1 - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1)) = \frac{1}{2\pi} \int w(X|k\mu_1, 0, k\nu_1, 0) e^{i(X-kX_1)} dk dX, \quad (9)$$

where  $\hat{\rho}_1 = \text{Tr}_2 \hat{\rho}$  is the density operator of the first subsystem, obtained by taking the partial trace of the density operator  $\hat{\rho}$  of the whole system over the second subsystem.

Let us introduce the concept of separable and entangled probability distributions for center-of-mass tomography. The symplectic tomogram of a separable state of a system, which consists of two subsystems, is represented by the convex sum of symplectic tomograms of subsystems [46],

$$w^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \sum_k p_k w_1^{(k)}(X_1|\mu_1, \nu_1) w_2^{(k)}(X_2|\mu_2, \nu_2), \quad (10)$$

where  $p_k$  are probabilities, i.e.,  $p_k \geq 0$  and  $\sum_k p_k = 1$ . The probability distribution  $w^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2)$  is called the entangled probability distribution if it cannot be presented as the convex sum of the form (10) [46]. For the center-of-mass tomogram, the relation between tomograms of the system and its subsystems for a separable state follows from (10); it has the form

$$w(X|\mu_1, \mu_2, \nu_1, \nu_2) = \sum_k p_k \int w_1^{(k)}(X_1|\mu_1, \nu_1) w_2^{(k)}(X - X_1|\mu_2, \nu_2) dX_1. \quad (11)$$

The probability distribution  $w(X|\mu_1, \mu_2, \nu_1, \nu_2)$  is said to be the entangled probability distribution if it cannot be cast in the form (11). The generalization of formula (11) to the case of systems with many degrees of freedom is given in Appendix A.

In Section 3 just below, we consider examples of separable and entangled probability distributions for center-of-mass tomography.

### 3. Examples of Entangled Probability Distribution

Let us consider an entangled state of a two-dimensional oscillator, which is a superposition of the ground state  $\psi_0(q) = \pi^{-1/4} e^{-q^2/2}$  and the first excited state  $\psi_1(q) = \pi^{-1/4} \sqrt{2} q e^{-q^2/2}$  of the form

$$\psi_{\text{ent}}(q_1, q_2) = \frac{1}{\sqrt{2}} (\psi_0(q_1)\psi_1(q_2) + \psi_1(q_1)\psi_0(q_2)) = \frac{q_1 + q_2}{\sqrt{\pi}} \exp\left(-\frac{q_1^2}{2} - \frac{q_2^2}{2}\right). \quad (12)$$

The center-of-mass tomogram of this state follows from the general relation (5); it reads

$$w_{\text{ent}}(X|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{e^{-X^2/\sigma}}{\sqrt{\pi\sigma}} \left( \frac{1}{2} - \frac{\mu_1\mu_2 + \nu_1\nu_2}{\sigma} + \frac{X^2}{\sigma} \left( 1 + \frac{2(\mu_1\mu_2 + \nu_1\nu_2)}{\sigma} \right) \right), \quad (13)$$

where  $\sigma = \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2$ . We call this probability distribution the entangled probability distribution, since it determines the entangled state. To compare it with the center-of-mass tomogram of a separable state, we consider the following wave function:

$$\psi_{\text{sep}}(q_1, q_2) = \psi_0(q_1)\psi_1(q_2) = \frac{\sqrt{2}q_2}{\sqrt{\pi}} \exp\left(-\frac{q_1^2}{2} - \frac{q_2^2}{2}\right). \quad (14)$$

The corresponding tomogram is given by

$$w_{\text{sep}}(X|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{e^{-X^2/\sigma}}{\sqrt{\pi\sigma^3}} \left( \mu_1^2 + \nu_1^2 + \frac{2X^2}{\sigma} (\mu_2^2 + \nu_2^2) \right). \quad (15)$$

Tomograms of the separable and entangled states of the first subsystem considered above are

$$w_1^{\text{sep}}(X_1|\mu_1, \nu_1) = \frac{1}{\sqrt{\pi(\mu_1^2 + \nu_1^2)}} \exp\left(-\frac{X_1^2}{\mu_1^2 + \nu_1^2}\right), \tag{16}$$

$$w_1^{\text{ent}}(X_1|\mu_1, \nu_1) = \frac{1}{\sqrt{\pi(\mu_1^2 + \nu_1^2)}} \left(\frac{1}{2} + \frac{X_1^2}{\mu_1^2 + \nu_1^2}\right) \exp\left(-\frac{X_1^2}{\mu_1^2 + \nu_1^2}\right). \tag{17}$$

