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Applications of Supersymmetric Polynomials in Statistical Quantum Physics

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Abstract: We propose a correspondence between the partition functions of ideal gases consisting of both bosons and fermions and the algebraic bases of supersymmetric polynomials on the Banach space of absolutely summable two-sided sequences $\ell_1(\mathbb{Z}_0)$. Such an approach allows us to interpret some of the combinatorial identities for supersymmetric polynomials from a physical point of view. We consider a relation of equivalence for $\ell_1(\mathbb{Z}_0)$, induced by the supersymmetric polynomials, and the semi-ring algebraic structures on the quotient set with respect to this relation. The quotient set is a natural model for the set of energy levels of a quantum system. We introduce two different topological semi-ring structures into this set and discuss their possible physical interpretations.

Keywords: quantum ideal gas; grand partition function; supersymmetric polynomials on Banach spaces; algebraic basis; topological semi-ring; tropical semi-ring

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1. Introduction

Symmetric polynomial variables and relations between the bases of the algebra of symmetric polynomials are widely used in algebra, combinatorics (see [1]), and, in particular, in statistical quantum mechanics. In [2,3], Schmidt and Schnack proposed some correspondence between the relations in the algebra of symmetric polynomials and partition functions of bosons and fermions. Under this correspondence, one basis of symmetric polynomials is responsible for bosons and another for fermions. Such an approach was applied and developed for different cases by many authors (see, e.g., [4–8]). On the other hand, recently, some new results for the algebras of symmetric analytic functions on infinite-dimensional Banach spaces were obtained [9–14]. The infinite number of variables of the underlying space allows us to introduce some interesting algebraic operations on the spectra of such algebras that may have a physical meaning. In addition, in the infinite-dimensional case, we can consider the behavior of the ideal gas "at infinity" if, for example, the number of particles grows to infinity while the total energy of the system is bounded.

In [15–17], supersymmetric polynomials and analytic functions of abstract Banach spaces were considered. The supersymmetric polynomials of several variables were studied in [18–20]. It seems to be that some bases of supersymmetric polynomials give us a tool for the investigation of a quantum ideal gas consisting of both bosons and fermions. Moreover, supersymmetric polynomials define a relation of equivalence on the underlying vector space and the quotient set with respect to this relation, which looks like the most natural model for the set of energy levels of a given quantum system. Such a set admits some algebraic semi-ring structures that are related, in particular, to tropical (idempotent) mathematics.

In this paper, we discuss the relations between the algebras of supersymmetric polynomials on Banach spaces and the partition functions of bosons and fermions and consider some new algebraic structures on the set of energy levels of the corresponding quantum systems.

In Section 2, we gather the basic known information about the algebraic bases of symmetric polynomials on the Banach space, ℓ_1, and their relations to the partition functions of ideal quantum gases. In Section 3, we consider the algebraic bases of supersymmetric polynomials and discuss their relations to the partition functions of ideal gases consisting, simultaneously, of bosons and fermions. In Section 4, we construct two different semi-ring structures on a set of energy levels. The first one is related to the algebraic operations that were introduced in [17] for a more general case. The second is related to the idempotent operation, max, and looks like an infinite-dimensional generalization of the tropical semi-ring: $\mathbb{R} \cup +\infty$ (c.f. [21]).

General information on the polynomials and analytic functions on abstract Banach spaces can be found in [22,23]. Idempotent analysis and tropical semi-rings are considered in [24,25].

2. Preliminary Results for Symmetric Polynomials and Partition Functions

2.1. Symmetric Polynomials

Let $\mathbb{N}$ be the set of all positive integers, and ℓ_1 be the Banach space of all absolutely summing complex sequences $x = (x_1, \ldots, x_n, \ldots)$, with a norm of $\|x\| = \sum_{n=1}^{\infty} |x_n|$. The function f on ℓ_1 is called symmetric if

$$f\big((x_{\sigma(1)}, x_{\sigma(2)}, \ldots)\big) = f((x_1, x_2, \ldots))$$

for every $(x_1, x_2, \ldots) \in \ell_1$ and every bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

Let us define the following symmetric polynomials on ℓ_1. Let the polynomial F_n be defined by

$$F_n((x_1, x_2, \ldots)) = \sum_{i=1}^{\infty} x_i^n, \tag{1}$$

where $n \in \mathbb{N}$. The polynomial F_n is called a power sum symmetric polynomial. Let us define polynomial B_n as

$$B_n((x_1, x_2, \ldots)) = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \tag{2}$$

where $n \in \mathbb{N}$. The polynomial B_n is called a complete symmetric polynomial. Let the polynomial G_n be defined by

$$G_n((x_1, x_2, \ldots)) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \tag{3}$$

where $n \in \mathbb{N}$. The polynomial G_n is called an elementary symmetric polynomial.

Definition 1. *A linear combination of the finite products of the powers (zero powers are also allowed) for the elements of an algebra is called an algebraic combination of these elements.*

A subset of an algebra is called algebraically independent if zero elements of the algebra cannot be represented as a nontrivial algebraic combination of the elements of this subset.

An algebraically independent subset of an algebra is called an algebraic basis of this algebra if every element of the algebra can be represented as an algebraic combination of the elements of the subset. Due to algebraic independence, every such representation is unique.

