engineering proceedings

Proceeding Paper

Sum of Exponential Model for Fitting Data †

Ting-Cheng Chang *, Min-Hao Wu and Ying Lin

Department of Computer Science, College of Information Engineering, Guangzhou Panyu Polytechnic, Guangzhou 511483, China; minhao1616@126.com (M.-H.W.); ying.lins@outlook.com (Y.L.)
* Correspondence: tcchang0615@gmail.com

† Presented at the 3rd IEEE International Conference on Electronic Communications, Internet of Things and Big Data Conference 2023, Taichung, Taiwan, 14–16 April 2023.

Abstract: As an approach to feature estimation, exponential fitting has attracted research interests in mathematical modeling. Semantic networks are used for numerous applications in computers, physics, and biology. However, such applications may have fitting troubles with various mathematical tools. Therefore, we present a novel method of fitting $2n$ data points of a signal to a sum of n exponential functions. The experiments proved that the proposed method operated well for linear and nonlinear functions, as its algorithm was straightforward, practical, and easy to determine. At the same time, the computational intricacy was considerably low, which has specific worth in use.

Keywords: fitting $2n$ data points; a sum of n; exponential function

check for updates

Citation: Chang, T.-C.; Wu, M.-H.; Lin, Y. Sum of Exponential Model for Fitting Data. *Eng. Proc.* **2023**, *38*, 87. https://doi.org/10.3390/engproc2023038087

Academic Editors: Teen-Hang Meen, Hsin-Hung Lin and Cheng-Fu Yang

Published: 25 July 2023

1. Introduction

Relevant information with the linear combinations of real and complex exponentials is pervasive in science and engineering applications. Given that Gaspard Riche de Prony developed an approach [1] to resolve the problem for equally spaced samples, numerous advancements, and applications have been proposed. We surveyed the most effective ones to explain their applications and experiences and to allow their application in various fields. A linear combination of exponentials was used in regular differential equations to explain the different physical processes. After being modeled by the remedy of a formula, a combination of exponentials provided valuable information such as decay rates or product residential or commercial properties in a physical system. Likewise, the exponential fitting had an excellent approximation on the compact of the domain with Fourier transformation in complex exponentials [2–6].

The purpose of this study was to present a method of fitting real signal data sampled at a period T in a set of $2n$ data points $\{x(0),\ x(T),\ x(2T),\ \cdots,\ x([2n-1]T)\}$. The data points to the s curve were composed of n exponential functions with unknown weights and exponents. Mathematically, this involved the solution of the following Equations (1)–(5).

$$x(kT) = \sum_{i=1}^{n} c_i e^{p_i(kT)} \tag{1}$$

For the unknown C_i and P_i in the complex conjugate pairs (P_i is an imaginary number), Equation (1) represents a sum of sinusoids. This curve fitting can have many applications. For example, if $x(t)$ represents the impulse response of a linear time-invariant system, and the Laplace transform of Equation (1) yields the transfer function of an n^{th}-order model of the system.

2. Curve Fitting Method

We let ϕ_i denote $e^{p_i(kT)}$ and x_i denote $x(kT)$. Then, Equation (1) could be rewritten as:

$$\begin{array}{llll} c_1 & +c_2 & +\cdots+c_n & =x_0 \\ c_1\phi_1 & +c_2\phi_2 & +\cdots+c_n\phi_n & =x_1 \\ c_1\phi_1^2 & +c_2\phi_2^2 & +\cdots+c_n\phi_n^2 & =x_2 \\ & & \vdots & \\ c_1\phi_1^{2n-1} & +c_2\phi_2^{2n-1} & +\cdots+c_n\phi_n^{2n-1} & =x_{2n-1} \end{array} \tag{2}$$

Equation (2) is explained by the following theorem.