Symplectic tomograms of the separable and entangled states read

$$w_{\text{sep}}^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{2X_2^2}{\pi(\mu_1^2 + \nu_1^2)^{1/2}(\mu_2^2 + \nu_2^2)^{3/2}} \exp\left(-\frac{X_1^2}{\mu_1^2 + \nu_1^2} - \frac{X_2^2}{\mu_2^2 + \nu_2^2}\right), \tag{18}$$

$$w_{\text{ent}}^s(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{1}{\pi(\mu_1^2 + \nu_1^2)^{1/2}(\mu_2^2 + \nu_2^2)^{1/2}} \times \left(\frac{X_1^2}{\mu_1^2 + \nu_1^2} + \frac{X_2^2}{\mu_2^2 + \nu_2^2} + \frac{2X_1X_2(\mu_1\mu_2 + \nu_1\nu_2)}{(\mu_1^2 + \nu_1^2)(\mu_2^2 + \nu_2^2)}\right) \exp\left(-\frac{X_1^2}{\mu_1^2 + \nu_1^2} - \frac{X_2^2}{\mu_2^2 + \nu_2^2}\right). \tag{19}$$

#### 4. Dynamics of Tomograms for Hamiltonians Quadratic in the Position and Momentum Operators

In this Section, we consider the evolution of the center-of-mass tomogram of systems with Hamiltonians, which are quadratic in the position and momentum operators. The integrals of motion of such systems are linear in the position and momentum operators [57]. This allows one to obtain the time dependence of the center-of-mass tomogram describing the quantum state. Indeed, the density operator evolves as

$$\hat{\rho}(t) = \hat{u}(t)\hat{\rho}(0)\hat{u}^\dagger(t), \tag{20}$$

where  $\hat{u}(t) = \exp(-it\hat{H})$  is the evolution operator. The center-of-mass tomogram corresponding to the state  $\hat{\rho}(t)$  is expressed in terms of the position and momentum operators,  $\hat{q}^H(t) = \hat{u}^\dagger(t)\hat{q}\hat{u}(t)$  and  $\hat{p}^H(t) = \hat{u}^\dagger(t)\hat{p}\hat{u}(t)$ ; in the Heisenberg representation, it is

$$w(X|\mu_1, \mu_2, \nu_1, \nu_2; t) = \text{Tr}(\hat{\rho}(t)\delta(X - \mu_1\hat{q}_1 - \nu_1\hat{p}_1 - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)) = \text{Tr}\left(\hat{\rho}(0)\delta\left(X - \mu_1\hat{q}_1^H(t) - \nu_1\hat{p}_1^H(t) - \mu_2\hat{q}_2^H(t) - \nu_2\hat{p}_2^H(t)\right)\right). \tag{21}$$

As the first example, let us consider a two-dimensional harmonic oscillator,

$$\hat{H} = \frac{\hat{p}_1^2}{2} + \frac{\hat{q}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \frac{\hat{q}_2^2}{2}. \tag{22}$$

The position and momentum operators in the Heisenberg representation have the form [46]

$$\begin{aligned} \hat{q}_1^H(t) &= \hat{q}_1 \cos t + \hat{p}_1 \sin t, & \hat{q}_2^H(t) &= \hat{q}_2 \cos t + \hat{p}_2 \sin t, \\ \hat{p}_1^H(t) &= -\hat{q}_1 \sin t + \hat{p}_1 \cos t, & \hat{p}_2^H(t) &= -\hat{q}_2 \sin t + \hat{p}_2 \cos t. \end{aligned} \tag{23}$$

The center-of-mass tomogram can be rewritten as

$$w(X|\mu_1, \mu_2, \nu_1, \nu_2; t) = w(X|\mu_1^H(t), \mu_2^H(t), \nu_1^H(t), \nu_2^H(t), t = 0), \tag{24}$$

where the time-dependent parameters are [46]

$$\begin{aligned} \mu_1^H(t) &= \mu_1 \cos t - \nu_1 \sin t, & \mu_2^H(t) &= \mu_2 \cos t - \nu_2 \sin t, \\ \nu_1^H(t) &= \mu_1 \sin t + \nu_1 \cos t, & \nu_2^H(t) &= \mu_2 \sin t + \nu_2 \cos t. \end{aligned} \tag{25}$$

