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Review

Classicalization of Quantum Mechanics: Classical Radiation Damping Without the Runaway Solution

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Abstract: In this paper, we review a new treatment of classical radiation damping, which resolves a known contradiction in the Abraham–Lorentz equation that has long been a concern. This radiation damping problem has already been solved in quantum mechanics by the method introduced by Friedrichs. Based on Friedrichs' treatment, we solved this long-standing problem by classicalizing quantum mechanics by replacing the canonical commutation relation from quantum mechanics with the Poisson bracket relation in classical mechanics.

Keywords: classical radiation damping; resonance singularity; non-Hilbert space; the Abraham–Lorentz equation; classicalization of quantum mechanics; cyclotron motion; waveguide; classical Van Hove singularly; anomalous damping

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1. Introduction

The radiation of light from electrons as they accelerate or change their motion plays an essential role in a wide range of physical phenomena. This includes cyclotron motion, which appears in diverse contexts such as astrophysics, plasma physics, and accelerator light source technology.

However, in the known Abraham–Lorentz equation proposed to describe such classical decay processes, a term appears in the force equation for the charged particle that contains the third-order derivative of the particle position with respect to time. Therefore, it conflicts with basic principle of physics, and the theoretical underpinnings of this phenomenon have remained unresolved [1–5]. Specifically, we encounter *runaway solutions* caused by the third-order differential terms, and *acausal behavior* resulting from Dirac initial conditions proposed to solve the first problem [2].

This problematic behavior in the Abraham–Lorentz equation can be traced back to the fact that the Liénard–Wiechert potentials are used to derive the electromagnetic field emitted by the particle as the latter accelerates. The problem can be revealed when one considers the energy associated with the field emitted by the charged particle in comparison with the energy lost by the particle itself. In principle, these should be balanced. However, let us note that the emitted field consists of two components: the field component that escapes without further interacting with the particle, and the self-interacting component that instead attaches itself to the particle and forms the particle-field dressing [6].

The derivation of the Abraham–Lorentz equation ignores the energy associated with the field dressing, and hence cannot provide a solution that is fully consistent with basic physical principles. Indeed, we have shown in Ref. [7] that when the particle decays, there is also energy release from the field that dresses the particle.

Furthermore, as discussed in this paper, the essence of this problem is related to one of the most fundamental problems in physics, i.e., how to derive a damping solution that

breaks time symmetry from the basic equation of motion that is symmetric in time. Therefore, we cannot expect a theoretical solution to this problem unless the exact mechanical basis for the breaking of symmetry in time is clarified. One of the authors of this paper has worked on the problem of symmetry breaking for many years together with Ilya Prigogine, and has established that the essence lies in the following four points [8–10]:

(1) A resonance singularity may appear in the dynamical solutions for which the frequency denominator becomes zero (see also relevant studies [11,12]). This is the so-called *small denominator problem* related to non-integrability and instability, which may lead to chaos of dynamical systems.
(2) However, even if the denominator becomes zero, if the frequency has a continuous spectrum, by analytically continuing the frequency in the denominator into the complex plane, the contribution from the part divided by zero can be mathematically consistently treated as a δ function. (Note that because the δ function has even parity with respect to its variable, the appearance of this distribution breaks the parity symmetry with respect to the sign reversal of the frequency denominator as an odd function.)
(3) The direction of this analytical continuation defines two Riemann surfaces, one when analytically continuing from the lower half of the complex frequency plane, and the other when analytically continuing from the upper half. Then, the spectrum on one of these Riemann surfaces gives a solution that is consistent with the future, and the other will give the time-reversed solution.
(4) As a result, solutions that break time symmetry can be obtained in a manner that is mathematically consistent with the fundamental laws of physics.

In reaching the above conclusion, what was decisive was Kurt Friedrichs' historical discovery regarding the emission of photons in quantum mechanics [13]. Let us consider the problem of an atom decaying into the vacuum of free space. Friedrichs' surprising discovery was that if a resonance singularity occurs because the discrete spectrum of energy in the unperturbed system overlaps with the continuous spectrum, then when the perturbation is applied to that system, the discrete spectrum can disappear from the spectrum of the total Hamiltonian that includes the interaction and, further, that a complete set of the spectrum can be obtained that consists *only* of the continuous spectrum. As a result of the disappearance of the discrete spectrum from the complete set, it has become clear that the initially prepared state, which was stable in the unperturbed system, decays exponentially and breaks time-symmetry without any contradiction with the fundamental laws in quantum mechanics under the influence of the resonance interaction. Therefore, the description of quantum radiation damping based on the fundamental laws has been established in quantum mechanics, in contrast to the situation in classical mechanics.