Let $\mathcal{P}_s(\ell_1)$ denote the algebra of all continuous symmetric complex-valued polynomials on ℓ_1. Every set of polynomials, $\{F_n\colon n \in \mathbb{N}\}$, $\{B_n\colon n \in \mathbb{N}\}$, and $\{G_n\colon n \in \mathbb{N}\}$, is an algebraic basis in $\mathcal{P}_s(\ell_1)$ (see, e.g., [9,13]). There are so-called Newton recurrent formulas connecting different algebraic bases:

$$mG_m = \sum_{k=1}^{m} (-1)^{k-1} G_{m-k} F_k, \quad m \in \mathbb{N}, \tag{4}$$

$$mB_m = \sum_{k=1}^{m} B_{m-k} F_k, \quad m \in \mathbb{N}, \tag{5}$$

$$G_m = \sum_{k=1}^{m} (-1)^{k-1} G_{m-k} B_k, \quad m \in \mathbb{N}, \tag{6}$$

and

$$B_m = \sum_{k=1}^{m} (-1)^{k-1} B_{m-k} G_k, \quad m \in \mathbb{N}. \tag{7}$$

Let $\mathcal{B}(x)(t)$ and $\mathcal{G}(x)(t)$ be the so-called generating functions for polynomials B_n and G_n, respectively, defined as the following formal series:

$$\mathcal{B}(x)(t) = \sum_{n=0}^{\infty} t^n B_n(x), \quad B_0 = 1 \tag{8}$$

and

$$\mathcal{G}(x)(t) = \sum_{n=0}^{\infty} t^n G_n(x), \quad G_0 = 1. \tag{9}$$

The following relations are well-known ([1], p. 3):

$$\mathcal{G}(x)(t) = \exp\left(- \sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n} \right) \quad \text{and} \quad \mathcal{B}(x)(t) = \exp\left(\sum_{n=1}^{\infty} t^n \frac{F_n(x)}{n} \right), \tag{10}$$

and they immediately imply that

$$\mathcal{G}(x)(t)\mathcal{B}(-x)(t) = 1. \tag{11}$$

Here, the equality holds for every $x \in \ell_1$ and for every t in the common domain of convergence. Note that $\mathcal{G}(x)(t)$ is a well-defined analytic function of $x \in \ell_1$ for every fixed $t \in \mathbb{C}$ and an exponential-type function of t for every fixed x [26].

2.2. Partition Functions

The canonical partition function plays a fundamental role in statistical mechanics since most thermodynamic functions can be derived from it [3]. It is defined by

$$Z_N(\beta) = \operatorname{Tr} \exp(-\beta H), \tag{12}$$

where H denotes the Hamiltonian of the system, N is the number of particles, and

$$\beta = \frac{1}{k_B T}$$

denotes the inverse temperature (k_B is the Boltzmann constant, and T is the temperature). In other words, H is a self-adjointed operator such that $\exp(-\beta H)$ is a trace class operator for $\beta \in \mathbb{R}$.

The grand canonical partition function is defined by

$$Z(z,\beta) = \sum_{N=0}^{\infty} Z_N(\beta) z^N, \tag{13}$$

where the variable z is physically interpreted as the fugacity of the system, i.e., $z = \exp(\mu/(k_B T))$ (μ is the chemical potential). It describes the system in which the number of particles can be changed. The physical interpretation implies that z must be non-negative.

Note that the partition function completely defines all possible states of the system. Moreover, it can be used for deriving the likelihood of states.

Consider the ideal gas consisting of non-interacting identical particles (bosons or fermions). In this case, the Hamiltonian H is the sum of N identical single-particle Hamiltonians:

$$H = \sum_{n=1}^{N} h_n.$$

Let E_i be single-particle energy eigenvalues counted in such a way that several E_i have the same value in the case of degeneracy. In [27], it is shown that

$$Z_N(\beta) = B_N((x_1, x_2, \ldots)) \tag{14}$$

for the system of bosons and

$$Z_N(\beta) = G_N((x_1, x_2, \ldots)) \tag{15}$$

for the system of fermions, where B_N is defined by (2), G_N is defined by (3), and

$$x_i = \exp(-\beta E_i). \tag{16}$$

Note that Z_N is a symmetric function between energy levels, not between particles. The ordering of levels needed for (14) and (15) is simple for one-dimensional systems, but this is potentially difficult in higher dimensions due to the degeneracies of energy levels and the use of multi-indices to characterize them.

According to (8), (9), (13), (14), and (15), the grand canonical partition function can be represented in the form

$$Z(z,\beta) = \mathcal{B}((x_1, x_2, \ldots))(z)$$

for bosons and

$$Z(z,\beta) = \mathcal{G}((x_1, x_2, \ldots))(z)$$

for fermions, where x_i are defined by (16). In addition, according to [2], the co-ordinates $(x_1, x_2, \ldots)$ of $x \in \ell_1$ correspond to the abstract energy levels of the system; a monomial $x_1^{n_1} \cdots x_m^{n_m}$, $n_1 + \cdots + n_m = N$ in a partition function corresponds to the possible occupation of levels $x_1, \ldots, x_m$ by N particles. Moreover, there exists a so-called *fundamental symmetry* ω of $\mathcal{P}_s(\ell_1)$, which is defined as an algebra homomorphism from $\mathcal{P}_s(\ell_1)$ to itself such that $\omega(F_n) = (-1)^{n-1} F_n$ $n \in \mathbb{N}$. In other words, for every n, $\big(\omega(F_n)\big)(x) = -F_n(x)$. Note that ω is an involution in the sense that ω^2 is the unity operator. It is known that $\omega G_n = B_n$ and $\omega B_n = G_n$ for every $n \in \mathbb{N}$ ([1], p. 4). In [2], it was observed that Newton's identity (4) corresponds to Landsberg's identity in physics [28], and equation (11) is related to a Bose-Fermi symmetry. Some specific examples for the mentioned Bose-Fermi symmetry can be found in [29–31].

2.3. Note about the Banach Space ℓ_1

As we mentioned above, $\exp(-\beta H)$ is a trace class operator, and so its eigenvalues $x_i = \exp(-\beta E_i)$ are summable, that is, $x = (x_1, x_2, \ldots) \in \ell_1$. On the other hand, in [2], it was observed that for the case $n = \infty$,, the evaluations $\mathcal{G}$ and $\mathcal{B}$ lead to corresponding grand canonical partition functions only if these series converge. Since all $x_i \geq 0$, the vector x must be in ℓ_1. Thus, the space of absolutely summable sequences, ℓ_1, is the most natural domain for vectors $x = (x_1, x_2, \ldots)$, and $\mathcal{P}_s(\ell_1)$ is the most natural algebra of the symmetric polynomials for $n = \infty$. However, it is possible to consider symmetric polynomials in the general case ℓ_p, $1 \leq p < \infty$ and even for the case of "continual" numbers of variables if $x \in L_p$, $1 \leq p \leq \infty$ (see [13,32–34] and the references therein).