In this way, the evolution of the center-of-mass tomogram for quadratic Hamiltonians can be obtained by the corresponding time-dependent transformation of the parameters of the initial tomogram. If the initial state of the system is the entangled state (12), then after the evolution, the state is described by

$$w_{\text{ent}}(X|\mu_1, \mu_2, \nu_1, \nu_2; t) = \frac{e^{-X^2/\sigma}}{\sqrt{\pi\sigma}} \times \left( \frac{1}{2} - \frac{\mu_1^H(t)\mu_2^H(t) + \nu_1^H(t)\nu_2^H(t)}{\sigma} + \frac{X^2}{\sigma} \left( 1 + \frac{2(\mu_1^H(t)\mu_2^H(t) + \nu_1^H(t)\nu_2^H(t))}{\sigma} \right) \right). \tag{26}$$

The other example is a two-dimensional inverted oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2} - \frac{\hat{q}_1^2}{2} + \frac{\hat{p}_2^2}{2} - \frac{\hat{q}_2^2}{2}. \tag{27}$$

The center-of-mass tomogram of this system, which is initially in the entangled state, has the form (26), where the parameters are given by [46]

$$\begin{aligned} \mu_1^H(t) &= \mu_1 \cosh t + \nu_1 \sinh t, & \mu_2^H(t) &= \mu_2 \cosh t + \nu_2 \cosh t, \\ \nu_1^H(t) &= \mu_1 \sinh t + \nu_1 \cosh t, & \nu_2^H(t) &= \mu_2 \sinh t + \nu_2 \cosh t. \end{aligned} \tag{28}$$

### 5. Cluster Tomography

States of quantum systems with several degrees of freedom can be described by cluster tomograms. The cluster tomogram for a system with three degrees of freedom is defined as follows [31]:

$$w^{\text{cl}}(X, X_3|\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = \text{Tr}(\hat{\rho} \delta(X - \mu_1\hat{q}_1 - \nu_1\hat{p}_1 - \mu_2\hat{q}_2 - \nu_2\hat{p}_2) \times \delta(X_3 - \mu_3\hat{q}_3 - \nu_3\hat{p}_3)). \tag{29}$$

It is a conditional probability distribution of variables  $X$  and  $X_3$  related to the center-of-mass positions of the first and second subsystems with two and one degrees of freedom, respectively. The positions are measured in rotated and scaled reference frames determined by parameters  $\mu_1, \nu_1, \mu_2, \nu_2, \mu_3,$  and  $\nu_3$ . The cluster tomogram for a pure state with the wave function  $\psi$  reads

$$w^{\text{cl}}(X, X_3|\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = \int dY_1 dY_2 \frac{\delta(X - Y_1 - Y_2)}{4\pi^2|\nu_1\nu_2|} \times \left| \int dq_1 dq_2 dq_3 \psi(q_1, q_2, q_3) \exp\left(\frac{i\mu_1}{2\nu_1}q_1^2 + \frac{i\mu_2}{2\nu_2}q_2^2 + \frac{i\mu_3}{2\nu_3}q_3^2 - \frac{iY_1}{\nu_1}q_1 - \frac{iY_2}{\nu_2}q_2 - \frac{iX_3}{\nu_3}q_3\right) \right|^2. \tag{30}$$

Now, we calculate the cluster tomogram of the  $W$  state, which is an entangled state of a three-dimensional oscillator of the form

$$\begin{aligned} \psi_W(q_1, q_2, q_3) &= \frac{1}{\sqrt{3}} \left( \psi_0(q_1)\psi_0(q_2)\psi_1(q_3) + \psi_0(q_1)\psi_1(q_2)\psi_0(q_3) \right. \\ &\quad \left. + \psi_1(q_1)\psi_0(q_2)\psi_0(q_3) \right). \end{aligned} \tag{31}$$