In this paper, we show that by replacing the canonical commutation relation for annihilation and creation operators (in the second quantized formalism of quantum mechanics) with the Poisson bracket relation for classical normal modes (in classical mechanics), we can similarly describe classical radiation damping without any contradiction with the fundamental laws of classical mechanics. This replacement procedure is, one can say, a *classicalization* of the corresponding quantum system. This idea has already been used in Ref. [14], but this is the first time it has been discussed in the context of the Abraham–Lorentz equation.

As an application of the classical treatment of radiation damping, we briefly mention the anomalous damping that occurs when electrons in a waveguide experience cyclotron motion with a frequency close to the lower limit of the continuous modes in a waveguide (the *cutoff frequency*), which we recently studied [15]. This anomalous damping is a consequence of a singularity in the density of states at the cutoff in frequency space, corresponding to the Van Hove singularity that is known in quantum systems [16–19]. This anomalous damping cannot be described using the existing Abraham–Lorentz equations.

2. Issues with the Abraham–Lorentz Equation

The Abraham–Lorentz equation is given by

$$m\frac{d^2x}{dt^2} = F_{\text{ext}}(x) + m\tau\frac{d^3x}{dt^3}. \tag{1}$$

Here, x is a coordinate, t denotes the time, F_{ext} is the external force, and $\tau \equiv 2e^2/(3mc^3)$, which can be considered the time required for light to pass the classical electron radius with c, e, and m denoting the speed of light, the elementary charge, and the mass of the electron, respectively. This equation has the following formal solution for the acceleration $a(t) \equiv d^2x/dt^2$:

$$a(t) = \left[a(0) - \frac{1}{m\tau}\int_0^t dt' F_{\text{ext}}\big(x(t')\big)e^{-t'/\tau}\right]e^{t/\tau}. \tag{2}$$

Therefore, even in the case of $F_{\text{ext}}(t) = 0$, there is a runaway solution that immediately diverges with time when $a(0) \neq 0$.

Paul Dirac showed that this runaway solution can be removed by choosing the initial condition,

$$a(0) = \frac{1}{m\tau}\int_0^\infty dt' F_{\text{ext}}\big(x(t')\big)e^{-t'/\tau}. \tag{3}$$

However, because future information about F_{ext} determines this initial condition, the latter violates causality.

One solution to this complication is a treatment proposed by Lev Landau and Evgeny Lifshitz [1,20,21]. However, as explained in detail in Section 4 below, this method does not eliminate the unphysical elements of the Abraham–Lorentz equation, but instead accepts the problematic theoretical description as it is and attempts to derive a damping solution. In contrast, the method suggested by us, which is explained below, avoids such unphysical features from the beginning.

3. Classic Friedrichs' Model

Let us consider a one-dimensional model where a dipole molecule, which has a charge of $Z_a e = Ze$ on one end and charge $Z_b e = -Ze$ on the other end (with Z denoting the atomic number), is oscillating on the x-axis. These charges couple to the scalar field in free space. One can write the Hamiltonian for this model as follows:

$$\begin{aligned} H = \sum_{i=a,b}\tfrac{1}{2m_i}\left[p_i + Z_i e\sqrt{\mu_0}\tfrac{\omega_1}{c}\phi(x_i)\right]^2 + \tfrac{1}{2}m_1\omega_1^2(x_b - x_a)^2 \\ + \int_{-\infty}^{\infty} dx\left[\tfrac{c^2}{2}\pi^2(x) + \tfrac{1}{2}\left(\tfrac{\partial\phi(x)}{\partial x}\right)^2\right], \end{aligned} \tag{4}$$

where i represents the species of the particle at the dipole endpoints, and m_i and p_i are the mass and x-component momentum of these particles, respectively. The quantity x_i is the position where the dipole molecule and the field interact. Additionally, $m_1 \equiv m_a m_b/(m_a + m_b)$ is the reduced mass, μ_0 denotes the the vacuum permeability, and ω_1 is the frequency of the dipole. Furthermore, $\phi(x,t)$ and $\pi(x,t) \equiv (1/c^2)\partial\phi(x,t)/\partial t$ are the scalar field and its conjugate generalized momentum density, respectively. For simplicity, we set $Z_a = Z$ and $Z_b = -Z$.

