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Article

Highly Dispersive Optical Solitons in Birefringent Fibers of Complex Ginzburg–Landau Equation of Sixth Order with Kerr Law Nonlinear Refractive Index

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Abstract: In this paper, we derived optical soliton solutions with a highly dispersive nonlinear complex Ginzburg–Landau (CGL) equation in birefringent fibers that have Kerr law nonlinearity. We applied two mathematical methods, namely the addendum Kudryashov's method and the unified Riccati equation expansion method. Straddled solitary solutions, bright soliton, dark soliton and singular soliton solutions were obtained.This model represents the propagation of a dispersive optical soliton through a birefringent fiber. This happens when pulses propagating through an optical fiber split into two pulses.

Keywords: unified Riccati equation expansion method; addendum to Kudryashov's (AK) method; soliton; CGL equation

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1. Introduction

It is well recognized that nonlinear partial differential equations play a vital role in physics and engineering challenges, where they are commonly utilized to represent many complicated phenomena, such as optical fibers, plasma physics, fluid mechanics, biology, chemical kinetics, and so on. In fact, explicit analytical solutions of these equations can be found using several mathematical methods. There are numerous systems, including the Lakshmanan–Porsezian–Daniel (LPD) model, the Radhakrishnan–Kundu–Lakshmanan (RKL) equation, the Gabitov–Turisyn (GT) equation, the Schrodinger–Hirota (SH) equation, the complex Ginzburg–Landau (CGL) equation, the Fokas–Lenell's (FL) equation [1–36] and so on. These are well-known examples of fibers that maintain polarization. The Manakov equation and the Thirring model are two examples of common models for studying split pulses in birefringent fibers. The underlying dynamics of soliton wave propagation for each of these models is the existence of a careful balance between self-phase modulation (SPM) and chromatic dispersion (CD). When the CD is low, an unreasonable result can happen, pending the transmission of optical fiber pulses. To get around this crisis, a variety of technology backup plans have been put into action. One suggestion is to use Bragg gratings with dispersive reflectivity to make up for the low CD count. Pure-quartic solitons were introduced as a different strategy throughout .

In this case, the CD is mutated by 4th-order dispersion (4OD). The disadvantage of such a model is that, for pure-quartic NLSE, one can only recover stationary optical solitons analytically and it can only be analyzed numerically. As a result, pure-quartic solitons were never widely adopted as a substitute paradigm to deal with the dilemma. Then, the fundamental idea of cubic-quartic (CQ) solitons emerged, replacing CD with fourth-order dispersion (4OD) and third-order dispersion (3OD) jointly [21,22]. The advent of the torrent of analytical results in the flow is proof that these topics have recently attracted considerable interest. In this context, sixth-order dispersion (6OD)), fifth-order dispersion (5OD), fourth-order dispersion (4OD), third-order dispersion (3OD)) and the inter- modal dispersion (IMD) terms are considered, in addition to the pre-existing CD, which, together, make up the HD solitons that provide the necessary delicate balance between self-phase modulation (SPM) and CD for the solitons to sustain the inter-continental distance propagation.

Governing Model

The highly dispersive perturbed complex Ginzburg–Landau equation with the Kerr law refractive index in polarization-preserving fiber is well known [28] and reads as:

$$\begin{aligned} i\,q_t + i\,a_1 q_x + a_2 q_{xx} + i\,a_3 q_{xxx} + a_4 q_{xxxx} + i\,a_5 q_{xxxxx} + a_6 q_{xxxxxx} + c\,|q|^2 q \\ = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left\{2|q|^2\left(|q|^2\right)_{xx} - \left[\left(|q|^2\right)_x\right]^2\right\} + \gamma q \\ +i\ \left[\lambda(|q|^{2m} q)_x + \mu\,(|q|^{2m})_x\,q + v\ \ |q|^{2m} q_x\right] \end{aligned} \tag{1}$$

where $q(x,t)$ is the wave profile of the solution, which is a complex-valued function, q^* is the complex conjugate and $i^{\,2} = -1$. Here, $a_j(j = 1-6), c, \alpha, \beta, \gamma, \lambda, \mu, v$ and m are real constants. The first term in Equation (1) is the linear temporal evolution. The constant a_1 is the inter-model dispersion (IMD), the constant a_2 is the chromatic dispersion (CD), the constant a_3 is the 3OD, the constant a_4 is the 4OD, the constant a_5 is the 5OD and the constant a_6 is the 6OD. The constant c is the coefficient of the Kerr law of nonlinearity. The constants α, β, μ, v are the coefficients of nonlinear dispersions. Finally, γ and λ are the coefficients of self-detuning and self-steepening terms, respectively. The objective of this article is to study the following two couples of CGLE in birefringent fibers, which are written for the first time as follows:

$$\begin{aligned} i\,u_t + i\,a_1 u_x + a_2 u_{xx} + i\,a_3 u_{xxx} + a_4 u_{xxxx} + i\,a_5 u_{xxxxx} + a_6 u_{xxxxxx} + (c_1\,|u|^2 + d_1|v|^2)u \\ = \frac{1}{u^*}\left[\alpha_1|u_x|^2 + \beta_1|v_x|^2 + \sigma_1\left(|u|^2\right)_{xx} + \delta_1\left(|v|^2\right)_{xx}\right] \\ -\frac{1}{(\mu_1|u|^2+\zeta_1|v|^2)u^*}\left\{\theta_1\left[\left(|u|^2\right)_x\right]^2 + \gamma_1\left(|u|^2\right)_x\left(|v|^2\right)_x + \rho_1\left[\left(|v|^2\right)_x\right]^2\right\} + \psi_1 u \\ +i\ \ \left[\lambda_1(|u|^{2m}u)_x + E_1\ (|u|^{2m})_x\ u + v_1\ |u|^{2m}u_x\right], \end{aligned} \tag{2}$$

and

$$\begin{aligned} i\,v_t + i\,b_1 v_x + b_2 v_{xx} + i\,b_3 v_{xxx} + b_4 v_{xxxx} + i\,b_5 v_{xxxxx} + b_6 v_{xxxxxx} + (c_2\,|v|^2 + d_2|u|^2)v \\ = \frac{1}{v^*}\left[\alpha_2|v_x|^2 + \beta_2|u_x|^2 + \sigma_2\left(|v|^2\right)_{xx} + \delta_2\left(|u|^2\right)_{xx}\right] \\ -\frac{1}{(\mu_2|v|^2+\zeta_2|u|^2)v^*}\left\{\theta_2\left[\left(|v|^2\right)_x\right]^2 + \gamma_2\left(|v|^2\right)_x\left(|u|^2\right)_x + \rho_2\left[\left(|u|^2\right)_x\right]^2\right\} + \psi_2 v \\ +i\ \ \left[\lambda_2(|v|^{2m}v)_x + E_2\ (|v|^{2m})_x\ v + v_2\ |v|^{2m}v_x\right], \end{aligned} \tag{3}$$