Theorem 1. *The nonlinear equation, such as Equation (2), possesses a unique solution {c_k, N_k} (k = 1, 2, . . . , n) if and only with the following $n \times n$ matrix, which is nonsingular.*

$$A \equiv \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_{n-1} & x_n & \cdots & x_{2n-2} \end{bmatrix} \tag{3}$$

The solution for N_k is given by the n distinct roots of the n^{th} degree polynomial equation.

$$\det \begin{bmatrix} 1 & x_0 & x_1 & \cdots & x_{n-1} \\ \phi_k & x_1 & x_2 & \cdots & x_n \\ \phi_k^2 & x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_k^n & x_n & x_{n+1} & \cdots & x_{2n-1} \end{bmatrix} = 0 \tag{4}$$

The solution for c_k can then be given by:

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_k & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_k^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_k^{2n-1} & \phi_2^{2n-1} & \phi_3^{2n-1} & \cdots & \phi_n^{2n-1} \end{bmatrix}^{-1} \tag{5}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = A \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{6}$$

3. Proof

3.1. Sufficiency Part

It could be supposed that A was nonsingular; the first $n1$ equation of Equation (2) could be arranged as:

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_k & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_k^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_k^{2n-1} & \phi_2^{2n-1} & \phi_3^{2n-1} & \cdots & \phi_n^{2n-1} \end{bmatrix} \tag{7}$$

$$B\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{8}$$

We let the columns of the left-most matrix in Equation (8) be denoted as:

$$v_k \equiv \begin{bmatrix} 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^n \end{bmatrix}^T, k = 1,\ 2,\ \cdots, n \tag{9}$$

and let $x_0 \equiv \begin{bmatrix} x_0 & x_1 & \cdots & x_n \end{bmatrix}^T$. Then, Equation (8) showed that x_0 was a linear combination of $\{v_1, v_2, \ldots, v_n\}$.

Next, if the set of n + 1 consecutive equations in Equation (2) was considered as the starting point with the second equation, they could be rearranged as:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^n & \phi_2^n & \phi_3^n & \cdots & \phi_n^n \end{bmatrix} \begin{bmatrix} c_1\phi_1 \\ c_2\phi_2 \\ c_3\phi_3 \\ \vdots \\ c_n\phi_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n+1} \end{bmatrix} \tag{10}$$

Equation (10) shows that $x_1 \equiv [x_1,\ x_2,\ \ldots,\ x_{n+1}]^T$ was a linear combination of $\{v_1, v_2, \ldots, v_n\}$.

Similarly, if we considered the set of n + 1 consecutive equations in Equation (2) starting with the third equation, we could see that $x_2 \equiv [x_2,\ x_3,\ \ldots,\ x_{n+2}]^T$ was a linear combination of $\{v_1, v_2, \ldots, v_n\}$.

This continued until the last n + 1 of Equation (2) was taken, from which it was shown that $x_{n-1} \equiv [x_{n-1}, x_n, \ldots x_{2n+1}]^T$ was a linear combination of $\{v_1, v_2, \ldots, v_n\}$.

Equation (3) of the Theorem implies that the vectors $\{x_0, x_1, \ldots x_{n-1}\}$ are linear and independent of each other. Hence, they span an n-dimensional subspace in an (n + l) dimensional Euclidean space. This subspace must be the same as the one spanned by the vectors $\{v_1, v_2, \ldots, v_n\}$ since each x_i, i = 0, 1, …, $n-1$ is a linear combination of the set $\{v_1, v_2, \ldots, v_n\}$. It follows that the vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent and that, from Equation (7) of v_k (k = 0, 1, …, n), they must be distinct.