Note that in [35], some of the relations between a trace class operator A and the (infinite-dimensional) Fredholm determinant $\det(I - A)$, were considered, where I is the identity operator. In particular, if A is self-adjoint with eigenvalues x_i,, then

$$\det(I - tA) = \mathcal{G}(x)(t) \quad \text{and} \quad (\det(I - A))^{-1} = \mathcal{B}(-x)(t).$$

Applications of determinants of the form $\det(I - A)$ for partition functions can be found in [36].

3. Supersymmetric Polynomials and Partition Functions for Mixed Systems of Bosons and Fermions

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. By $\ell_1(\mathbb{Z}_0)$, we denote the Banach space of all absolutely summing complex sequences indexed by the elements of $\mathbb{Z}_0$ (two-sided sequences). Every element of $\ell_1(\mathbb{Z}_0)$ can be represented in the form

$$(y|x) = (\ldots, y_2, y_1|x_1, x_2, \ldots)$$

with

$$\|(y|x)\| = \sum_{i=1}^{\infty} (|x_i| + |y_i|),$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ belong to ℓ_1.

For every $n \in \mathbb{N}$, we define the polynomials T_n, $n \in \mathbb{N}$ on $\ell_1(\mathbb{Z}_0)$ by

$$T_n((y|x)) = F_n(x) - F_n(y),$$

where F_n is defined by (1).

A polynomial on $\ell_1(\mathbb{Z}_0)$ is called *supersymmetric* (see [17]) if it can be represented as an algebraic combination of elements of the set $\{T_n \colon n \in \mathbb{N}\}$. Let us denote $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$ as the algebra of all supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$. Note that the set $\{T_n \colon n \in \mathbb{N}\}$ is the algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. Let us define another important supersymmetric polynomial on $\ell_1(\mathbb{Z}_0)$, which also forms the algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. For $n \in \mathbb{N}$, let $W_n \colon \ell_1(\mathbb{Z}_0) \to \mathbb{C}$ be defined by

$$W_n((y|x)) = \sum_{k=0}^{n} G_k(x) B_{n-k}(-y). \tag{17}$$

Note that polynomial W_n can be obtained if we substitute in Newton's Formula (4) for polynomials T_n instead of F_n [17]. In other words,

$$mW_m((y|x)) = \sum_{k=1}^{m} (-1)^{k-1} W_{m-k}((y|x)) T_k((y|x)), \quad m \in \mathbb{N}. \tag{18}$$

From (18), in particular, it follows that all polynomials, W_n, are supersymmetric and form the algebraic basis in $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$.

Let $\mathcal{W}((y|x))(t)$ be the formal series

$$\mathcal{W}((y|x))(t) = \sum_{n=0}^{\infty} t^n W_n((y|x)), \quad W_0 = 1, \tag{19}$$

that is, $\mathcal{W}$ is the generating function for polynomial W_n. According to ([17], Theorem 2),

$$\mathcal{W}((y|x))(t) = \frac{\mathcal{G}(x)(t)}{\mathcal{G}(y)(t)}, \tag{20}$$

the equality is true on the common domain of convergence.

Consider a mixed system of bosons and fermions. In [27], it is shown that the partition function for the system, where the total number, N, of bosons and fermions is fixed, can be represented in the form

$$Z_N(\beta) = \sum_{k=0}^{N} G_k\Big(\Big(x_1^{(F)}, x_2^{(F)}, \dots\Big)\Big) B_{N-k}\Big(\Big(x_1^{(B)}, x_2^{(B)}, \dots\Big)\Big), \tag{21}$$

where

$$x_i^{(F)} = \exp\Big(-\beta E_i^{(F)}\Big) \quad \text{and} \quad x_i^{(B)} = \exp\Big(-\beta E_i^{(B)}\Big),$$

$E_i^{(F)}$ and $E_i^{(B)}$ are the single-particle energies of fermions and bosons, respectively.

Let $\widetilde{W}_n : \ell_1(\mathbb{Z}_0) \to \mathbb{C}$ be defined by

$$\widetilde{W}_n((y|x)) = W_n((-x|-y)), \tag{22}$$

where W_n is defined by (17). According to (17) and (22),

$$\widetilde{W}_n((y|x)) = \sum_{k=0}^{n} G_k(-y) B_{n-k}(x). \tag{23}$$

According to (21) and (23),

$$Z_N(\beta) = \widetilde{W}_N((\tilde{y}|\tilde{x})), \tag{24}$$

where

$$\tilde{y} = \Big(-x_1^{(F)}, -x_2^{(F)}, \dots\Big) \tag{25}$$

and

$$\tilde{x} = \Big(x_1^{(B)}, x_2^{(B)}, \dots\Big). \tag{26}$$

If the sequences are finite, we complete them with an infinite number of zeros. Note that the equality (24) makes sense only if $\tilde{x}$ and $\tilde{y}$ belong to ℓ_1. Otherwise, we can only consider (24) as the formal equality.