where $u(x,t)$ and $v(x,t)$ are complex-valued functions that reflect the wave profiles and $a_j, b_j(j = 1-6), c_k, d_k, \alpha_k, \beta_k, \sigma_k, \delta_k, \theta_k, \gamma_k, \rho_k, \mu_k, \zeta_k, \psi_k, \lambda_k, E_k, v_k(k = 1,2)$ are real-valued constants. Its c_k, d_k $(k = 1,2)$ are the self-phase modulation (SPM) coefficients and the cross-phase modulation (XPM) coefficients, respectively. The terms owing to α_k, β_k, δ_k, σ_k, μ_k, ζ_k, θ_k, γ_k, ρ_k, ψ_k $(k = 1,2)$ are the perturbation effect; in particular, ψ_k is obtained by the detuning effect, λ_k are the self-steepening (SS) coefficient terms and the nonlinear terms coefficients used to discuss optical solutions are E_k and v_k. The objective of this article is to solve Equations (2) and (3) utilizing the addendum Kudryashov method and the unified Riccati equation expansion approach.

The current paper is an analysis of HD-CGLE but in birefringent fibers. It is worth mentioning that the model is addressed in a single channel but in birefringent fibers with the Kerr law nonlinear refractive index. These are the familiar erbium-doped fibers with the presence of a 3OD effect, in addition to CD and STD. This is a differential group delay (DGD) effect after the occurrence of pulse splitting. The accumulation of such a DGD leads to the effect of birefringence. Thus, HD-CGLE in birefringent fibers is the focus of attention in this work. The above system (2) and (3) is more general than that obtained in the articles [10–18,21–23,27,28], and our results are different and new.

The structure of this article is ordered as follows: the introduction is in Section 1. Section 2 discusses the mathematical foundations. In order to determine the optical soliton solutions for the system (1) and (2), we use the addendum to Kudryashov's in Section 3 and the unified Riccati equation expansion approach in Section 4. Numerical simulations are presented in Section 5. Conclusions are demonstrated in Section 6.

2. Mathematical Preliminaries

We will consider the transformations of traveling waves as

$$u(x,t) = \phi_1(\xi)e^{i(-\kappa x+wt+\theta_0)}, \tag{4}$$
$$v(x,t) = \phi_2(\xi)e^{i(-\kappa x+wt+\theta_0)}, \tag{5}$$

where $\phi_j(\xi)$ $(j=1,2)$ are real-valued functions representing the traveling waves' amplitudes, $\xi = x - Vt$. Here, V is the velocity of the solitons, κ its frequency, w is the wave number and θ_0 is the phase constant. Substituting (4) and (5) into Equations (2) and (3) and separating the real and imaginary parts, we obtain the real parts as follows:

$$\begin{gathered}
-k(\lambda_1+\upsilon_1)\phi_1^{2m+2}(\zeta_1\phi_2^2+\mu_1\phi_1^2)+d_1\zeta_1\phi_1^2\phi_2^4-\beta_1k^2\phi_1^2(\zeta_1\phi_2^2+\mu_1\phi_1^2)+(c_1\zeta_1+d_1\mu_1)\phi_1^4\phi_2^2\\
-2\delta_1(\zeta_1\phi_2^2+\mu_1\phi_1^2)\phi_2\phi_2''+a_6(\zeta_1\phi_2^2+\mu_1\phi_1^2)\phi_1\phi_1^{(6)}+(-15a_6k^2+5a_5k+a_4)(\zeta_1\phi_2^2+\mu_1\phi_1^2)\phi_1\phi_1^{(4)}\\
+(15a_6k^4-10a_5k^3-6a_4k^2+3a_3k+a_2-2\sigma_1)(\zeta_1\phi_2^2+\mu_1\phi_1^2)\phi_1\phi_1''+(-a_6k^6+a_5k^5+a_4k^4\\
-a_3k^3-a_2k^2-\alpha_1k^2+a_1k-\psi_1-w)\phi_1^2(\zeta_1\phi_2^2+\mu_1\phi_1^2)-[(\alpha_1\zeta_1+2\sigma_1\zeta_1)\phi_2^2\\
+(\alpha_1\mu_1+2\sigma_1\mu_1-4\theta_1)\phi_1^2](\phi_1')^2+4\gamma_1\phi_1\phi_2\phi_1'\phi_2'-[(\beta_1\zeta_1+2\delta_1\zeta_1-4\rho_1)\phi_2^2\\
+(\beta_1\mu_1+2\delta_1\mu_1)\phi_1^2](\phi_2')^2+c_1\mu_1\phi_1^6=0,
\end{gathered}\tag{6}$$

$$\begin{gathered}
-k(\lambda_2+\upsilon_2)\phi_2^{2m+2}(\zeta_2\phi_1^2+\mu_2\phi_2^2)+d_2\zeta_2\phi_2^2\phi_1^4-\beta_2k^2\phi_2^2(\zeta_2\phi_1^2+\mu_2\phi_2^2)+(c_2\zeta_2+d_2\mu_2)\phi_2^4\phi_1^2\\
-2\delta_1(\zeta_2\phi_1^2+\mu_2\phi_2^2)\phi_1\phi_1''+b_6(\zeta_2\phi_1^2+\mu_2\phi_2^2)\phi_2\phi_2^{(6)}+(-15b_6k^2+5b_5k+b_4)(\zeta_2\phi_1^2+\mu_2\phi_2^2)\phi_2\phi_2^{(4)}\\
+(15b_6k^4-10b_5k^3-6b_4k^2+3b_3k+b_2-2\sigma_2)(\zeta_2\phi_1^2+\mu_2\phi_2^2)\phi_2\phi_2''+(-b_6k^6+b_5k^5+b_4k^4\\
-b_3k^3-b_2k^2-\alpha_2k^2+b_1k-\psi_2-w)\phi_2^2(\zeta_2\phi_1^2+\mu_2\phi_2^2)-[(\alpha_2\zeta_2+2\sigma_2\zeta_2)\phi_1^2\\
+(\alpha_2\mu_2+2\sigma_2\mu_2-4\theta_2)\phi_2^2](\phi_2')^2+4\gamma_2\phi_2\phi_1\phi_2'\phi_1'-[(\beta_2\zeta_2+2\delta_2\zeta_2-4\rho_2)\phi_1^2\\
+(\beta_2\mu_2+2\delta_2\mu_2)\phi_2^2](\phi_1')^2+c_2\mu_2\phi_2^6=0,
\end{gathered}\tag{7}$$

