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Resolution of Systems of Difference Equations and Its Implications for the VAR Model †

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Abstract: Systems of difference equations frequently present dynamically unstable solutions in the long term, which could imply the appearance of complications in the application of vector autoregressive (VAR) models in the Johansen sense, regardless of the precision required. In this work the necessary conditions are presented to guarantee the dynamical convergence of the solutions from the approach of the systems in discrete time series with the stochastic processes. The main aim is to show the importance of dynamic stability in structural-type models with respect to estimator bias.

Keywords: VAR model; systems of differential equations; cointegration in the long run

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1. Introduction

In order to analyze the structure and/or forecast realizations of a stochastic process, or an observation within the time series, the models currently developed by econometricians have a certain degree of complexity. Thus, econometric estimation and the analysis of stochastic processes to explain economic phenomena through models are increasingly relevant, mainly in the context of time series, taking into consideration that various processes focus on understanding the dynamic structure of the series and on the possibility of forecasting its dynamic pattern of temporal behavior or the extrapolation of a stochastic process [1–4] where the lags of the variables involved play a key role in terms of the autoregressive models, as is the case in the estimation of autoregressive vector models (VARs), the central theme in this work.

These models are expressed through differential equations, since each variable is explained by the lags of both itself and the remaining variables.

It should be noted that each of the variables involved must meet the assumption of stationarity as a particular state of statistical equilibrium, where their probability distributions remain stable over time [5,6]. This implies that once the system is interrupted by some type of shocks, it will adjust back to equilibrium [7] or the shocks gradually disappear.

In estimating these models, it is accepted and often required that, if the estimators meet the tests, then they are the best linearly unbiased estimators (BLUEs). But what happens if the tests applied to the model are not fulfilled? Are the estimators not valid for the analysis? Does the model have to be scrapped?

2. Empirical Obtaining of Estimators in a VAR Model

It is important to point out that when there are large samples, the assumptions of normality, homoscedasticity, and the absence of autocorrelation in the errors are hardly fulfilled. This occurs regularly when using short duration data, for example monthly, quarterly, as well as long periods in the analysis. This could mean a limitation of the model that leads to strong criticism in this regard; however, as Wooldridge mentioned in his modern approach, given the law of large numbers, an asymptotic normality is assumed, due to the size of the sample and in terms of the homoscedasticity and absence of autocorrelation in the errors, the results obtained allow themselves to be the best linearly unbiased estimators, pointed out by Guarati and Porter [8].

In this regard, it has agreed in common that for the forecasting purposes, the VAR models are required to fulfill the assumptions of the estimation: normality, homoscedasticity, and the absence of autocorrelation in the errors. However, if the estimated model is used only to analyze the structural changes of the economic variables, the requirement could be relaxed to the stability of the solutions; that is, the convergence in terms of the dynamic analysis, which can be determined by estimating the inverse roots of the characteristic polynomial of the autoregressive vector.

It should be noted that, since these are unrestricted models, the main advantage is that there will be no specification errors in the empirical estimation, in addition to the fact that the long-term cointegration solution is exempt from the problem of spuriousness or meaningless regressions, as it is defined by Granger and Newbold [9], with the initial idea owed to Yule [10].

Therefore, the VAR models not only provide a better estimate of forecasts compared to static ones, but also could be analyzed in a dynamic and structural manner where the importance of a shock of one variable on the others is revealed, with the relaxation of the related assumptions.

To apply the structural analysis of systems of simultaneous differential equations derived from the VAR approach, the series corresponding to Mexican exports to the US market and imports of Chinese origin are used with an annual periodicity from 2001 to 2020 (Figure 1).