In this paper, we follow the standard approximations by neglecting the field–field interaction terms that are proportional to the square of $\phi(x_i)$, and by assuming the interaction between the field and the charged particle occurs at the center of the dipole molecule x_c (dipole approximation), so that $\phi(x_i) \simeq \phi(x_c)$. Furthermore, introducing new variables as $P_1 \equiv p_a + p_b$, $M_1 \equiv m_a + m_b$, $x_1 \equiv x_b - x_a$, and $p_1 \equiv m_1 p_b/m_b - m_1 p_a/m_a$, and dropping

the kinetic energy $P_1^2/(2M)$ for the free motion of the center of the dipole molecule, one obtains the following approximation:

$$H \simeq \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2 x_1^2 + \int_{-\infty}^{\infty} dx \left[\frac{c^2}{2}\pi^2(x) + \frac{1}{2}\left(\frac{\partial\phi(x)}{\partial x}\right)^2\right] - \frac{Ze\sqrt{\mu_0}}{m_1}\frac{\omega_1}{c}p_1\phi(x_c). \quad (5)$$

Let us further rewrite the approximated Hamiltonian (5) by introducing the following normal modes of the harmonic oscillator and the fields, respectively:

$$\phi(x,t) = \frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_k}}\left(q_k e^{ikx} + q_k^* e^{-ikx}\right), \quad (6)$$

and

$$\pi(x,t) = \frac{-i}{2\sqrt{\pi}c}\int_{-\infty}^{\infty} dk \sqrt{\omega_k}\left(q_k e^{ikx} - q_k^* e^{-ikx}\right), \quad (7)$$

with $\omega_k = c|k|$, k denoting the light mode,

$$q_1 \equiv \sqrt{\frac{m_1\omega_1}{2}}\left(x_1 + i\frac{p_1}{m_1\omega_1}\right), \quad (8)$$

and

$$x_1 = \sqrt{\frac{1}{2m_1\omega_1}}(q_1 + q_1^*), \qquad p_1 = -i\sqrt{\frac{m_1\omega_1}{2}}(q_1 - q_1^*). \quad (9)$$

Similarly, for each mode of light k, one can introduce the coordinates and momentum of the corresponding harmonic oscillator using the relations

$$x_k = \sqrt{\frac{1}{\omega_k}}(q_k + q_k^*), \qquad p_k = -i\sqrt{\frac{\omega_k}{2}}(q_k - q_k^*). \quad (10)$$

Then, we rewrite the right-hand side of Equation (5) as

$$H \simeq \omega_1 q_1^* q_1 + \int_{-\infty}^{\infty} dk\, \omega_k q_k^* q_k + \lambda\int_{-\infty}^{\infty} dk (q_1 - q_1^*)(V_k q_k - V_k^* q_k^*), \quad (11)$$

where we defined the dimensionless coupling constant,

$$\lambda \equiv \sqrt{\frac{Z^2e^2\omega_1}{2m_1\varepsilon_0 c^3}}, \quad (12)$$

and the form factor,

$$V_k \equiv i\sqrt{\frac{\omega_1^2 c}{4\pi}}\frac{e^{ikx_c}}{\sqrt{\omega_k}}. \quad (13)$$

Let us note that the normal modes for the classical system obey the canonical form of the Poisson bracket:

$$\begin{aligned} \{q_1, q_1^*\} &= -i, \\ \{q_k, q_{k'}^*\} &= -i\delta(k-k'), \\ \{q_\alpha, q_\beta\} &= 0, \end{aligned} \quad (14)$$

where α, β are also 1 or k, but $\alpha \neq \beta$. $\delta(\cdot)$ is the Dirac delta function. Here, the Poisson bracket between any functions f and g that are expressed in terms of the normal modes is defined by the relation

$$\{f, g\} = -i \sum_{\alpha} \left(\frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial q_\alpha^*} - \frac{\partial f}{\partial q_\alpha^*} \frac{\partial g}{\partial q_\alpha} \right), \tag{15}$$

which is obtained from the usual definition of the the Poisson bracket,

$$\{f, g\} = \sum_{\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial x_\alpha} \right), \tag{16}$$

where the subscript α corresponds to the normal mode of the particle 1 or the field mode k. Notice that we use the conventional discrete notation as short-hand; for the continuous variable $\alpha = k$, the summation must be replaced with integration over k.