and the imaginary parts:

$$\begin{gathered}
-\zeta_1[2E_1m+2\lambda_1m+\lambda_1+\upsilon_1]\phi_1^{2m+1}\phi_2^2\phi_1'-\mu_1(2E_1m+2m\lambda_1+\lambda_1+\upsilon_1)\phi_1^{2m+3}\phi_1'\\
\zeta_1(-6a_6k^5+5a_5k^4+4a_4k^3-3a_3k^2-2a_2k-V+a_1)\phi_1\phi_2^2\phi_1'\\
+\zeta_1(-6a_6k+a_5)\phi_1\phi_2^2\phi_1^{(5)}+\zeta_1(20a_6k^3-10a_5k^2-4a_4k+a_3)\phi_1\phi_2^2\phi_1'''\\
+\mu_1(-6a_6k^5+5a_5k^4+4a_4k^3-3a_3k^2-2a_2k-V+a_1)\phi_1^3\phi_1'\\
+\mu_1(20a_6k^3-10a_5k^2-4a_4k+a_3)\phi_1^3\phi_1'''+\mu_1(-6a_6k+a_5)\phi_1^3\phi_1^{(5)}=0,
\end{gathered}\quad . \tag{8}$$

$$\begin{gathered}
-\zeta_2[2E_2m+2\lambda_2m+\lambda_2+\upsilon_2]\phi_2^{2m+1}\phi_1^2\phi_2'-\mu_2(2E_2m+2m\lambda_2+\lambda_2+\upsilon_2)\phi_2^{2m+3}\phi_2'\\
+\zeta_2(-6b_6k^5+5b_5k^4+4b_4k^3-3b_3k^2-2b_2k-V+b_1)\phi_2\phi_1^2\phi_2'\\
+\zeta_2(-6b_6k+b_5)\phi_2\phi_1^2\phi_2^{(5)}+\zeta_2(20b_6k^3-10b_5k^2-4b_4k+b_3)\phi_2\phi_1^2\phi_2'''\\
+\mu_2(-6b_6k^5+5b_5k^4+4b_4k^3-3b_3k^2-2b_2k-V+b_1)\phi_2^3\phi_2'\\
+\mu_2(20b_6k^3-10b_5k^2-4b_4k+b_3)\phi_2^3\phi_2'''+\mu_2(-6b_6k+b_5)\phi_2^3\phi_2^{(5)}=0,
\end{gathered}\tag{9}$$

respectively. Set

$$\phi_2 = \chi\phi_1, \tag{10}$$

where χ is a nonzero constant and $\chi \neq 1$. Putting (10) into (6)–(9) allows us to express the real parts as follows:

$$
\begin{gathered}
-k(\lambda_1+v_1)\phi_1^{2m+4}(\zeta_1\chi^2+\mu_1)+[c_1\mu_1+(c_1\zeta_1+d_1\mu_1)\chi^2+d_1\zeta_1\chi^4]\phi_1^6 \\
\phi_1^4[(-a_6k^6+a_5k^5+a_4k^4-a_3k^3-a_2k^2-\alpha_1k^2+a_1k-\psi_1-w-\beta_1k^2)(\zeta_1\chi^2+\mu_1)] \\
+a_6(\zeta_1\chi^2+\mu_1)\phi_1^3\phi_1^{(6)}+(-15a_6k^2+5a_5k\ +a_4)(\zeta_1\chi^2+\mu_1)\phi_1^3\phi_1^{(4)} \\
[-2\delta_1\chi^2+15a_6k^4-10a_5k^3-6a_4k^2+3a_3k+a_2-2\sigma_1](\zeta_1\chi^2+\mu_1)\phi_1^3\phi_1'' \\
-\phi_1^2(\phi_1')^2\{(\beta_1\zeta_1+2\delta_1\zeta_1-4\rho_1)\chi^4+(\alpha_1\zeta_1+\beta_1\mu_1+2\sigma_1\zeta_1+2\delta_1\mu_1+4\gamma_1)\chi^2 \\
+(\alpha_1\mu_1+2\sigma_1\mu_1-4\theta_1)\}=0,
\end{gathered}
\tag{11}
$$

and

$$
\begin{gathered}
-k(\lambda_2+v_2)\chi^{2m+2}\phi_1^{2m+4}(\zeta_2+\mu_2\chi^2)+[d_2\zeta_2+(c_2\zeta_2+d_2\mu_2)\chi^2+c_2\mu_2\chi^4]\chi^2\phi_1^6 \\
+\{-\beta_2k^2-b_6k^6+b_5k^5+b_4k^4-b_3k^3-b_2k^2-\alpha_2k^2+b_1k-\psi_2-w\}\chi^2(\zeta_2+\mu_2\chi^2)\phi_1^4 \\
+b_6(\zeta_2+\mu_2\chi^2)\chi^2\phi_1^2\phi_1^{(6)}+(-15b_6k^2+5b_5k\ +b_4)(\zeta_2+\mu_2\chi^2)\chi^2\phi_1^3\phi_1^{(4)} \\
[-2\delta_1+(15b_6k^4-10b_5k^3-6b_4k^2+3b_3k+b_2-2\sigma_2)\chi^2](\zeta_2+\mu_2\chi^2)\phi_1^3\phi_1'' \\
-\phi_1^2(\phi_1')^2\{(\beta_2\zeta_2+2\delta_2\zeta_2-4\rho_2)+(\alpha_2\zeta_2+\beta_2\mu_2+2\sigma_2\zeta_2+2\delta_2\mu_2-4\gamma_2)\chi^2 \\
+(\alpha_2\mu_2+2\sigma_2\mu_2-4\theta_2)\chi^4\}=0,
\end{gathered}
\tag{12}
$$

and the imaginary parts as:

$$
\begin{gathered}
-(2E_1m+2m\lambda_1+\lambda_1+v_1)\phi_1^{2m+3}\phi_1'+(20a_6k^3-10a_5k^2-4a_4k+a_3)\phi_1^3\phi_1''' \\
+(-6a_6k+a_5)\phi_1^3\phi_1^{(5)}+(-6a_6k^5+5a_5k^4+4a_4k^3-3a_3k^2-2a_2k-V+a_1)\phi_1^3\phi_1'=0,
\end{gathered}
\tag{13}
$$

$$
\begin{gathered}
-(2E_2m\ +2\lambda_2m\ +\lambda_2+v_2)\chi^{2m}\phi_1^{2m+3}\phi_1'+(20b_6k^3-10b_5k^2-4b_4k+b_3)\phi_1^3\phi_1''' \\
+(-6b_6k+b_5)\phi_1^3\phi_1^{(5)}+(-6b_6k^5+5b_5k^4+4b_4k^3-3b_3k^2-2b_2k-V+b_1)\phi_1^3\phi_1'=0,
\end{gathered}
\tag{14}
$$

respectively. The linear independence assumption on Equations (13) and (14) gives the soliton's velocity:

$$
V=-6a_6k^5+5a_5k^4+4a_4k^3-3a_3k^2-2a_2k+a_1=-6b_6k^5+5b_5k^4+4b_4k^3-3b_3k^2-2b_2k+b_1 \tag{15}
$$

and the soliton's frequency:

$$
\kappa=\frac{a_5}{6a_6}=\frac{b_5}{6b_6}, \tag{16}
$$

and the constraint conditions

$$
\begin{aligned}
&2E_1m+(2m+1)\lambda_1+v_1=0, \\
&2E_2m\ +(2m\ +1)\lambda_2+v_2=0, \\
&20a_6k^3-10a_5k^2-4a_4k+a_3=0, \\
&20b_6k^3-10b_5k^2-4b_4k+b_3=0.
\end{aligned}
\tag{17}
$$

Equations (11) and (12) are equivalent under the constraints:

$$
\begin{aligned}
&\frac{(\lambda_1+v_1)(\zeta_1\chi^2+\mu_1)}{(\lambda_2+v_2)\chi^{2m+2}(\zeta_2+\mu_2\chi^2)}=\frac{c_1\mu_1+(c_1\zeta_1+d_1\mu_1)\chi^2+d_1\zeta_1\chi^4}{[d_2\zeta_2+(c_2\zeta_2+d_2\mu_2)\chi^2+c_2\mu_2\chi^4]\chi^2}=\frac{a_6(\zeta_1\chi^2+\mu_1)}{b_6(\zeta_2+\mu_2\chi^2)\chi^2} \\
&=\frac{(-15a_6k^2+5a_5k\ +a_4)(\zeta_1\chi^2+\mu_1)}{(-15b_6k^2+5b_5k\ +b_4)(\zeta_2+\mu_2\chi^2)\chi^2}=\frac{(-a_6k^6+a_5k^5+a_4k^4-a_3k^3-a_2k^2-\alpha_1k^2+a_1k-\psi_1-w-\beta_1k^2)(\zeta_1\chi^2+\mu_1)}{(-b_6k^6+b_5k^5+b_4k^4-b_3k^3-b_2k^2-\alpha_2k^2+b_1k-\psi_2-w-\beta_2k^2)\chi^2(\zeta_2+\mu_2\chi^2)} \\
&=\frac{[-2\delta_1\chi^2+15a_6k^4-10a_5k^3-6a_4k^2+3a_3k+a_2-2\sigma_1](\zeta_1\chi^2+\mu_1)}{[-2\delta_1+(15b_6k^4-10b_5k^3-6b_4k^2+3b_3k+b_2-2\sigma_2)\chi^2](\zeta_2+\mu_2\chi^2)} \\
&=\frac{(\beta_1\zeta_1+2\delta_1\zeta_1-4\rho_1)\chi^4+(\alpha_1\zeta_1+\beta_1\mu_1+2\sigma_1\zeta_1+2\delta_1\mu_1+4\gamma_1)\chi^2+(\alpha_1\mu_1+2\sigma_1\mu_1-4\theta_1)}{\{(\beta_2\zeta_2+2\delta_2\zeta_2-4\rho_2)+(\alpha_2\zeta_2+\beta_2\mu_2+2\sigma_2\zeta_2+2\delta_2\mu_2-4\gamma_2)\chi^2+(\alpha_2\mu_2+2\sigma_2\mu_2-4\theta_2)\chi^4\}}
\end{aligned}
\tag{18}
$$

Now, Equation (11) can be written in the form:

$$
\phi_1^3\phi_1^{(6)}+\Delta_1\phi_1^3\phi_1^{(4)}+\Delta_2\phi_1^3\phi_1''+\Delta_3\phi_1^2(\phi_1')^2+\Delta_4\phi_1^4+\Delta_5\phi_1^6+\Delta_6\phi_1^{2m+4}=0, \tag{19}
$$

where

$$\begin{aligned}
\Delta_1 &= \frac{(-15a_6k^2+5a_5k\ +a_4)}{a_6},\\
\Delta_2 &= \frac{[-2\delta_1\chi^2+(15a_6k^4-10a_5k^3-6a_4k^2+3a_3k+a_2-2\sigma_1)]}{a_6},\\
\Delta_3 &= -\frac{\{(\beta_1\zeta_1+2\delta_1\zeta_1-4\rho_1)\chi^4+(\alpha_1\zeta_1+\beta_1\mu_1+2\sigma_1\zeta_1+2\delta_1\mu_1+4\gamma_1)\chi^2+(\alpha_1\mu_1+2\sigma_1\mu_1-4\theta_1)\}}{a_6(\zeta_1\chi^2+\mu_1)},\\
\Delta_4 &= \frac{(-a_6k^6+a_5k^5+a_4k^4-a_3k^3-a_2k^2-\alpha_1k^2+a_1k-\psi_1-w-\beta_1k^2)}{a_6},\\
\Delta_5 &= \frac{[c_1\mu_1+(c_1\zeta_1+d_1\mu_1)\chi^2+d_1\zeta_1\chi^4]}{a_6(\zeta_1\chi^2+\mu_1)},\ \ \Delta_6 = \frac{-k(\lambda_1+v_1)}{a_6}.
\end{aligned} \tag{20}$$

provided $a_6(\zeta_1\chi^2+\mu_1) \neq 0$. Let us now solve Equation (19) using the following two methods.