Let us consider the grand canonical partition function. According to (13) and (24),

$$Z(z,\beta) = \sum_{N=0}^{\infty} z^N \widetilde{W}_N((\tilde{y}|\tilde{x})), \quad \widetilde{W}_0 = 1. \tag{27}$$

For $(y|x) \in \ell_1(\mathbb{Z}_0)$ and $t \in \mathbb{C}$, let $\widetilde{\mathcal{W}}((y|x))(t)$ be the formal series

$$\widetilde{\mathcal{W}}((y|x))(t) = \sum_{n=0}^{\infty} t^n \widetilde{W}_n((y|x)). \tag{28}$$

Evidently,

$$Z(z,\beta) = \widetilde{\mathcal{W}}((\tilde{y}|\tilde{x}))(z). \tag{29}$$

On the other hand, according to (28), (22), (20), and (19),

$$\begin{aligned}\widetilde{\mathcal{W}}((y|x))(t) &= \sum_{n=0}^{\infty} t^n \widetilde{W}_n((-x|-y)) \\ &= \mathcal{W}((-x|-y))(t) \\ &= \frac{\mathcal{G}(-y)(t)}{\mathcal{G}(-x)(t)} = \frac{\mathcal{B}(x)}{\mathcal{B}(y)}.\end{aligned} \tag{30}$$

Therefore, according to (29), (30), and (11),

$$Z(z,\beta) = \frac{\mathcal{G}(-\tilde{y})(t)}{\mathcal{G}(-\tilde{x})(t)} = \frac{\mathcal{B}(\tilde{x})}{\mathcal{B}(\tilde{y})},$$

where $\tilde{y}$ and $\tilde{x}$ are defined by (25) and (26), respectively.

Thus, we have represented the grand canonical partition function of the mixed system of bosons and fermions via the generating functions $\mathcal{G}$ and $\mathcal{B}$ for elementary symmetric polynomials.

Let us observe that, if we apply the transformation $(y|x) \mapsto (-x|-y)$ to T_n for the case $y = 0$, we will obtain

$$F_n(x) = T_n((0|x)) \mapsto T_n((-x|0) = (-1)^{n-1} T_n((0|x)) = (-1)^{n-1} F_n(x) = (\omega(F_n))(x).$$

In other words, the involution ω on $\mathcal{P}_s(\ell_1)$ can be extended to $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$, setting $(\omega(P))((y|x)) = P((-x|-y))$. In particular, $\omega(W_n) = \widetilde{W}_n$. Applying the homomorphism ω to (18), we obtain

$$m\widetilde{W}_m((y|x)) = \sum_{k=1}^{m} \widetilde{W}_{m-k}((y|x)) T_k((y|x)), \quad m \in \mathbb{N},$$

that is, $\widetilde{W}_n$ can be obtained if we substitute T_n instead of F_n into the Newton Formula (5); therefore, we have another representation for $\widetilde{W}_n$, which can be interpreted as another realization of Landsberg's identity. In addition, from (6), (7), we can obtain

$$\widetilde{W}_m = \sum_{k=1}^{m} (-1)^{k-1} \widetilde{W}_{m-k} W_k, \quad m \in \mathbb{N}.$$

Example 1. *Let us compute $Z_N(\beta) = \widetilde{W}_N(\widetilde{y}, \widetilde{x})$ for $N = 4$, $\widetilde{x} = (x_1, x_2)$, $\widetilde{y} = (-y_1, -y_2, -y_3)$. According to (23),*

$$\begin{aligned}\widetilde{W}_N(\widetilde{y}, \widetilde{x}) &= B_4(x) + G_1(-y)B_3(x) + G_2(-y)B_2(x) + G_3(-y)B_1(x) + G_4(-y) \\ &= x_1^4 + x_2^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + (y_1 + y_2 + y_3)(x_1^3 + x_2^3 + x_1^2x_2 + x_1x_2^2) \\ &+ (y_1y_2 + y_1y_3 + y_2y_3)(x_1^2 + x_2^2 + x_1x_2) + y_1y_2y_3(x_1 + x_2).\end{aligned}$$

4. Semi-ring Structures on the Set of Variables

4.1. The Ring $\mathcal{M}_0$

First we consider the dense linear subspace c_{00} of ℓ_1. Let c_{00} be the vector space of all the eventual zero sequences of complex numbers. Let $c_{00}(\mathbb{Z}_0)$ be the subspace of $\ell_1(\mathbb{Z}_0)$ consisting of all $(y|x) \in \ell_1(\mathbb{Z}_0)$ such that $x, y \in c_{00}$. In order to shorten the notation, we will write the elements of c_{00} as $(x_1, \ldots, x_n)$ instead of $(x_1, \ldots, x_n, 0, \ldots)$. Correspondingly, we will write the elements of $c_{00}(\mathbb{Z}_0)$ as $(y_1, \ldots, y_m|x_1, \ldots, x_n)$.

Let us define the following equivalence relation on $c_{00}(\mathbb{Z}_0)$. For $a, b \in c_{00}(\mathbb{Z}_0)$, let $a \sim b$ if and only if $T_n(a) = T_n(b)$ for every $n \in \mathbb{N}$. Let $\mathcal{M}_0 = c_{00}(\mathbb{Z}_0)/\sim$. Note that we have two types of equivalent elements:

$$(y_1, \dots, y_m | x_1, \dots, x_n) \sim \big(y_{\tau(1)}, \dots, y_{\tau(m)} | x_{\sigma(1)}, \dots, x_{\sigma(n)}\big),$$

where τ and σ are permutations on sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$ respectively, and

$$(y_1, \dots, y_m, c | c, x_1, \dots, x_n) \sim (y_1, \dots, y_m | x_1, \dots, x_n).$$

Consequently, every element of $\mathcal{M}_0$ has the representative $(y|x)$, where $x, y \in c_{00}$, such that the multi-sets of the nonzero elements of x and y are disjointed. On the other hand, every pair of disjointed finite multi-sets of nonzero complex numbers defines some element of $\mathcal{M}_0$. So, we have the bi-jection between $\mathcal{M}_0$ and the set of all pairs of the disjointed finite multi-sets of nonzero complex numbers. Let us define the ring operations on $\mathcal{M}_0$. First, we define some auxiliary operations on c_{00}. Let

$$(x_1, \dots, x_n) \bullet (x'_1, \dots, x'_m) = (x_1, \dots, x_n, x'_1, \dots, x'_m)$$

and

$$\begin{aligned}(x_1, \dots, x_n) \diamond (x'_1, \dots, x'_m)& \\ &= \big(x_1x'_1, x_1x'_2, \dots, x_1x'_m, x_2x'_1, x_2x'_2, \dots, x_2x'_m, \dots, x_nx'_1, x_nx'_2, \dots, x_nx'_m\big)\end{aligned}$$

for $(x_1, \dots, x_n), (x'_1, \dots, x'_m) \in c_{00}$.