If one changes the Poisson bracket to the canonical commutation relation, then the second quantized form of the Hamiltonian is obtained. Furthermore, because the Hamiltonian (11) is in bilinear form in terms of the normal modes, it should be diagonalizable by a suitable Bogoliubov transformation. However, in the case of optical emission, the procedure is not as straightforward because of the resonance singularity. To explain this, we use a simpler Hamiltonian called the Friedrichs model given in Equation (17) below, which was first used to solve the problem of the resonance singularity in quantum mechanics [13]. That solution presented by Friedrichs [13] makes this model historically important in that the phenomenological quantum jump proposed by Niels Bohr was first strictly derived from the Schrödinger equation, which is based on the fundamental laws of physics (however, in Friedrichs' paper [13], this system is discussed in the form of the ordinary Schrödinger equation, rather than second quantized form, which would include both point spectra and continuous spectra). As explained below, the Friedrichs model has a non-trivial solution under the influence of the resonance singularity.

By focusing on this resonance singularity, we find that a solution with broken time symmetry can be obtained from the time-symmetric equation of motion without mathematical contradiction [8].

The Friedrichs model [13] is obtained by applying the rotating wave approximation to the above Hamiltonian (11), which gives

$$H = \omega_1 q_1^* q_1 + \int_{-\infty}^{\infty} dk\, \omega_k q_k^* q_k - \lambda \int_{-\infty}^{\infty} dk (V_k^* q_1 q_k^* + V_k q_1^* q_k). \tag{17}$$

The mathematical advantage of this model is that the Bogoliubov transformation to diagonalize it is much simpler than that for the Hamiltonian (11) above, making it strightforward to see the essence of our treatment of the radiation damping problem [22]. However, once one understands the essence of the solution to this problem from the simpler model, one can deal with the more complicated form of the Hamiltonian in Equation (11) as well. In Section 4, we briefly discuss our method in the context of the classical radiation damping of electrons undergoing cyclotron motion inside a waveguide, which we studied in our recent paper [15]. In this case, even while retaining the counter-rotating terms in Equation (11), we found several interesting results that could not be obtained within the Abraham–Lorentz description.

The idea that we propose here in solving the classical radiation damping problem is to replace the quantum mechanical commutation relation with the Poisson bracket relation and, can be said, to classicalize the quantum system, since the Poisson bracket relation and the commutation relation are algebraically isomorphic. This means that an exact solution in quantum mechanics can be directly rewritten as an exact solution in classical mechanics. Because our present classical theory can be treated precisely in parallel with the quantum case, we refer to the present model as the *classical Friedrichs model*.

To clarify the role of the resonance singularity, let us first formally diagonalize the Hamiltonian (17) by the simplified Bogoliubov transformation (at this stage, since this is a formal solution, the analytic continuation with respect to the Cauchy integral has been intentionally left vague and unspecified), given by the following:

$$Q_1 = N_1\left(q_1 + \lambda\int_{-\infty}^{\infty} dk \frac{V_k}{\omega_k - \tilde{\omega}_1} q_k\right), \tag{18}$$

$$Q_k = q_k + \frac{-\lambda V_k^*}{\eta(\omega_k)}\left(q_1 + \int_{-\infty}^{\infty} dk' \frac{-\lambda V_{k'}}{\omega_k - \omega_{k'}} q_{k'}\right), \tag{19}$$

where

$$\eta(\zeta) \equiv \zeta - \omega_1 - \int_{-\infty}^{\infty} dk \frac{\lambda^2 |V_k|^2}{\zeta - \omega_k}, \tag{20}$$

with ζ denoting the variable with the dimension of frequency.

The third term of Equation (20) is the equivalent of the self-energy function in the quantum formalism [23]. Furthermore, N_1 is a normalization constant given by

$$N_1 \equiv \left[\frac{d\eta(\zeta)}{d\zeta}\right]^{1/2}_{\zeta=\tilde{\omega}_1}. \tag{21}$$

The renormalized real frequency $\tilde{\omega}_1$ associated with the mode Q_1 is determined by

$$\eta(\tilde{\omega}_1) = 0. \tag{22}$$

Then, one obtains the formal diagonalized Hamiltonian,

$$H = \tilde{\omega}_1 Q_1^* Q_1 + \int_{-\infty}^{+\infty} dk\, \omega_k Q_k^* Q_k. \tag{23}$$

However, the above diagonalization is a formal procedure that does not take into account the resonant singularity that occurs when passing through the point $\omega_k - \tilde{\omega}_1 = 0$ that appears in the energy denominator of the k-integral. In fact, no matter how one analytically continues the above formulation, the existence of this resonant singularity causes the above Q_1 to lead to a divergence during the diagonalization process of the Hamiltonian. This was a significant issue before Friedrichs's solution: did the resonant singularity in the energy denominator make it impossible to diagonalize the Hamiltonian describing the optical emission?