The cluster tomogram of the state  $W$  reads



$$w_W^{cl}(X, X_3 | \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = \frac{2}{3\pi} \frac{e^{-\frac{X_3^2}{\sigma_3} - \frac{X^2}{\sigma_{12}}}}{\sqrt{\sigma_3 \sigma_{12}}} \left( \frac{X_3^2}{\sigma_3} + X^2 \frac{(\mu_1 + \mu_2)^2 + (\nu_1 + \nu_2)^2}{\sigma_{12}^2} + 2 X X_3 \frac{\mu_1 \mu_3 + \nu_1 \nu_3 + \mu_2 \mu_3 + \nu_2 \nu_3}{\sigma_3 \sigma_{12}} + \frac{(\mu_1 - \mu_2)^2 + (\nu_1 - \nu_2)^2}{2 \sigma_{12}} \right), \quad (32)$$

where  $\sigma_3 = \mu_3^2 + \nu_3^2$  and  $\sigma_{12} = \mu_1^2 + \nu_1^2 + \mu_2^2 + \nu_2^2$ . The tomogram of the state  $W$  satisfies the normalization condition

$$\int w_W^{cl}(X, X_3 | \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) dX dX_3 = 1. \quad (33)$$

### 6. Conclusions

To conclude, let us point out the main results of this study.

We considered the tomographic picture of quantum mechanics, where the states of quantum systems are described by tomograms. The tomograms are fair probability distribution functions, in contrast to quasi-probability functions like the Wigner function. This fact allows one to transfer some properties of quantum systems to classical probability theory. In this paper, we focused on center-of-mass tomographic probability distribution [30]. We introduced entangled probability distributions for center-of-mass tomography; they correspond to entangled states of quantum systems. Such probability distributions have not been discussed in classical probability theory. We considered examples of two-dimensional usual harmonic and inverted oscillators. Also, we studied symplectic and cluster tomographic probability distributions [31] for the oscillator states. We used the method of integrals of motion [32] to determine the time evolution of tomograms.

We constructed new kinds of probability distribution functions, which earlier had not been known in the probability theory describing the classical randomness phenomena. One of such new probability distributions is the center-of-mass probability distribution, introduced for the description of states of quantum harmonic oscillators. The other new probability distribution is the cluster tomographic probability distribution. The introduced probability distribution functions have a specific property. They describe systems with many subsystems. For center-of-mass tomography, only one variable is enough; however, for cluster tomography, few random variables are needed. Also, these probability distributions are able to describe the quantum phenomenon of entanglement, which is not available for classical systems. In view of these circumstances, we call these probability distributions the entangled probability distributions. Such entangled probability distributions had not been known in classical probability theory.

The approach to finding and studying entangled probability distributions is based on constructing probability representations of quantum states. This involves using wave functions with the property that linear superpositions of these functions belong to a set that determines the probability distributions. In classical mechanics, the superpositions of classical trajectories do not have such a property. The superposition of two solutions of Newton equation is not the solution to the Newton equation. Thus, only in the quantum world can we obtain entangled probability distributions that describe the system states, which are represented by superpositions of the system’s wave functions.

In quantum mechanics, the system states are separable and entangled. For separable states, density operators determining the states are expressed as convex sums of direct products of density operators of the subsystems. After constructing the probability representation of system separable states, we see that the probability distribution is equal to the convex sum of the products of probability distributions determining the subsystem states. For entangled states, density operators cannot be expressed as the convex sum of the products of probability distributions determining the subsystem states. This means that the probability distributions determining the entangled quantum states of the systems cannot

be represented as convex sums of the products of probability distributions determining the probability distributions of the subsystems. Simple examples illustrating these properties are presented in Ref. [46].

Some examples of entangled states, which are Schrödinger cat states constructed for two qubits and two oscillating qubits, are given in Ref. [58]; see also the following recent statement [59]. Entangled probability distributions determining these Schrödinger cat states were not discussed in conventional probability theory. In a future publication, we aim to consider the possibility of constructing classical analogs of quantum center-of-mass probability distributions and study their properties. Also, we consider to study the entropic characteristics of center-of-mass probability distributions introduced for both quantum and classical systems.

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### Appendix A

Hereafter, we consider a system, which consists of two subsystems with numbers of degrees of freedom  $N_1$  and  $N_2$ . In order to find the relation between tomograms of the system and its subsystems, we consider the density operator of a separable state,

$$\hat{\rho}_{12} = \sum_k p_k \hat{\rho}_1^{(k)} \otimes \hat{\rho}_2^{(k)}, \tag{A1}$$

where  $p_k \geq 0$  and  $\sum_k p_k = 1$ . The corresponding symplectic tomogram is given by the convex sum of the symplectic tomograms of the subsystems; it has the form

$$w^s(\vec{X}_1, \vec{X}_2 | \vec{\mu}_1, \vec{\mu}_2, \vec{v}_1, \vec{v}_2) = \sum_k p_k w_{1,k}^s(\vec{X}_1 | \vec{\mu}_1, \vec{v}_1) w_{2,k}^s(\vec{X}_2 | \vec{\mu}_2, \vec{v}_2). \tag{A2}$$