Let

$$[z] + [z'] = \big[(y \bullet y' | x \bullet x')\big]$$

and

$$[z][z'] = \Big[\Big((y \diamond x') \bullet (x \diamond y') | (y \diamond y') \bullet (x \diamond x')\Big)\Big]$$

for $z = (y|x), z' = (y'|x') \in c_{00}(\mathbb{Z}_0)$, where $x, y, x', y' \in c_{00}$. According to [17], the $\mathcal{M}_0$ with these operations is a ring, where $-[(y|x)] = [(x|y)]$. Note that $\mathcal{M}_0$ is not a linear space, so it is not an algebra [17].

Let $a \sim b$. Since $T_n(a) = T_n(b)$ for every $n \in \mathbb{N}$, it follows that $f(a) = f(b)$ for every supersymmetric function f. That is, the value of a supersymmetric function does not depend on the choice of a representative of a class. So, we can set

$$f([a]) = f(a)$$

for a supersymmetric function f and for $[a] \in \mathcal{M}_0$.

Let us consider how our ring operations interplay with the algebraic basis T_n and the partition function $\widetilde{\mathcal{W}}(z)(t)$. According to [17],

$$T_n([z][z']) = T_n([z])T_n([z']) \quad \text{and} \quad T_n([z] + [z']) = T_n([z]) + T_n([z']) \tag{31}$$

for every $n \in \mathbb{N}$ and $[z], [z'] \in \mathcal{M}_0$. In other words, each T_n is a ring homomorphism from $\mathcal{M}_0$ to $\mathbb{C}$. Moreover, it is easy to check (c.f. [17]) that

$$\mathcal{W}([z] + [z'])(t) = \mathcal{W}([z])(t)\mathcal{W}([z'])(t)$$

and

$$\widetilde{\mathcal{W}}([z] + [z'])(t) = \widetilde{\mathcal{W}}([z])(t)\widetilde{\mathcal{W}}([z'])(t).$$

The following example may be interesting for evaluating grand canonical partition functions "at infinity".

Example 2. *Let λ and μ be positive numbers. Set*

$$z^{(n)} = \Big(\underbrace{-\frac{\mu}{n},\ldots,-\frac{\mu}{n}}_{n}\,\Big|\,\underbrace{\frac{\lambda}{n},\ldots,\frac{\lambda}{n}}_{n}\Big).$$

Taking into account ([17], pp. 7–8) and the relations between $\mathcal{W}$ and $\widetilde{\mathcal{W}}$, we can see that if $n \to \infty$, then both $\mathcal{W}(z^{(n)})(t)$ and $\widetilde{\mathcal{W}}(z^{(n)})(t)$ approach the function $e^{(\lambda+\mu)t}$. Moreover, at the "limit point", $Z_1(\beta) = \lambda + \mu$, and $Z_N(\beta) = 0$ for every $N > 1$.

Consider the case when sequences $\tilde{x}$ and $\tilde{y}$, defined by (26) and (25), respectively, have only a finite number of nonzero elements, i.e., $\tilde{x}, \tilde{y} \in c_{00}$. Then, $(\tilde{y}|\tilde{x}) \in c_{00}(\mathbb{Z}_0)$. So, $[(\tilde{y}|\tilde{x})] \in \mathcal{M}_0$. Since the functions $\widetilde{W}_n$, used in the representations (24) and (27) of partition functions are supersymmetric, it follows that values $\widetilde{W}_n(u)$ do not depend on the choice of the representative $u \in [(\tilde{y}|\tilde{x})]$. So, it is natural to consider the partition functions as functions on such equivalence classes. Note that all the elements of the sequence $\tilde{x}$ are non-negative and all the elements of the sequence $\tilde{y}$ are non-positive. So, the equivalence class $[(\tilde{y}|\tilde{x})]$ belongs to the subset $\mathcal{M}_0^{\pm}$ of $\mathcal{M}_0$, defined in the following way. According to $\mathcal{M}_0^{\pm}$, let us denote the set of elements $[u]$, where u is of the form

$$u = (-y_1, \ldots, -y_m|x_1, \ldots, x_n), \quad x_i \geq 0, \quad y_j \geq 0.$$

Note that $\mathcal{M}_0^{\pm}$ can be completed with respect to a ring norm on $\mathcal{M}_0$ (see [15,17]). In Section 4.2, we consider such completions in more detail.

For every $[u] \in \mathcal{M}_0^{\pm}$ and odd number k,

$$T_k(u) = F_k(x) + F_k(y) \geq 0,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, and it is equal to zero if and only if $u = 0$.

It is known that $\mathcal{M}_0$ contains divisors of zero. For example,

$$\begin{aligned}
[(-1|1)][(0|1,-1)] &= \big[\big((-1)\diamond(1,-1)\big)\bullet\big((1)\diamond(0)\big)\big|\big((1)\diamond(1,-1)\big)\bullet\big((-1)\diamond(0)\big)\big] \\
&= \big[(-1,1)\bullet(0)\big|(1,-1)\bullet(0)\big] \\
&= \big[(-1,1,0|1,-1,0)\big] \\
&= \big[(0|0)\big] \\
&= 0.
\end{aligned}$$

Proposition 1. *The set $\mathcal{M}_0^{\pm}$ is a commutative semi-ring with respect to the ring operations in $\mathcal{M}_0$, without divisors of zero.*