This situation is reminiscent of Poincaré's known non-integrability theorem in classical mechanics in systems with a small number of degrees of freedom [8]. However, when the spectrum of light is continuous as considered here, Friedrich discovered [13] that by making the appropriate analytic continuation, the Hamiltonian of this system can be diagonalized exactly without divergence as

$$H = \int_{-\infty}^{+\infty} dk\, \omega_k Q_k^* Q_k, \tag{24}$$

where the renormalized particle mode Q_1 is now absent. In this case, however, the Bogoliubov transformation and its inverse transformation is expressed as

$$Q_k = q_k + \frac{-\lambda V_k^*}{\eta^+(\omega_k)}\left(q_1 + \int_{-\infty}^{\infty} dk' \frac{-\lambda V_{k'}}{\omega_k - \omega_{k'} + i\varepsilon} q_{k'}\right), \tag{25}$$

where

$$q_1 = \int_{-\infty}^{\infty} dk \frac{-\lambda V_k}{\eta^-(\omega_k)} Q_k, \tag{26}$$

and

$$q_k = Q_k + \lambda^2 V_k^* \int_{-\infty}^{\infty} dk' \frac{V_{k'}}{\eta^-(\omega_{k'})(\omega_{k'} - \omega_k - i\varepsilon)} Q_{k'}, \tag{27}$$

where ε is a positive infinitesimal number. For brevity, we leave the limit $\varepsilon \to 0+$ implicit from this point forward. We use the analytic continuation indicated in Equation (25) for $t > 0$. Here, $\eta^{\pm}(\zeta)$ is analytically continued function to two Riemann sheets, defined as

$$\eta^{\pm}(\zeta) \equiv \zeta - \omega_1 - \int_{-\infty}^{\infty} dk \frac{\lambda^2 |V_k|^2}{[\zeta - \omega_k]^{\pm}}, \tag{28}$$

where the $\pm$ symbol of the notation $[\zeta - \omega_k]^{\pm}$ means that $\eta^{\pm}(\zeta)$ are consistent with $\eta^{\pm}(\omega) = \eta(\omega \pm i\varepsilon)$ with real ω. We can rewrite this expression with the integration over ω_k as

$$\eta^{\pm}(\zeta) \equiv \zeta - \omega_1 - \int_0^{\infty} d\omega_k \frac{dk}{d\omega_k} \frac{\lambda^2 |V_k|^2}{[\zeta - \omega_k]^{\pm}}, \tag{29}$$

where $dk/d\omega_k$ is the density of states.

Correspondingly, the renormalized normal modes (Q_α, Q_α^*) satisfy the Poisson brackets,

$$\{Q_k, Q_{k'}^*\} = -i\delta(k - k'), \tag{30}$$

$$\{Q_\alpha, Q_\beta\} = 0, \tag{31}$$

where the Poisson bracket can be expressed by replacing q_α by Q_α in Equation (15).

Note that if Q_1 did exist, then $Q_1^* Q_1$ would be an invariant of motion corresponding to $q_1^* q_1$ in the unperturbed system. Just as in the case of the classical non-integrable system with a small number of degrees of freedom, this invariant of the motion in the present case is destroyed by the resonance singularity. Friedrichs found that in an unstable system in which particles and fields interact, it is impossible to create canonical variables for renormalized particles and fields that behave independently of each other through a canonical transformation, and he discovered that the description can be complete without including the renormalized particles [13].

Now that the Hamiltonian has been diagonalized, the classical radiation damping problem for this system can be solved, without any runaway solutions, as follows. To solve the classical mechanics problem, we use the Poisson bracket and first introduce Liouvillian operator, L_H, by the following definition,

$$L_H f \equiv i\{H, f\}, \tag{32}$$

where the imaginary factor i has been introduced to ensure that the Liouvillian acts as a Hermitian operator within the Hilbert space. Then, the state function, $\rho(t)$, in the phase space obeys the Liouville equation,

$$i\frac{\partial}{\partial t}\rho(t) = L_H \rho(t), \tag{33}$$

and any dynamical variable $A(t)$ satisfies the following equation,

$$-i\frac{d}{dt}A(t) = L_H A(t). \tag{34}$$

Equation (33) in classical system under consideration corresponds to the Schrödinger representation, while Equation (34) corresponds to the Heisenberg representation, respectively. Note the sign of these two equations is different.