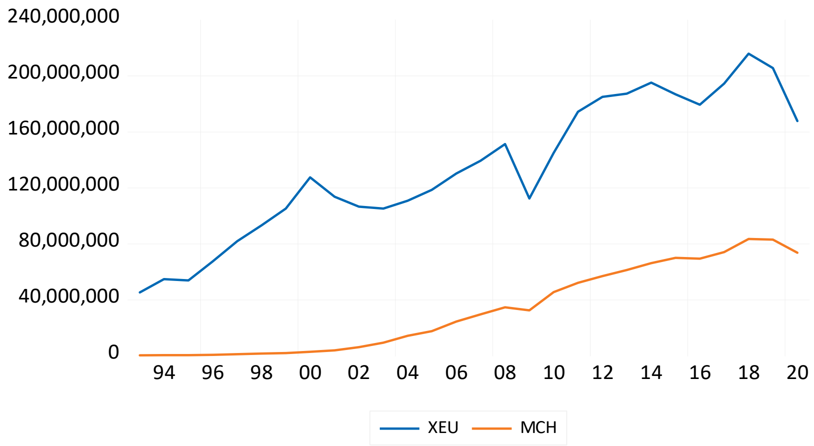

Figure 1. Mexican exports to the United States and Chinese imports to Mexico in millions of dollars.

It should be noted that the study Mexican trade with its two most important trading partners during the last 20 years is of great importance given the two facts. Firstly, China's entry into the World Trade Organization, an unprecedented event that contributed to the expansion of the Chinese products around the world markets, and eventually has become the second largest trading partner for Mexico since 2003. Secondly, at the beginning of 2020 COVID-19 pandemic has generated an important structural change in the trading among the three countries.

It is evident that both series are non-stationary since they have a trend. In order to estimate a VAR model, it is necessary for the series to be integrated in the same order to be transformed into stationariy; in this particular case, they are I(1). Figure 2 shows the stationarity of the two series after the respective first difference, which was confirmed by augmented Dickey–Fuller [11] and Phillips–Perron [12] tests.

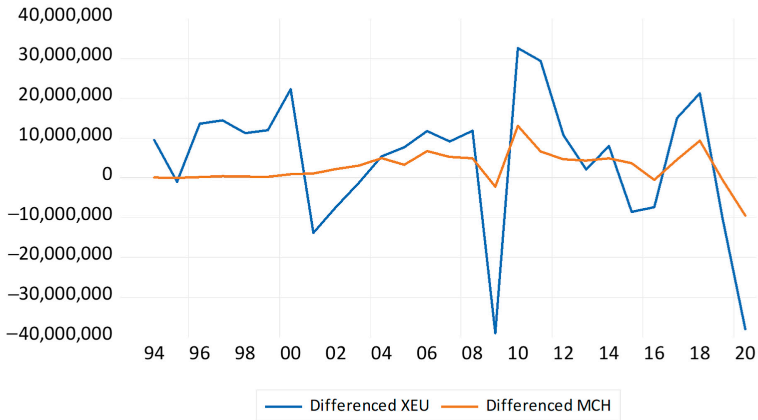

Figure 2. Mexican exports to the United States and Chinese imports to Mexico by first differences.

Subsequently, the VAR models were estimated with one and two lags respectively, estimations that accredited the stability tests within the unit circle and the residuals model's presented normality, homoscedasticity and an absence of autocorrelation.

2.1. Solution of the VAR System of Equations with One Lag

The equation derived from the normalized equation estimated in the Eviews 12 Student Version Lite of S&P Global, New York, NY, USA with a model that shows stability within the unit circle (inverse roots of the autoregressive characteristic polynomial) is as follows:

$$XEU_t = 0.0613XEU_{t-1} + 1.142MCH_{t-1} + 98,954,092.05 \quad (1)$$

$$MCH_t = -0.1816XEU_{t-1} + 1.1944MCH_{t-1} + 23,399,484.6987 \quad (2)$$

$$XEU_t = 1.76\ MCH_t \quad (3)$$

To simplify the notation, suppose that $XEU_t = A_t$ and $MCH_t = B_t$, so that

$$A_t = 0.0613A_{t-1} + 1.142B_{t-1} + 98,954,092.05 \quad (4)$$

$$B_t = -0.1816A_{t-1} + 1.1944B_{t-1} + 23,399,484.6987 \quad (5)$$

This implies that.