Proof. It is easy to check that if $[u]$ and $[v]$ are in $\mathcal{M}_0^{\pm}$, then both $[u] + [v]$ and $[u][v]$ are in $\mathcal{M}_0^{\pm}$. But for a given $[u] = [(-y_1, \ldots, -y_m|x_1, \ldots, x_n)] \in \mathcal{M}_0^{\pm}$, $u \neq 0$, the element $-[u] = [(x_1, \ldots, x_n| -y_1, \ldots, -y_m)]$ does not belong to $\mathcal{M}_0^{\pm}$. Thus, $\mathcal{M}_0^{\pm}$ is a semi-ring but not a ring. If $[u][v] = 0$, then, according to (31), $T_1(u)T_1(v) = 0$. So either $T_1(u) = 0$ or $T_1(v) = 0$. Thus, either $u = 0$ or $v = 0$. □

The semi-ring $\mathcal{M}_0^{\pm}$ has the following important property:

$$[(-y_1, \ldots, -y_m|x_1, \ldots, x_n)] = \big[(-y'_1, \ldots, -y'_{m'}|x'_1, \ldots, x'_{n'})\big]$$

if and only if $m = m'$, $n = n'$ and there are permutations σ and τ such that

$$(-y_1, \ldots, -y_m|x_1, \ldots, x_n) = \big(-y'_{\tau(1)}, \ldots, -y'_{\tau(m)}|x'_{\sigma(1)}, \ldots, x'_{\sigma(n)}\big).$$

Let $\sharp[u]$ be a pair (m,n) such that in the representation $[u] = [(-y_1,\dots,-y_m|x_1,\dots,x_n)]$ in $\mathcal{M}_0^{\pm}$, the number of nonzero elements y_j is equal to m and the number of nonzero elements x_i is equal to n. From the definition of the ring operations in $\mathcal{M}_0$, we have this if $\sharp[u] = (m,n)$ and $\sharp[v] = (k,s)$, then $\sharp([u]+[v]) = (m+k, n+s)$ and $\sharp([u][v]) = (ms+nk, mk+ns)$. In particular, $\sharp[u]^2 = (2mn, m^2+n^2)$.

Proposition 2. *Every invertible element in $\mathcal{M}_0^{\pm}$ is of the form $(0|x)$ for some $x > 0$ or $(-y|0)$ for some $y > 0$. Every idempotent $[u]$ in $\mathcal{M}_0^{\pm}$ is of the form $[u] = [(0|1)]$ or $[u] = [(-1|0)]$.*

Proof. Let $[u][v] = [(0|1)]$, then, $\sharp([u][v]) = (0,1)$, and so, $\sharp[u] = (0,1)$ and $\sharp[v] = (0,1)$ or $\sharp[u] = (1,0)$ and $\sharp[v] = (1,0)$. Consequently, $u = (0|x)$ and $v = (0|1/x)$ for some $x > 0$ or $u = (-y|0)$ and $v = (-1/y|0)$ for some $y > 0$.

Let $[u]$ be an idempotent in $\mathcal{M}_0^{\pm}$, that is, $[u]^r = [u]$ for some positive integer $r > 1$. Then, $\sharp[u]^r = \sharp[u]$ only if $\sharp[u] = (1,0)$ or $\sharp[u] = (0,1)$. Elements of the form $[(-a|0)$ and $[(0|a)]$, $a > 0$, are idempotents only if $a = 1$. □

Proposition 3. *Elements of the form $[(-x_1,\dots,-x_n|x_1,\dots,x_n)]$, $x_i > 0$ can be represented as*

$$[(-x_1,\dots,-x_n|x_1,\dots,x_n)] = [(-1|1)][(-x_1,\dots,-x_k|x_{k+1},\dots,x_n)]$$

for every integer k, $0 \le k \le n$.

Proof. The straightforward computation. □

From the proposition, it follows that we have no multiplicative cancelation in $\mathcal{M}_0^{\pm}$, that is, the equalities $[u][v] = [w]$ and $[u][v'] = [w]$ do not imply $[v] = [v']$.

4.2. A Tropical Semi-Ring Structure

We now introduce another semi-ring structure on $\mathcal{M}_0^{\pm}$, which is related to tropical mathematics. Some of the applications of tropical semi-rings for quantum mechanics can be found in [37]. Let us recall that the *min tropical semi-ring* is the semi-ring $(\mathbb{R}\cup\{+\infty\}, \underline{\oplus}, \odot)$, where the operations $\underline{\oplus}$ and $\odot$ are defined by

$$x\underline{\oplus}y = \min\{x,y\}, \quad x\odot y = x+y, \quad x,y\in\mathbb{R}\cup\{+\infty\}.$$

The operations $\underline{\oplus}$ and $\odot$ are called the *tropical addition* and the *tropical multiplication*, respectively. The unit for $\underline{\oplus}$ is $+\infty$, and the unit for $\odot$ is 0.

Similarly, the *max tropical semi-ring* is the semi-ring $(\mathbb{R}\cup\{-\infty\}, \overline{\oplus}, \odot)$ such that

$$x\overline{\oplus}y = \max\{x,y\}, \quad x\odot y = x+y, \quad x,y\in\mathbb{R}\cup\{-\infty\}.$$

In this semi-ring, the unit for $\overline{\oplus}$ is $-\infty$, and the unit for $\odot$ is 0. The semi-rings are isomorphic with respect to the mapping $x \mapsto -x$. The usual metric $\rho(a,b) = |a-b|$ on $\mathbb{R}$ can be extended to $\mathbb{R}\cup\{-\infty\}$ by setting $\rho(a,-\infty) = 1$ for every $a\in\mathbb{R}$. Similarly, $\rho(a,+\infty) = 1$, $a\in\mathbb{R}$, for the case $\mathbb{R}\cup\{+\infty\}$.