Since Q_k is a dynamical variable, we use Equation (34). Then, one obtains

$$Q_k(t) = e^{+iL_H t} Q_k(0). \tag{35}$$

In the representation that diagonalizes the Hamiltonian (24), the Liouvillian is given by

$$L_H = \sum_k \omega_k \left(Q_k^* \frac{\partial}{\partial Q_k^*} - Q_k \frac{\partial}{\partial Q_k} \right). \tag{36}$$

Using Equation (35), the equation of motion for the renormalized field mode Q_k reads

$$-i\frac{dQ_k}{dt} = L_H Q_k = -\omega_k Q_k. \tag{37}$$

Equation (37) shows that Q_k is an eigenfunction of the Liouvillian with the eigenvalue $-\omega_k$. We have the solution of this equation as

$$Q_k(t) = e^{-i\omega_k t} Q_k(0). \tag{38}$$

One can substitute this solution into the inverse Bogoliubov transformation (26) and perform the contour integral with respect to the continuous variable k to obtain the original normal mode q_1 for the harmonic oscillator. In doing so, one finds a contribution to the integration from the pole at $\zeta = \zeta_1$ coming from the factor $1/\eta^-(\zeta)$. Hence, we find that the original normal mode $q_1(t)$ of the electron has a contribution containing the factor $\exp[-i\zeta_1 t]$ that decays and oscillates according to the complex frequency $\zeta_1 = \tilde{\omega_1} - i\gamma$, where ζ_1 is the solution of the dispersion equation $\eta^-(\zeta_1) = 0$ in the lower-half complex plane in the second Riemann sheet with the decay rate $\gamma > 0$. This is the resonance eigenvalue $\zeta_1 = \tilde{\omega_1} - i\gamma$.

However, note that since the dispersion equation is an integral equation in the unknown variable ζ_1 and, furthermore, a nonlinear equation in ζ_1, it is generally quite a complicated task to compute the value of ζ_1 exactly if the result of the k-integration in (28) cannot be expressed as a function of ζ. Even in such a case, one can find the value of ζ_1 by the recurrence equation

$$\zeta_1^{(n+1)} = \omega_1 + \int_{-\infty}^{\infty} dk \frac{\lambda^2 |V_k|^2}{[\zeta_1^{(n)} - \omega_k]^-}, \tag{39}$$

where $\zeta_1^{(n)}$ is the n-th approximation starting with $\zeta_1^{(0)} = \omega_1$. For example,

$$\begin{aligned} \zeta_1^{(1)} &= \omega_1 + \int_{-\infty}^{\infty} dk \frac{\lambda^2 |V_k|^2}{\omega_1 + i\varepsilon - \omega_k} \\ &= \omega_1 + \int_{-\infty}^{\infty} dk\, \mathcal{P} \frac{\lambda^2 |V_k|^2}{\omega_1 - \omega_k} - i\pi \int_{-\infty}^{\infty} dk\, \lambda^2 |V_k|^2 \delta(\omega_1 - \omega_k), \end{aligned} \tag{40}$$

where $\mathcal{P}$ has a meaning of the principal part of the integration. The last term of Equation (40) gives the decay rate proportional to λ^2, which corresponds to the result obtained by the Fermi golden rule in quantum mechanics.

In addition, Equation (29) shows that there is a non-exponential contribution in time due to the branch point coming from the lower limit of the integration over ω_k [24–30]. This branch cut contribution cannot be evaluated by the Abraham–Lorentz equation. Furthermore, as discussed in more detail in Section 4 below, the contribution from the branch point effect gives rise to light emitted from the dressing attached to the charged particle in addition to the pole contribution mentioned above.

The branch point contribution becomes dominant on a very short and very long time scale compared with the relaxation time defined from the resonance eigenvalue by $t_r = 1/\gamma$ [24,31]. The short time contribution corresponds to the quantum Zeno effect in classical radiation damping [32].

In this way, one can describe the process of radiation damping without contradiction with the fundamental principles of mechanics represented by the runaway solution occurring in the Abraham–Lorentz equation. This means that, as far as the motion of the original modes interacting with each other is concerned, the phenomenon of radiation damping can be described in a manner that is consistent with fundamental principles such as energy conservation.

4. Discussion and Concluding Remarks

In this Section, we analyze in detail the derivation of the Abraham–Lorentz equation, compare it with our treatment of classical radiative damping here, and discuss what is incorrect with the Abraham–Lorentz equation that leads to unphysical results. Furthermore, we explain that although Landau and Lifshitz proposed a way to avoid the runaway solution, their approach does not solve the deeper problem that the Abraham–Lorentz formalism violates basic physical principles.

As known, in deriving the Abraham–Lorentz equations, the Liénard–Wiechert potential is used to derive the electromagnetic field emitted by the charged particle as the latter accelerates. In this case, the energy lost by the particle and the energy of the emitted field should be balanced. Note, however, that the emitted field consists of two components [2].