Moreover, each v_k is a linear combination of $\{x_0, x_1, \ldots x_{n+1}\}$. This implies:

$$v_k = \begin{bmatrix} 1 \\ \phi_k \\ \phi_k^2 \\ \vdots \\ \phi_k^n \end{bmatrix} = d_1 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + d_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n+1} \end{bmatrix} + \cdots + d_n \begin{bmatrix} x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \\ x_{2n-1} \end{bmatrix} \tag{11}$$

Equation (11) could be rearranged as an (n + 1) × (n + 1) equation system.

$$\begin{bmatrix} 1 & x_0 & x_1 & \cdots & x_{n-1} \\ \phi_k & x_1 & x_2 & \cdots & x_n \\ \phi_k^2 & x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_k^n & x_n & x_{n+1} & \cdots & x_{2n-1} \end{bmatrix} \begin{bmatrix} -1 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = 0 \tag{12}$$

Since the solution of Equation (12) was nontrivial, the determinant of the square matrix had to vanish, leading to Equation (4) which was an n^{th}-degree polynomial equation in N_k because the coefficient of the n^{th} power term of N_k could be seen from Equations (3) and (12) to be $(-1)^n$ det A, which was assumed to be nonzero. The n roots N_k of Equation (4) must be distinct because each N_k had to satisfy Equation (4) and be distinct. Having obtained the distinct values of N_k, $k = 0, 1, \ldots, n$, c_k could be given by the first n equations (Equation (2)), which led to Equation (6) in the Theorem.

As for the uniqueness of the solution, since every solution $\{c_k, N_k\}$ ($k = 1, 2, \ldots, n$) had to satisfy Equation (4), according to the above arguments, Equation (4) produced exactly n distinct values for N_k, and the solution of Equation (2) was unique.

3.2. Necessity Part

Suppose Equation (2) has a unique solution $\{c_k, N_k\}$ ($k = 1, 2, \ldots, n$). This implies the following.

(i) N_k must be distinct from each other; otherwise, non-unique combinations of c_k in Equation (2) exist and are fulfilled.

(ii) None of c_k vanishes; otherwise, the value of N_k associated with a vanishing c_k becomes non-unique.

The first n of Equation (2) gave:

$$C = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \phi_3^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix}^{-1} \tag{13}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = C \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{14}$$

The next n of Equation (2), starting with the second equation, gave:

$$D = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \phi_3^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix}^{-1} \tag{15}$$

$$\begin{bmatrix} c_1\phi_1 \\ c_2\phi_2 \\ c_3\phi_3 \\ \vdots \\ c_n\phi_n \end{bmatrix} = D \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \tag{16}$$

This proceeded until the set of n in the consecutive Equation (2), starting with the n^{th} equation, was reached.

$$E = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \phi_3^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix}^{-1} \tag{17}$$

$$\begin{bmatrix} c_1\phi_1^{n-1} \\ c_2\phi_2^{n-1} \\ c_3\phi_3^{n-1} \\ \vdots \\ c_n\phi_n^{n-1} \end{bmatrix} = E \begin{bmatrix} x_{n-1} \\ x_2 \\ x_{n+1} \\ \vdots \\ x_{2n-2} \end{bmatrix} \tag{18}$$

Combining Equations (14) to (18) yielded:

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \phi_3^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix}^{-1} \tag{19}$$

$$G = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{n-1} & x_n & x_{n+1} & \cdots & x_{2n-2} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} c_1 & c_1\phi_1 & \cdots & c_1\phi_1^{n-1} \\ c_2 & c_2\phi_2 & \cdots & c_2\phi_2^{n-1} \\ c_3 & c_3\phi_3 & \cdots & c_3\phi_3^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_n & c_n\phi_n & \cdots & c_n\phi_n^{n-1} \end{bmatrix} = FG \tag{21}$$