Let $(-y_m,\dots,-y_1|x_1,\dots,x_n)$ be a representation of $[u]\in\mathcal{M}_0^{\pm}$. We say that this representation is *ordered* if $x_1\ge x_2\ge\dots\ge x_n$ and $-y_1\le -y_2\le\dots\le -y_m$. The ordered representation of $[u]$ is unique, and we denote it by $(-y_m,\dots,-y_1|x_1,\dots,x_n)_o$. Let us denote by $\mathfrak{e}$ as the formal element

$$\mathfrak{e} = (\dots,+\infty,\dots,+\infty,+\infty|-\infty,-\infty,\dots,-\infty,\dots).$$

Definition 2. *Let us define a tropical semi-ring $\mathcal{M}_0^{\oplus}$ as the set $\mathcal{M}_0^{\pm} \cup \{\mathfrak{e}\}$ with operations $\oplus$ and $\odot$ such that*

$$
\begin{aligned}
&(-y_m, \ldots, -y_1|x_1, \ldots, x_n)_o \oplus (-d_m, \ldots, -d_1|b_1, \ldots, b_n)_o \\
&\quad = \big((-y_m)\underline{\oplus}(-d_m), \ldots, (-y_1)\underline{\oplus}(-d_1)|x_1\overline{\oplus}b_1, \ldots, x_n\overline{\oplus}b_n\big)_o \\
&\quad = \big(\min\{-y_m, -d_m\}, \ldots, \min\{-y_1, -d_1\}|\max\{x_1, b_1\}, \ldots, \max\{x_n, b_n\}\big)_o
\end{aligned}
$$

and

$$
\begin{aligned}
&(-y_m, \ldots, -y_1|x_1, \ldots, x_n)_o \odot (-d_m, \ldots, -d_1|b_1, \ldots, b_n)_o \\
&\quad = \big((-y_m) \odot (-d_m), \ldots, (-y_1) \odot (-d_1)|x_1 \odot b_1, \ldots, x_n \odot b_n\big)_o \\
&\quad = (-y_m - d_m, \ldots, -y_1 - d_1|x_1 + b_1, \ldots, x_n + b_n)_o.
\end{aligned}
$$

Proposition 4. *$\mathcal{M}_0^{\pm} \cup \{\mathfrak{e}\}$ is a semi-ring, and the unit for $\oplus$ is $\mathfrak{e}$, and the unit for $\odot$ is* 0.

Proof. Let us check the distributive law. From the distributive laws in the min. tropical semi-ring and the max. tropical semi-ring,

$$
\begin{aligned}
&(-c_m, \ldots, -c_1|a_1, \ldots, a_n)_o \odot \big((-y_m, \ldots, -y_1|x_1, \ldots, x_n)_o \oplus (-d_m, \ldots, -d_1|b_1, \ldots, b_n)_o\big) \\
&= \big(-c_m \odot \big((-y_m)\underline{\oplus}(-d_m)\big), \ldots, -c_1 \odot \big((-y_1)\underline{\oplus}(-d_1)\big)|a_1 \odot \big(x_1\overline{\oplus}b_1\big), \ldots, a_n \odot \big(x_n\overline{\oplus}b_n\big)\big)_o \\
&= (-c_m, \ldots, -c_1|a_1, \ldots, a_n)_o \odot (-y_m, \ldots, -y_1|x_1, \ldots, x_n)_o \\
&\quad \oplus(-c_m, \ldots, -c_1|a_1, \ldots, a_n)_o \odot (-d_m, \ldots, -d_1|b_1, \ldots, b_n)_o.
\end{aligned}
$$

□

Let X be a Banach space with an unconditional Schauder basis (e_n), $n \in \mathbb{N}$. Then, any vector $x \in X$ can be represented as

$$
x = \sum_{n=1}^{\infty} x_n e_n = (x_1, \ldots, x_n, \ldots).
$$

Denote the ring of elements as $\mathcal{M}_X$

$$
[u] = [(\ldots, y_m, \ldots, y_1|x_1, \ldots, x_n, \ldots)], \quad x_i, y_j, \in \mathbb{C}
$$

such that $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_m, \ldots)$ are in X, endowed with the following ring norm:

$$
\|[u]\| = \inf\big(\|x\|_X + \|y\|_X\big),
$$

where the infimum is taken over all representations $u = (\ldots, y_m, \ldots, y_1|x_1, \ldots, x_n, \ldots)$. It is known that this norm generates a metric $d([u], [v]) = \|[u] - [v]\|$, and $\mathcal{M}_X$ is a complete metric space with respect to the metric. Moreover, the ring operations in $\mathcal{M}_X$ are continuous, and $\mathcal{M}_0$ is a dense subring in $\mathcal{M}_X$ [15,17].

According to $\mathcal{M}_X^{\pm}$, let us denote the closed subset in $\mathcal{M}_X$, consisting of the elements

$$
[u] = [(\ldots, -y_m, \ldots, -y_1|x_1, \ldots, x_n, \ldots)], \quad x_i \geq 0, \quad y_j \geq 0.
$$

Thus, $\mathcal{M}_X^{\pm}$ is a complete metric space and a topological semi-ring.

We can extend the metric to $\mathcal{M}_X^{\oplus} = \mathcal{M}_X^{\pm} \cup \{\mathfrak{e}\}$ by setting $d(u, \mathfrak{e}) = 1$ for every $u \in \mathcal{M}_X^{\oplus}$. Note that $\mathcal{M}_X^{\oplus} \setminus \{\mathfrak{e}\}$ is a commutative group with respect to "$\odot$" and

$$
\big\|[u]^{\odot k}\big\| = \Big\|\underbrace{[u] \odot \cdots \odot [u]}_{k}\Big\| = \|[ku]\| = k\|u\|, \quad k \in \mathbb{N}.
$$

Theorem 1. *For any Banach space X with an unconditional basis, the following statements are true:*

1. *The tropical operations are continuous in $\mathcal{M}_X^{\oplus}$;*
2. *The mappings*
$$\Phi^+[u] = \max_i x_i \quad \text{and} \quad \Phi^-[u] = \min_j(-y_j)$$
are continuous semi-ring homomorphisms from $\mathcal{M}_X^{\oplus}$ to the max. tropical semi-ring $(\mathbb{R} \cup \{+\infty\}, \overline{\oplus}, \odot)$ and to the min. tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, \underline{\oplus}, \odot)$, respectively.