(a) The field component that escapes into space without further interaction with the particle (in the three-dimensional case, the energy of these fields are proportional to $1/R$, where R is the distance from the particle).
(b) The field component that dresses the particle (the energy of these fields are proportional to $1/R^2$).

In the case of the Friedrichs model [13], which can be solved exactly using the normal modes we discuss here, the dressing fields (b) are obtained as a contribution from the branch point that comes from the lower limit of the integral with respect to ω_k in the self-frequency function of Equation (29).

The field energy from the type (b) field component is neglected in deriving the Abraham–Lorentz equations, because this dressing field is considered incapable of carrying energy away to infinity due to its R-dependence. However, using the Friedrichs model, which allows us to rigorously predict the process of light emission, we have revealed in Ref. [7] that due to the interference effect between the branch point contribution and the pole contribution, the dressing field associated with the branch-point effect decays exponentially along with the exponential decay of the excited state of the bare particle. Since the dressing field decays exponentially in time, the emitted light can escape to infinity after all. At the same time, we have also shown [7] that the light energy from the dressing field emission is a higher-order effect with respect to the coupling constant λ than the light energy that originates solely from the pole contribution to the self-frequency function in Equation (29). Moreover, we have shown that when such higher-order effects can be ignored because $\lambda \ll 1$, then the result from the Fermi's golden rule calculated in Equation (40) provides a relatively good approximation, and it is consistent with the result for the decay rate obtained by Landau and Lifshitz based on the Abraham–Lorentz equation [1,20,21], as will be explained below.

However, if the frequency of the charged particle takes a value close to this branch point, the decay rate of the particle motion cannot be calculated by the perturbation expansion, even if $\lambda \ll 1$, because the effect from the branch point cannot be ignored. An interesting example that can be compared with an actual experiment is the light emission from an electron undergoing cyclotron motion in a waveguide. As we have shown in detail in Ref. [15], in the case of this cyclotron motion, the integration of the self-frequency

function with respect to ω_k can be performed explicitly. Further, in this case, the light disperses according to

$$\omega_k = \sqrt{c^2k^2 + \omega_c^2}. \tag{41}$$

Here, ω_c is the cut-off frequency, and the light in that mode behaves as if it had a mass determined by the cut-off frequency. The density of states for the electromagnetic field is given by

$$\frac{dk}{d\omega_k} = \frac{\omega_k}{c\sqrt{\omega_k^2 - \omega_c^2}}. \tag{42}$$

Equation (42) gives a branch point in complex ω_k-space with a divergence occurring at $\omega_k = \omega_c$ at the lower limit of the integration over ω_k in the self-frequency function. This is equivalent to the Van Hove singularity in solid state physics [16,17].

As has been shown in detail in Ref. [15], an interesting consequence of the Van Hove singularity is that when the frequency ω_1 of the particle is tuned near the cut-off frequency ω_c, the decay rate of the cyclotron motion (absolute value of the non-zero imaginary part of ζ_1) is intensely magnified, assuming a coupling constant is small: $\lambda \ll 1$. Indeed, the exact calculation reveals [15] that the decay rate acquires a $\lambda^{4/3}$ dependence which is non-analytic for $\lambda = 0$ near the Van Hove singularity. If one then gradually increases the value of ω_1 so that ω_1 takes a value significantly above the Van Hove singularity, the decay rate gradually decreases and eventually takes on the usual λ^2 dependence in agreement with Fermi's golden rule. Indeed, near the cut-off frequency, the amplification ratio is $\lambda^{4/3}/\lambda^2 = \lambda^{-2/3} \simeq 10^4 \gg 1$ for the typical case of electron cyclotron motion with $\lambda \sim 10^{-6}$.

At the same time, in paper [15], we have shown that when the frequency ω_1 of the particle is tuned near the cut-off frequency ω_c, the effect of the Van Hove singularity not only significantly changes the value of the decay rate of the cyclotron motion, but also significantly influences the effect of the deviation from exponential decay due to the branch point effect, and that this deviation cannot be ignored compared to the exponential decay.

Due to the non-analyticity at $\lambda = 0$, one cannot obtain this result by the standard perturbation analysis. Furthermore, since the emission of light from the type (b) component is ignored when deriving the Abraham–Lorentz equation, such interesting results are overlooked within that treatment of the problem.