$$A_t - 0.0613A_{t-1} - 1.142B_{t-1} = 98,954,092.05 \quad (6)$$

$$B_t + 0.1816A_{t-1} - 1.1944B_{t-1} = 23,399,484.7 \quad (7)$$

Particular solutions that can be supposed for the two variables are $A_t = k_1$ and $B_t = k_2$, which implies that $A_{t-1} = k_1$ and $B_{t-1} = k_2$; then,

$$k_1 - 0.0613k_1 - 1.142k_2 = 98,954,092.05 \quad (8)$$

$$k_2 + 0.1816k_1 - 1.1944k_2 = 23,399,484.7 \quad (9)$$

Reducing yields are the following:

$$0.9387k_1 - 1.142k_2 = 98,954,092.05 \quad (10)$$

$$0.1816k_1 - 0.1944k_2 = 23,399,484.7 \quad (11)$$

Solving the system yields $k_1 = 300,576,585.66$ and $k_2 = 160,417,818.7$, while normalizing yields the following:

$$\frac{k_1}{k_2} = \frac{300,576,585.66}{160,417,818.7} = 1.8 \approx 1.76$$

Note that 1.8 is very close to the solution provided by the software (1.76).

To obtain the complementary solutions, the following can been carried out:

We assume that $A_t = \gamma\beta^t$ and $B_t = \delta\beta^t$ where γ and δ are constants; therefore, $A_{t-1} = \gamma\beta^{t-1}$ and $B_{t-1} = \delta\beta^{t-1}$.

Substituting into the difference equations that are now homogeneous,

$$\gamma\beta^t - 0.0613\gamma\beta^{t-1} - 1.142\delta\beta^{t-1} = 0 \tag{12}$$

$$\delta\beta^t + 0.1816\gamma\beta^{t-1} - 1.1944\delta\beta^{t-1} = 0 \tag{13}$$

Multiply everything by β^{1-t}, arriving at the following:

$$\gamma\beta - 0.0613\gamma - 1.142\delta = 0 \tag{14}$$

$$\delta\beta + 0.1816\gamma - 1.1944\delta = 0 \tag{15}$$

Grouping then yields

$$(\beta - 0.0613)\gamma - 1.142\delta = 0$$

$$0.1816\gamma + (\beta - 1.1944)\delta = 0$$

Obtaining the following:

$$(\beta - 0.0613)(\beta - 1.1944) - (0.1816)(-1.1944) = 0 \tag{16}$$

$$\beta^2 - 1.2557\beta + 0.2805 = 0 \tag{17}$$

Solving the equation, the values of β are $\beta_1 = 0.965$ and $\beta_2 = 0.2907$.

The complementary solutions are as follows:

$$A_t = \gamma_1(0.965)^t + \gamma_2(0.2907)^t \tag{18}$$

$$B_t = \delta_1(0.965)^t + \delta_2(0.2907)^t \tag{19}$$

Consequently, the general solutions are as follows:

$$A_t = \gamma_1(0.965)^t + \gamma_2(0.2907)^t + 300,576,585.66 \tag{20}$$

$$B_t = \delta_1(0.965)^t + \delta_2(0.2907)^t + 160,417,818.7 \tag{21}$$

2.2. Solution of the VAR System of Equations with Two Lags

Regrouping yields the following:

$$A_t = 0.176A_{t-1} - 0.45\,1A_{t-2} + 1.48B_{t-1} + 0.064B_{t-2} + 131,598,419.78 \tag{22}$$

$$B_t = -0.1626A_{t-1} - 0.0724A_{t-2} + 1.236B_{t-1} + 0.023B_{t-2} + 28,642,607.5672 \tag{23}$$

$$1.27k_1 - 1.544k_2 = 131,598,419.78 \tag{24}$$

$$0.235k_1 - 0.259k_2 = 28,642,607.5672 \tag{25}$$

Solving the system results in $k_1 = 308,456,389.5$ and $k_2 = 169,284,339.7$.
Normalizing then results in the following:

$$\frac{k_1}{k_2} = \frac{308,456,389.5}{169,284,339.7} = 1.8$$

In this case, the solution given by the software is 1.67; therefore, it is consistent.