3. Addendum to Kudryashov's Method

Recently, Kudryashov [4] proposed a new approach, while Zayed et al. [28] generalized this approach and called it the addendum to Kudryashov's method. In this section, we apply this addendum method to solve Equation (19). For this purpose, we see that Equation (19) is integrable when $m = 1$, which is written as:

$$\phi_1^3\phi_1^{(6)} + \Delta_1\phi_1^3\phi_1^{(4)} + \Delta_2\phi_1^3\phi_1^{''} + \Delta_3\phi_1^2(\phi_1')^2 + \Delta_4\phi_1^4 + (\Delta_5+\Delta_6)\phi_1^6 = 0, \tag{21}$$

We postulate that (20) has the solution shape:

$$\phi_1(\xi) = \sum_{s=0}^{N} A_s\,[T(\xi)]^s\ , \tag{22}$$

where $A_s(s = 0, 1, 2, \ldots, N)$ are constants that can be computed later, $A_s \neq 0$, whereas $T(\xi)$ achieves the nonlinear ODE:

$$T'^2(\xi) = T^2(\xi)[1-\zeta T^{2h}(\xi)][\ln K]^2,\ \ 0 < K \neq 1, \tag{23}$$

whenever ζ is an arbitrary constant. It is well known the Equation (23) has the solution:

$$T(\xi) = \left[\frac{4A}{4A^2\,exp_K(h\xi) + \zeta\,\exp_K(-h\xi)}\right]^{1/h}, \tag{24}$$

where A is a nonzero constant, h is a natural number and $exp_K(h\xi) = K^{h\xi}$.

From balancing the nonlinear terms ϕ_1^6 and $\phi_1^3\phi_1^{(6)}$ in Equation (21), we obtain:

$$N = 3h \tag{25}$$

Now, we will dispute the next cases:

Case-1. When we choose $h = 1$, $N = 3$ posteriorly. Therefore, we deduce that Equation (21) has the following solution:

$$\phi(\xi) = A_0 + A_1T(\xi) + A_2\,T^2(\xi) + A_3\,T^3(\xi), \tag{26}$$

where A_0, A_1, A_2 and A_3 are to be calculated constants such that $A_3 \neq 0$. Inserting (26) along with (23) into Equation (21), one may combine all of the coefficients of $[T(\xi)]^l[T'(\xi)]^f$, $(l = (0, 1, 2, \ldots, 10, f = 0, 1)$ and set them to zero. Then, we arrive at a system of algebraic equations that Maple can solve to obtain the following results:

$$A_0 = A_1 = A_2 = 0,\ A_3 = 24\varepsilon[\ln K]^3\zeta\sqrt{\frac{35\zeta}{\Delta_5+\Delta_6}}, \tag{27}$$

and

$$\Delta_1 = -83 \ln^2 K, \quad \Delta_2 = 1891 \ \ln^4 K - \frac{3}{4}\Delta_3, \ \Delta_4 = -11025 \ \ln^6 K - \frac{9}{4}\Delta_3 \ln^2 K \tag{28}$$

provided that $\zeta(\Delta_5 + \Delta_6) > 0$ and $\varepsilon = \pm 1$.

Substituting (27) with (24) into Equation (26), we acquire the straddled solitary solution to Equations (2) and (3) as:

$$u(x,t) = \frac{1536\varepsilon[\ln K]^3\zeta \ A^3\sqrt{\frac{35\zeta}{\Delta_5+\Delta_6}}}{[4A^2 \ exp_K(x - Vt) + \zeta \exp_K(-(x - Vt))]^3} \ e^{i(-\ \kappa x + wt + \theta_0)}, \tag{29}$$

and

$$v(x,t) = \chi \left[\frac{1536\varepsilon[\ln K]^3\zeta \ A^3\sqrt{\frac{35\zeta}{\Delta_5+\Delta_6}}}{[4A^2 \ exp_K(x - Vt) + \zeta \exp_K(-(x - Vt))]^3}\right] e^{i(-\ \kappa x + wt + \theta_0)}. \tag{30}$$

In the special case, if we set $\zeta = 4A^2$ in Equations (29) and (30) posteriorly, we construct the bright soliton solution to Equations (2) and (3) as

$$u(x,t) = 24\varepsilon \ln^3 K\sqrt{\frac{35}{\Delta_5 + \Delta_6}} \ e^{i(-\ \kappa x + wt + \theta_0)} \operatorname{sech}^3[(x - Vt) \ln K], \tag{31}$$

and

$$v(x,t) = \chi \left[24\varepsilon \ln^3 K\sqrt{\frac{35}{\Delta_5 + \Delta_6}} \ \operatorname{sech}^3[(x - Vt) \ln K]\right] e^{i(-\ \kappa x + wt + \theta_0)}, \tag{32}$$

provided that $(\Delta_5 + \Delta_6) > 0$, while, if we set $\zeta = -4A^2$ in Equations (29) and (30), we posteriorly obtain the singular soliton solution of Equations (2) and (3) as:

$$u(x,t) = 24\varepsilon \ln^3 K\sqrt{\frac{-35}{\Delta_5 + \Delta_6}} \ e^{i(-\ \kappa x + wt + \theta_0)} \operatorname{csch}^3[(x - Vt) \ln K], \tag{33}$$

and

$$v(x,t) = \chi \left[24\varepsilon \ln^3 K\sqrt{\frac{-35}{\Delta_5 + \Delta_6}} \ csch^3[(x - Vt) \ln K]\right] e^{i(-\ \kappa x + wt + \theta_0)}, \tag{34}$$

provided that $(\Delta_5 + \Delta_6) < 0$.

Remark 1. *Under the constraint circumstances (28), the solutions (29)–(34) exist.*

Case-2. When we choose $h = 2$, $N = 6$ posteriorly. Therefore, we deduce that Equation (21) has the following solution:

$$\phi_1(\xi) = A_0 + A_1 T(\xi) + A_2 \ T^2(\xi) + A_3 \ T^3(\xi) + A_4 \ T^4(\xi) + A_5 \ T^5(\xi) + A_6 \ T^6(\xi), \tag{35}$$

where $A_i(i = 0, 1, \ldots, 6), A_6 \neq 0$. Here, $T(\xi)$ is the solution of the equation:

$$T^{'2}(\xi) = T^2(\xi)[1 - \zeta T^4(\xi)] \ln^2 K, \ \ 0 < K, \ K \neq 1. \tag{36}$$

Inserting (35) along with (36) into Equation (21), one may combine all of the coefficients of $[T(\xi)]^l[T^{'}(\xi)]^f, (l = (0, 1, 2, \ldots, 24, f = 0, 1)$ and set them to zero. Then, we arrive at a system of algebraic equations that can be solved by Maple to obtain the results:

$$A_0 = A_1 = \ A_2 = A_3 = A_4 = A_5 = 0, \quad A_6 = 192\varepsilon\zeta \ln^3 K\sqrt{\frac{35\zeta}{\Delta_6 + \Delta_5}}, \tag{37}$$

and

$$\Delta_1 = -332\ \ln^2 K, \quad \Delta_2 = 30256 \ln^4 K - \frac{3}{4}\Delta_3,\ \Delta_4 = -705600 \ln^6 K - 9\Delta_3 \ln^2 K, \tag{38}$$

provided that $\zeta(\Delta_5 + \Delta_6) > 0$ and $\varepsilon = \pm 1$.