To contextualize our results obtained here, to note is that the exponential decay rate of the cyclotron motion corresponding to that obtained from our self-frequency function has previously been evaluated based on the Greens function method [33,34]. Let us emphasize the following two new points we have raised relative to those pioneering works. First, we have shown here that in situations where the Van Hove singularity plays a crucial role in cyclotron motion, not only the exponential decay, but also non-exponential dynamics arising from the branch-point effect becomes essential. Second, and most crucially, in the current paper, we have shown that such calculations can be justified by recognizing that when solving the problem of classical radiation damping, it is possible to obtain a complete set describing the system dynamics that is spanned only by the renormalized modes for the fields, while the renormalized normal mode for the particle is absent. This possibility follows from Friedrichs seminal solution [13] for the quantum problem after applying our classicalization method.

In relation to the above discussion, let us also mention the treatment proposed by Landau and Lifshitz [1], which is said to not lead to runaway solutions from the Abraham–Lorentz equation. Landau and Lifshitz discussed the method in terms of the relativistic Abraham–Lorentz equation, but here we discuss it based on the methods used by Christie Eliezer [20] and by George Ford and Robert O'Connell [21] in the non-relativistic case. This

is a relatively simple method, which involves substituting the left side of the Abraham–Lorentz equation into the right side and applying it iteratively, as

$$m\frac{d^2x}{dt^2} = F_{\text{ext}}(x) + \tau\frac{d}{dt}\left[F_{\text{ext}}(x) + m\tau\frac{d^2x}{dt^2}\right] = F_{\text{ext}}(x) + \tau v\frac{d}{dx}F_{\text{ext}}(x) + \mathcal{O}(\tau^2), \tag{43}$$

with $v = dx/dt$. Here, τ is proportional to the square of the charge e (see Equation (1)) and, therefore, to λ^2.

Notice that the second term in the last line of this equation contains the velocity v, and hence breaks time-inversion symmetry. Further, this term yields exponential decay with decay rate proportional to $\tau \propto \lambda^2$ (similar to a friction term in mechanics). This gives the same result as the calculation using the Fermi golden rule mentioned above. Therefore, if this procedure converges, then by repeating this procedure an infinite number of times, we can obtain a contribution of any order of λ to the decay rate.

However, this method has the following three issues.

(1) Since the Abraham–Lorentz equation has runaway solutions, one is not ensured that this iterative procedure converges.
(2) As shown just above, in general, there are cases when perturbation calculations are impossible.
(3) Even if the iteration procedure converges, there is no point to calculate higher-order terms related to λ in the Abraham–Lorentz equation, since the emission of light from field component (b) is ignored.

In contrast, the method we presented here does not include the runaway solution, can be applied even when perturbative expansion is not possible, and also takes into account the emission of light from field component (b), so these issues do not come up.

Let us now remark on some noteworthy aspects of our method that is based on the Friedrichs model [13]. The purpose of deriving the Abraham–Lorentz equation was to investigate how a particle emits the field and decays when the particle and the field interact with each other. In this case, one would naively have the instinct to write the field variables as functions of the particle variables in the set of equations among the particle and the field that follow the fundamental laws of physics, and then use them to derive a closed equation using only the particle variables. However, this method has so far only led to equations involving the unphysical runaway solution for particle. As explained in this paper, Friedrichs discovered in Ref. [13] that in an unstable system in which particles and fields interact, it is impossible to create canonical variables for renormalized particles and fields that behave independently of each other through a canonical transformation; and, even more surprisingly, Friedrichs discovered that the description may be complete without including the renormalized particles, and therefore, in contrast to the above, the motion of the system can be correctly described by instead eliminating the particles from the description of the motion. Hence, we credit Friedrichs and his "reverse thinking" about the problem for providing the key to solving the controversial problem of the runaway solution in classical radiative damping.

Several recent experiments have studied the radiation reaction of electrons passing through aligned crystals and compared the resulting data to both "quantum" and "classical" analyses [35–38]. An interesting direction for future work would be to compare the results obtained by the method presented here with these experiments. Finally, it is interesting to consider our discussion here in relationship with the recent works by Zoltán Tulipánt [39], and Anton Ilderton and Greger Torgrimsson [40] on "deriving" the Abraham–Lorentz equation from QED. As shown above, our method explicitly incorporates Poincaré's non-integrability theorem, i.e., the destruction of the invariants of motion due to the resonance singularities, which is one of the notable results in the recent development of chaos theory in classical mechanics, and shows that this is an important key to understanding the problem of classical radiation damping. In their treatments, it would also be interesting to

clarify how this non-integrability is taken into account in order to understand this classical damping more profoundly.

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