Proof. 1. If $[u]$ and $[v]$ are not equal to $\mathfrak{e}$, then
$$\|[u] \oplus [v]\| \leq \|[u] \odot [v]\| \leq \|[u] \bullet [v]\|$$
and we know that the operation "$\bullet$" is continuous.

2. Clearly, $\Phi^+([u]) = x_1$ and $\Phi^-([u]) = -y_1$, in particular, $\Phi^+(\mathfrak{e}) = +\infty$, and $\Phi^-(\mathfrak{e}) = -\infty$. Moreover,
$$\Phi^+([u] \oplus [u']) = \max(x_1, x_1') = \Phi^+([u]) \oplus \Phi^+([u'])$$
and
$$\Phi^+([u] \odot [u']) = x_1 + x_1' = \Phi^+([u]) \odot \Phi^+([u']).$$
Thus Φ^+ is a semi-ring isomorphism.

In order to show continuity, we observe that the function $x = (x_1, x_2, \ldots) \mapsto \max_n |x_n|$ is bounded (on bounded subsets) on every Banach space X with a Schauder basis (e_n). Indeed, if (π_n), $n \in \mathbb{N}$ be the sequence of projections,
$$\pi_n(x) = \sum_{k=1}^{n} x_k e_k,$$
then
$$\sup_n \|\pi_n\| = K < \infty$$
(see ([38], pp. 1–2)), and so $|x_n| = |\pi_{n+1}(x) - \pi_n(x)| \leq 2K\|x\|_X$. Hence, $|\Phi^+([u])| \leq 2K\|[u]\|$. The continuity of Φ^+ follows from ([39], Theorem 11.22) taking into account that Φ^+ is a bounded homomorphism of the multiplicative-normed group $\mathcal{M}_X^{\oplus} \setminus \mathfrak{e}$ such that $\left\|[u]^{\odot k}\right\| = k\|u\|$, $k \in \mathbb{N}$. The same works for Φ^-. $\square$

5. Discussions and Conclusions

Mathematical models of quantum field theory deal with densely defined self-adjoint operators on an appropriate Hilbert space $L_2(\Omega)$ in the framework of the von Neumann axioms (see [40]). The statistical approach to quantum mechanics uses a different language of canonical partition functions, which, as we observed, can be described by symmetric and supersymmetric polynomials and are well-defined in the domain of the Banach space ℓ_1.

In this paper, we continue to develop the ideas proposed by Schmidt and Schnack in [2,3] about involving symmetric polynomials for investigations into the partition functions of ideal quantum gases. The first goal of the paper was to find a correspondence between the algebraic bases of supersymmetric polynomials and the partition functions of ideal gases consisting of both bosons and fermions. We can see that the combinatorial relations in the algebra of supersymmetric polynomials have corresponding physical interpretations. By taking into account the fact that the two elements (vectors) z and z', in the set of possible energy levels, are equivalent if and only if $P(z) = P(z')$ for every supersymmetric polynomial P. It is natural to consider the quotient set with respect to the equivalence as a natural domain. For such a quotient set, the usual vector operations are not valid, and we introduced new ring operations (addition and multiplication) on the quotient set $\mathcal{M}_0$. It seems to be that the new addition can be obtained using the direct sum of operators $\exp(-\beta H_1)$ and $\exp(-\beta H_2)$, and the new product leads to the tensor product of operators. Note that the elements of $\mathcal{M}_0$ have the physical interpretation if $x_i \geq 0$ and $y_j \leq 0$. Otherwise, we can obtain a system where the cancelation rule $[(y, a|a, x)] = [(x|y)]$ plays a nontrivial role and where we can obtain negative energy. It leads us to tachyonic particles that cannot exist because they are inconsistent with the known laws of physics. But such an approach can be interesting for tachyon condensation (for details on tachyon condensation, see [41]).

The fact that the energy on a level can not be negative suggests the use of elements in $\mathcal{M}_0$, which have a very specific form, $[z] = [(-y_1, ., -y_m|x_1, \ldots, x_n)]$, where all x_i and y_j are non-negative. The subset of such elements forms a semi-ring without divisions of zero, denoted by $\mathcal{M}_0{}^{\pm}$. We considered the algebraic properties of this semi-ring and its completions $\mathcal{M}_X^{\pm}$ with respect to the various metrics associated with different Banach spaces X. Moreover, we introduced new operations on $\mathcal{M}_X^{\oplus} = \mathcal{M}_X^{\pm} \cup \mathfrak{e}$ that lead to an infinite-dimensional analog of the so-called tropical semi-rings. We proved the continuity of the operations on $\mathcal{M}_X^{\oplus}$ and constructed some of the real-valued homomorphisms of $\mathcal{M}_X^{\oplus}$.

For further investigation, we are going to use block-symmetric (or MacMahon) and block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s)$ and explore their applications for the partition functions of quantum gases. The space $\ell_1(\mathbb{C}^s)$ can be defined as a vector space of sequences

$$\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}, \ldots)$$

such that every element $\mathbf{x}^{(n)} = (x_1^{(n)}, \ldots, x_s^{(n)})$ is a vector in $\mathbb{C}^s$, and

$$\|\mathbf{x}\| = \sum_{n=1}^{\infty} \|\mathbf{x}^{(n)}\|.$$

A polynomial is block-symmetric on $\ell_1(\mathbb{C}^s)$ if it is symmetric with respect to all permutations of the vectors (blocks) $\mathbf{x}^{(n)}$. We can expect that models based on block-symmetric (and maybe block-supersymmetric) polynomials can be useful for describing quantum gases with entanglement particles.

The combinatorial properties of block-symmetric polynomials were considered in [42]. The algebras of block-symmetric polynomials and the analytic functions and corresponding bases of the polynomials on $\ell_1(\mathbb{C}^s)$ were studied in [43–48]. Applications of block-symmetric polynomials for the quantum product of symmetric functions were proposed in [49].

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