Substituting (37) with (24) into Equation (35), we acquire the straddled solitary solution to Equations (2) and (3) as:

$$u(x,t) = \frac{12288\varepsilon\, \zeta A^3 \ln^3 K \sqrt{\frac{35\zeta}{\Delta_5+\Delta_6}}}{[4A^2\ exp_K[2(x-Vt)] + \zeta \exp_K[-2(x-Vt)]^3} e^{i(-\ \kappa x + wt + \theta_0)}, \tag{39}$$

and

$$v(x,t) = \chi \left[\frac{12288\varepsilon\, \zeta A^3 \ln^3 K \sqrt{\frac{35\zeta}{\Delta_5+\Delta_6}}}{[4A^2\ exp_K[2(x-Vt)] + \zeta \exp_K[-2(x-Vt)]^3} \right] e^{i(-\ \kappa x + wt + \theta_0)}. \tag{40}$$

In the special case, if we set $\zeta = 4A^2$ in Equations (39) and (40) posteriorly, we construct the bright soliton solution to Equations (2) and (3) as

$$u(x,t) = 192\varepsilon\ \ln^3 K \sqrt{\frac{35}{\Delta_5 + \Delta_6}}\ \text{sech}^3\ [2(x - Vt)\ln K] e^{i(-\ \kappa x + wt + \theta_0)}, \tag{41}$$

and

$$v(x,t) = \chi \left[192\varepsilon\ \ln^3 K \sqrt{\frac{35}{\Delta_5 + \Delta_6}}\ \text{sech}^3\ [2(x - Vt)\ln K] \right] e^{i(-\ \kappa x + wt + \theta_0)}. \tag{42}$$

provided that $\Delta_5 + \Delta_6 > 0$, while, if we set $\zeta = -4A^2$ posteriorly, one obtains the singular solitonsolution of Equations (2) and (3) as:

$$u(x,t) = 192\varepsilon \ln^3 K \sqrt{\frac{-35}{\Delta_5 + \Delta_6}}\ \ \text{csch}^3\ [2(x - Vt)\ln K]\ e^{i(-\ \kappa x + wt + \theta_0)}, \tag{43}$$

and

$$v(x,t) = \chi \left[192\varepsilon\ \ln^3 K \sqrt{\frac{-35}{\Delta_5 + \Delta_6}}\ \ \text{csch}^3\ [2(x - Vt)\ln K] \right] e^{i(-\ \kappa x + wt + \theta_0)}, \tag{44}$$

provided that $\Delta_5 + \Delta_6 < 0$.

Remark 2. *Under the constraint circumstances (38), the solutions (39)–(44) exist.*

Similarly, by changing the parameters h and N, one can obtain numerous solitary wave solutions of Equations (2) and (3).

4. Unified Riccati Equation Expansion Method

In this part, we apply the unified Riccati equation expansion approach to investigate the soliton solution to Equations (2) and (3). We assume that the solution to Equation (21) takes the following forms in this method:

$$\phi_1(\xi) = \sum_{i=0}^{N} B_i\ [Y(\xi)]^i, \tag{45}$$

where $B_i, i = 0, 1, 2, \ldots, N$ are constants while $Y(\xi)$ is the solution of Riccati equation:

$$Y'(\xi) = h_0 + h_1 Y + h_2 Y^2, \tag{46}$$

and $h_j, j = 0, 1, 2$ are arbitrary constants. Equation (21) now has a formal solution based on the balance number principle:

$$\phi_1(\xi) = B_0 + B_1 Y(\xi) + B_2 Y^2(\xi) + B_3 Y^3(\xi). \tag{47}$$

Equations (46) and (47) are substituted into Equation (21). Collecting the coefficients of $[Y(\xi)]^m, m = 0, 1, \ldots, 24$, we obtain a set of algebraic equations that may be solved using a computer software program to produce the following results:

$$\begin{aligned} B_0 &= \frac{105\epsilon h_1^3}{\sqrt{-35(\Delta_5+\Delta_6)}}, B_1 = \frac{360\epsilon h_1^2 h_2}{\sqrt{-35(\Delta_5+\Delta_6)}}, \\ B_2 &= \frac{1260\epsilon h_2^2 h_1}{\sqrt{-35(\Delta_5+\Delta_6)}}, B_3 = \frac{840\epsilon h_2^3}{\sqrt{-35(\Delta_5+\Delta_6)}}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} \Delta_1 = \tfrac{83}{2}\left(h_1^2 - 4h_2 h_0\right), \ \Delta_2 = \tfrac{1175}{2}\left(4h_2 h_0 - h_1^2\right)^2, \Delta_3 = -279\left(4h_2 h_0 - h_1^2\right)^2, \\ \Delta_4 = -315\left(4h_2 h_0 - h_1^2\right)^3, \ h_1^2 - 4h_2 h_0 \neq 0. \end{aligned} \tag{49}$$

provided that $\Delta_5 + \Delta_6 < 0$. Several other cases are removed for the sake of brevity. The precise solutions to the Riccati Equation (46) are provided by

$$Y(\xi) = \begin{cases} -\frac{h_1}{2h_2} - \frac{\sqrt{\Delta}}{2h_2} \frac{[r_1 \tanh(\frac{\sqrt{\Delta}}{2}\xi) + r_2]}{[r_1 + r_2 \tanh(\frac{\sqrt{\Delta}}{2}\xi)]} & \text{if} \quad \Delta > 0 \quad \text{and } r_1^2 + r_2^2 \neq 0, \\ -\frac{h_1}{2h_2} + \frac{\sqrt{-\Delta}}{2h_2} \frac{[r_3 \tan(\frac{\sqrt{-\Delta}}{2}\xi) - r_4]}{[r_3 + r_4 \tan(\frac{\sqrt{-\Delta}}{2}\xi)]} & \text{if} \quad \Delta < 0 \quad \text{and } r_3^2 + r_4^2 \neq 0, \end{cases} \tag{50}$$

where $r_j (j = 1, 2, \ldots, 5)$ are arbitrary constants and $\Delta = h_1^2 - 4h_0 h_2$.