4. Examples

Consider the signal:

$$x(t) = 2e^{-2t} - 3e^{t} \tag{22}$$

Sampling this signal at a sampling period T = 1 yielded Equation (2) with c_1 = 2, c_2 = −3, $N_1 = e^{-2}$ = 0.13533283, $N_2 = e^1$ = 2.718281828, x_0 = −1, x_1 = −7.88417491, x_2 = −22.130537, and x_3 = −60.2516532. Let us reverse this process. After sampling four consecutive points of the signal $x(t)$ at a uniform sampling period T = 1, we obtained the values of $\{x_0, x_1, x_2, x_3\}$, as indicated above, which could be solved for $\{c_1, c_2, N_1, N_2\}$. From Equation (2), we obtained:

$$c_1 + c_2 = -1 \tag{23}$$

$$c_1\phi_1 + c_2\phi_2 = -7.88417491 \tag{24}$$

$$c_1\phi_1^2 + c_2\phi_2^2 = -22.130537 \tag{25}$$

$$c_1\phi_1^3 + c_2\phi_2^3 = -60.2516532 \tag{26}$$

First, we could see that:

$$A = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \end{bmatrix} = \begin{bmatrix} -1 & -7.88417491 \\ -7.88417491 & -22.130537 \end{bmatrix} \tag{27}$$

was nonsingular. N_k was obtained as the root of Equation (4), which, in this case, led to:

$$\begin{aligned} &\det \begin{bmatrix} 1 & -1 & -7.88417491 \\ \phi_k & -7.88417491 & -22.130537 \\ \phi_k^2 & -22.130.537 & -60.2516532 \end{bmatrix} \\ &= -40.029677\phi_k^2 + 114.2293714\phi_k - 14.7260955 \\ &= 0 \end{aligned} \tag{28}$$

This yielded N_1 = 0.135335283 and N_2 = 2.718281828. Substituting these values into Equation (6) provided:

$$\begin{bmatrix} 1 & 1 \\ 0.13533528 & 2.71828182 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -7.88417491 \end{bmatrix} \tag{29}$$

from which we obtained $c_1 = 2$ and $c_2 = -3$. From known N_1 and N_2, $p_1 = -2$ and $p_2 = 1$ were obtained according to $\phi_i = e^{piT.}$

5. Conclusions

The problem of fitting *2n* data points to a curve consisting of *n* exponential functions was solved. The exponential functions were complex in general, with sinusoids being a special case. The curve-fitting problem was solved by a system of nonlinear algebraic equations. An example has been given to illustrate the procedure of this method.

Author Contributions: Conceptualization, methodology, T.-C.C.; writing—original draft preparation, M.-H.W.; validation, writing—review and editing, Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Guangdong College Research Platform and Research Project grant number 2021ZDZX1137 and 2019GKTSCX069, the Panyu Polytechnic Innovation Team grant number 2020CXTD003 (2011/210113263), the Panyu Polytechnic Research Project grant number 2021KJ04 (2011/210113263), and the Department of Education of Guangdong Province, China, grant number 2020KQNCX192.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Prony, R. Essai experimental et analytique sur les lois de la Dilatabilit_e des uideselastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, a differentes temperatures. *J. L'ecole Polytech. Floreal Prairial* **1795**, *3*, 24–76.
2. Istratovand, A.A.; Vyvenko, O.F. Exponential analysis in physical phenomena. *Rev. Sci. Instrum.* **1999**, *70*, 1233–1257. [CrossRef]
3. Naus, H.; van Stokkum, I.H.M.; Hogervorst, W.; Ubachs, W. Quantitative analysis of decay transients applied to a multimode pulsed cavity ringdown experiment. *Appl. Opt.* **2001**, *40*, 4416–4426. [CrossRef] [PubMed]
4. Stry, S.; Hering, P.; Mürtz, M. Portable difference-frequency laser-based cavity leak-out spectrometer for trace-gas analysis. *Appl. Phys. B* **2002**, *75*, 297–303. [CrossRef]
5. von Basum, G.; Dahnke, H.; Halmer, D.; Hering, P.; Mürtz, M. Online recording of ethane traces in human breath via infrared laser spectroscopy. *J. Appl. Physiol.* **2003**, *95*, 2583–2590. [CrossRef] [PubMed]
6. Booth, R.S. Fitting Data Containing Multiple Exponentials Using Functionals Derived from Finite Laplace Transforms of Time Moments of the Data. *Nucl. Technol.* **2017**, *198*, 217–227. [CrossRef]