Here, the variables are vectors with components  $\vec{X}_1 = (X_{1j})$ ,  $\vec{\mu}_1 = (\mu_{1j})$ ,  $\vec{v}_1 = (v_{1j})$ , where  $j = 1, 2, \dots, N_1$ , and  $\vec{X}_2 = (X_{2p})$ ,  $\vec{\mu}_2 = (\mu_{2p})$ ,  $\vec{v}_2 = (v_{2p})$ , where  $p = 1, 2, \dots, N_2$ .

The connection between the symplectic and the center-of-mass tomograms presented in Ref. [31] allows one to obtain the relation between the center-of-mass tomogram of the system and its subsystems. Indeed, the center-of-mass tomogram of the state (A1) reads

$$w(X | \vec{\mu}_1, \vec{\mu}_2, \vec{v}_1, \vec{v}_2) = \int w^s(\vec{X}_1, \vec{X}_2 | \vec{\mu}_1, \vec{\mu}_2, \vec{v}_1, \vec{v}_2) \delta\left(X - \sum_j X_{1j} - \sum_p X_{2p}\right) d\vec{X}_1 d\vec{X}_2, \tag{A3}$$

where the integral is taken over the components of the vectors  $\vec{X}_1$  and  $\vec{X}_2$ . Next, we use Equation (A2) to obtain

$$w(X | \vec{\mu}_1, \vec{\mu}_2, \vec{v}_1, \vec{v}_2) = \sum_k p_k \int \delta\left(X - \sum_j X_{1j} - \sum_p X_{2p}\right) \times w_{1,k}^s(\vec{X}_1 | \vec{\mu}_1, \vec{v}_1) w_{2,k}^s(\vec{X}_2 | \vec{\mu}_2, \vec{v}_2) d\vec{X}_1 d\vec{X}_2. \tag{A4}$$



We express the symplectic tomograms of the subsystem in terms of the center-of-mass tomograms, that is

$$w_{1,k}^s(\vec{X}_1|\vec{\mu}_1, \vec{v}_1) = \frac{1}{(2\pi)^{N_1}} \int w_{1,k}(\zeta_1|\vec{k}_1 \circ \vec{\mu}_1, \vec{k}_1 \circ \vec{v}_1) e^{i(\zeta_1 - \vec{k}_1 \cdot \vec{X}_1)} d\vec{k}_1 d\zeta_1, \tag{A5}$$

$$w_{2,k}^s(\vec{X}_2|\vec{\mu}_2, \vec{v}_2) = \frac{1}{(2\pi)^{N_2}} \int w_{2,k}(\zeta_2|\vec{k}_2 \circ \vec{\mu}_2, \vec{k}_2 \circ \vec{v}_2) e^{i(\zeta_2 - \vec{k}_2 \cdot \vec{X}_2)} d\vec{k}_2 d\zeta_2. \tag{A6}$$

Here,  $\vec{a} \circ \vec{b}$  denotes the vector with components  $\vec{a} \circ \vec{b} = (a_j b_j)$ , where  $\vec{a} = (a_j)$  and  $\vec{b} = (b_j)$ .

$$w(X|\vec{\mu}_1, \vec{\mu}_2, \vec{v}_1, \vec{v}_2) = \frac{1}{(2\pi)^N} \sum_k p_k \int w_{1,k}(\zeta_1|\vec{k}_1 \circ \vec{\mu}_1, \vec{k}_1 \circ \vec{v}_1) w_{2,k}(\zeta_2|\vec{k}_2 \circ \vec{\mu}_2, \vec{k}_2 \circ \vec{v}_2) \times \delta\left(X - \sum_j X_{1j} - \sum_p X_{2p}\right) e^{i(\zeta_1 + \zeta_2 - \vec{k}_1 \cdot \vec{X}_1 - \vec{k}_2 \cdot \vec{X}_2)} d\vec{X}_1 d\vec{X}_2 d\vec{k}_1 d\vec{k}_2 d\zeta_1 d\zeta_2, \tag{A7}$$

where  $N = N_1 + N_2$  is the number of degrees of freedom of the whole system.

We conclude that the center-of-mass tomogram of a separable state is presented in the form (A7).

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