4.1. Soliton Solutions

When $\Delta = h_1^2 - 4h_0 h_2 > 0$, Equations (48) and (50) lead to the solutions of Equation (21), which can be written as

$$\phi_1(\xi) = \frac{105\epsilon(\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}} \left[\frac{r_1 \tanh(\frac{\sqrt{\Delta}}{2}\xi) + r_2}{r_1 + r_2 \tanh(\frac{\sqrt{\Delta}}{2}\xi)}\right]^3. \tag{51}$$

Consequently, the solitary wave solutions of Equations (2) and (3) are given by

$$u(x,t) = \frac{105\epsilon(\Delta)^{3/2} e^{i(-\kappa x + wt + \theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}} \left[\frac{r_1 \tanh[\frac{\sqrt{\Delta}}{2}(x - Vt)] + r_2}{r_1 + r_2 \tanh[\frac{\sqrt{\Delta}}{2}(x - Vt)]}\right]^3, \tag{52}$$

and

$$v(x,t) = \chi \left\{ \frac{105\epsilon \ (\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}} \left[\frac{r_1 \tanh[\frac{\sqrt{\Delta}}{2}(x - Vt)] + r_2}{r_1 + r_2 \tanh[\frac{\sqrt{\Delta}}{2}(x - Vt)]}\right]^3 \right\} e^{i(-\kappa x + wt + \theta_0)}. \tag{53}$$

In particular, when $r_1 \neq 0$ and $r_2 = 0$ in Equations (52) and (53), the dark solitons that emerge are given by

$$u(x,t) = \frac{105\epsilon \ (\Delta)^{3/2} e^{i(-\kappa x + wt + \theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}} \tanh^3[\frac{\sqrt{\Delta}}{2}(x - Vt)] \tag{54}$$

and

$$v(x,t) = \chi\left[\frac{105\epsilon\ (\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \tanh^3[\frac{\sqrt{\Delta}}{2}(x-Vt)]\right]e^{i(-\ \kappa x+wt+\theta_0)}. \tag{55}$$

while, when $r_1 = 0$ and $r_2 \neq 0$ in Equations (52) and (53), the singular solutions are given by

$$u(x,t) = \frac{105\epsilon\ (\Delta)^{3/2}e^{i(-\ \kappa x+wt+\theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \coth^3[\frac{\sqrt{\Delta}}{2}(x-Vt)] \tag{56}$$

and

$$v(x,t) = \chi\left[\frac{105\epsilon\ (\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \coth^3[\frac{\sqrt{\Delta}}{2}(x-Vt)]\right]e^{i(-\ \kappa x+wt+\theta_0)}. \tag{57}$$

4.2. Periodic Wave Solutions

When $\Delta = h_1^2 - 4h_0h_2 < 0$, Equations (48) and (50) lead to the periodic solutions of Equations (2) and (3), which can be written as follows:

$$u(x,t) = \frac{105\epsilon(-\Delta)^{3/2}e^{i(-\ \kappa x+wt+\theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}}\left[\frac{r_3\tan[\frac{\sqrt{-\Delta}}{2}(x-Vt)]-r_4}{r_3+r_4\tan[\frac{\sqrt{-\Delta}}{2}(x-Vt)]}\right]^3, \tag{58}$$

and

$$v(x,t) = \chi\left\{\frac{105\epsilon(-\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}}\left[\frac{r_3\tan[\frac{\sqrt{-\Delta}}{2}(x-Vt)]-r_4}{r_3+r_4\tan[\frac{\sqrt{-\Delta}}{2}(x-Vt)]}\right]^3\right\}e^{i(-\ \kappa x+wt+\theta_0)}. \tag{59}$$

In particular, when $r_3 \neq 0$ and $r_4 = 0$ in Equations (58) and (59), the periodic solutions are given by

$$u(x,t) = \frac{105\epsilon(-\Delta)^{3/2}e^{i(-\ \kappa x+wt+\theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \tan^3[\frac{\sqrt{-\Delta}}{2}(x-Vt)], \tag{60}$$

and

$$v(x,t) = \chi\left[\frac{105\epsilon(-\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \tan^3[\frac{\sqrt{-\Delta}}{2}(x-Vt)]\right]e^{i(-\ \kappa x+wt+\theta_0)}, \tag{61}$$

while, when $r_3 = 0$ and $r_4 \neq 0$ in Equations (58) and (59), the periodic solutions are given by

$$u(x,t) = -\frac{105\epsilon(-\Delta)^{3/2}e^{i(-\ \kappa x+wt+\theta_0)}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \cot^3[\frac{\sqrt{-\Delta}}{2}(x-Vt)], \tag{62}$$

and

$$v(x,t) = \chi\left[-\frac{105\epsilon(-\Delta)^{3/2}}{\sqrt{-35(\Delta_5+\Delta_6)}}\ \cot^3[\frac{\sqrt{-\Delta}}{2}(x-Vt)]\right]e^{i(-\ \kappa x+wt+\theta_0)}. \tag{63}$$

Remark 3. *The solutions (52)–(63) exist under the constraint conditions (49).*

5. Numerical Simulations

In this section, we show the graphs (Figures 1 and 2) of some solutions to Equations (2) and (3). To accomplish this, we choose certain particular values for the obtained parameters of these solutions.

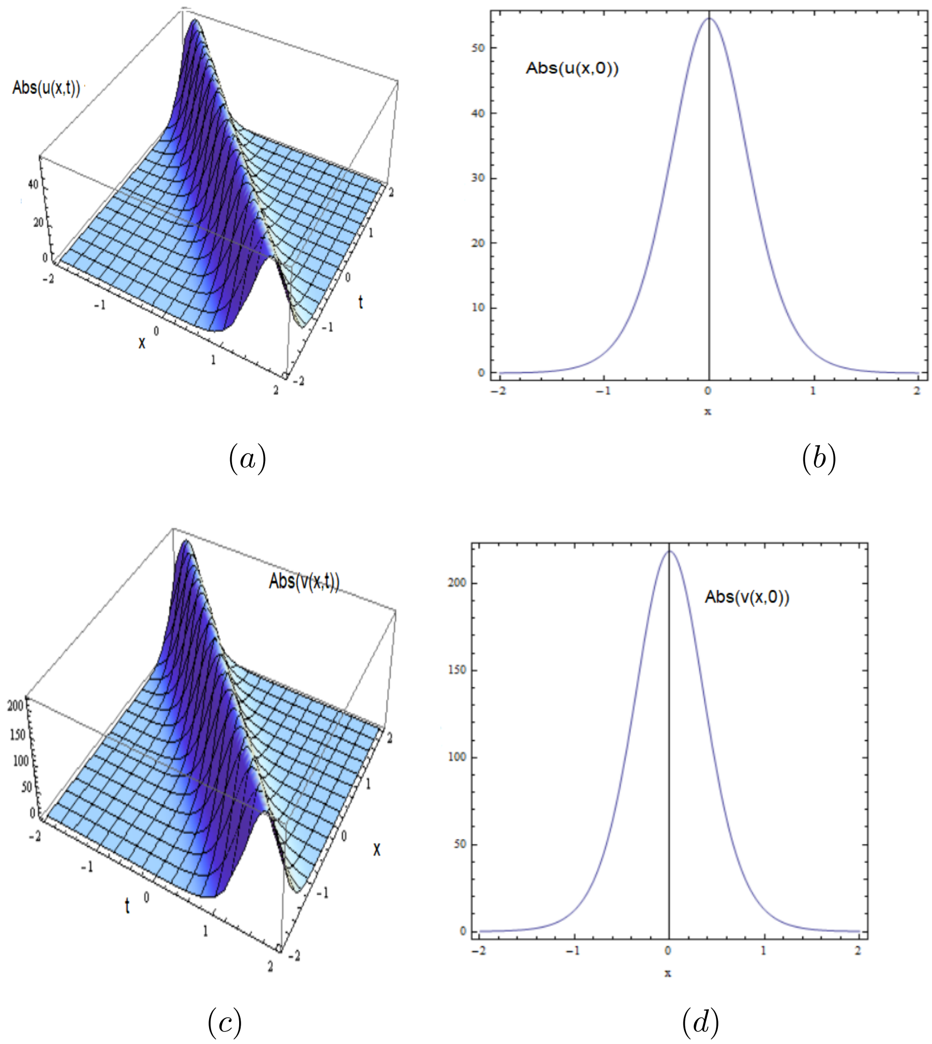


Figure 1. The numerical simulation of the bright soliton solutions (31) and (32) in 3D and its projection in 2D when $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 2, \lambda_1 = 4, \mu_1 = 3, \chi = 4, v_1 = -5, \zeta_1 = 2, K = 5, c_1 = 5$ and $d_1 = 7$.

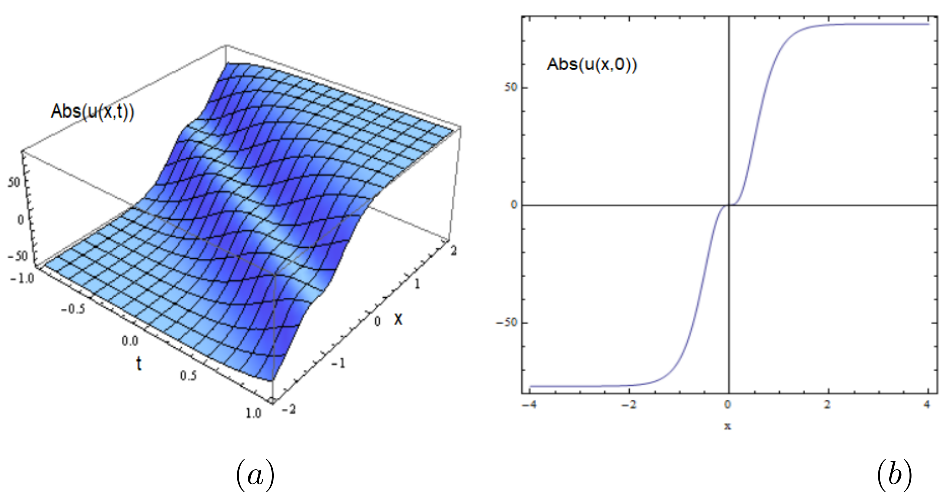
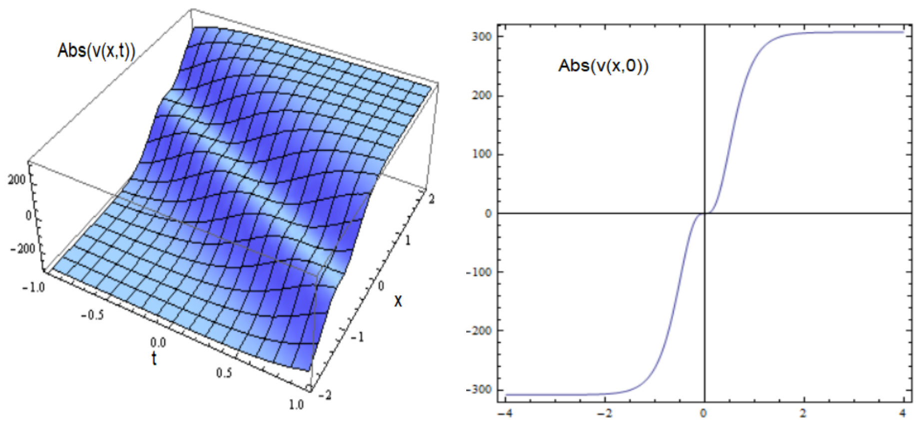


Figure 2. *Cont*.



(c) (d)

Figure 2. The numerical simulation of the dark soliton solutions (54) and (55) in 3D and its projection in 2D when $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 2, \lambda_1 = 4, \mu_1 = 3, \chi = 4, v_1 = -5, \zeta_1 = 2, c_1 = -5, h_1 = 5, h_0 = 2, h_2 = 3$ and $d_1 = -7$.

6. Conclusions

The highly dispersive nonlinear complex sixth-order Ginzburg–Landau (CGL) equation in birefringent fibers with Kerr law nonlinearity was studied in this work. Two integration methods were used via the addendum to Kudryashov's method and the unified Riccati equation expansion method. We found the bright, dark, singular soliton solutions for this model. In future, this work will be extended to fiber Bragg gratings and magneto-optic waveguides. In addition, we will study Equation (1) and the systems (2) and (3) with variable coefficients. Finally, all solutions of this article have been checked using Maple by putting them back into the original equations.

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