Following this reasoning, we could generalize as follows:
Let be a VAR model of j variables with i lags:

$$\begin{gathered}
X_{1\,t} = \alpha_{1\,1}\;X_{1\;\,t-1} + \alpha_{1\,2}\,X_{1\;\,t-2} + \cdots + \alpha_{1i}X_{1t-i} + \\
\alpha_{2\,1}X_{2\,t-1} + \alpha_{2\,2}\,X_{2\,t-2} + \cdots + \alpha_{2\,i}\,X_{2\;\,t-i} + \cdots + \\
\alpha_{j\,1}X_{j\,t-1} + \alpha_{j\,2}\,X_{j\,t-2} + \cdots + \alpha_{j\,i}\,X_{j\;\,t-i} + C_1 \\
X_{2\,t} = \beta_{1\,1}\;X_{1\;\,t-1} + \beta_{1\,2}\,X_{1\;\,t-2} + \cdots + \beta_{1i}X_{1t-i} + \\
\beta_{2\,1}X_{2\,t-1} + \beta_{2\,2}\,X_{2\,t-2} + \cdots + \beta_{2\,i}\,X_{2\;\,t-i} + \cdots + \\
\beta_{j\,1}X_{j\,t-1} + \beta_{j\,2}\,X_{j\,t-2} + \cdots + \beta_{j\,i}\,X_{j\;\,t-i} + C_2 \\
\vdots \\
X_{j\,t} = \gamma_{1\,1}\;X_{1\;\,t-1} + \gamma_{1\,2}\,X_{1\;\,t-2} + \cdots + \gamma_{1i}X_{1t-i} + \\
\gamma_{2\,1}X_{2\,t-1} + \gamma_{2\,2}\,X_{2\,t-2} + \cdots + \gamma_{2\,i}\,X_{2\;\,t-i} + \cdots + \\
\gamma_{j\,1}X_{j\,t-1} + \gamma_{j\,2}\,X_{j\,t-2} + \cdots + \gamma_{j\,i}\,X_{j\;\,t-i} + C_j
\end{gathered}$$

We express the VAR model in difference equations:

$$\begin{gathered}
X_{1\,t} - \alpha_{1\,1}\;X_{1\;\,t-1} - \alpha_{1\,2}\,X_{1\;\,t-2} - \cdots - \alpha_{1i}X_{1t-i} - \\
\alpha_{2\,1}X_{2\,t-1} - \alpha_{2\,2}\,X_{2\,t-2} - \cdots + \alpha_{2\,i}\,X_{2\;\,t-i} - \cdots - \\
\alpha_{j\,1}X_{j\,t-1} - \alpha_{j\,2}\,X_{j\,t-2} - \cdots - \alpha_{j\,i}\,X_{j\;\,t-i} = C_1 \\
X_{2\,t} - \beta_{1\,1}\;X_{1\;\,t-1} - \beta_{1\,2}\,X_{1\;\,t-2} - \cdots - \beta_{1i}X_{1t-i} - \\
\beta_{2\,1}X_{2\,t-1} - \beta_{2\,2}\,X_{2\,t-2} - \cdots - \beta_{2\,i}\,X_{2\;\,t-i} - \cdots - \\
\beta_{j\,1}X_{j\,t-1} - \beta_{j\,2}\,X_{j\,t-2} - \cdots - \beta_{j\,i}\,X_{j\;\,t-i} = C_2 \\
\vdots \\
X_{j\,t} - \gamma_{1\,1}\;X_{1\;\,t-1} - \gamma_{1\,2}\,X_{1\;\,t-2} - \cdots - \gamma_{1i}X_{1t-i} - \\
\gamma_{2\,1}X_{2\,t-1} - \gamma_{2\,2}\,X_{2\,t-2} - \cdots - \gamma_{2\,i}\,X_{2\;\,t-i} - \cdots - \\
\gamma_{j\,1}X_{j\,t-1} - \gamma_{j\,2}\,X_{j\,t-2} - \cdots - \gamma_{j\,i}\,X_{j\;\,t-i} = C_j
\end{gathered}$$

Supposing that $X_{1\,t} = k_1$, $X_{2\,t} = k_2$ and $X_{j\,t} = k_j$, then $X_{1\,t-1} = k_1$, $X_{2\,t-1} = k_2$, and $X_{j\,t-1} = k_j$.

Analogously, if $X_{1\,t-i} = k_1$, $X_{2\,t-2} = k_2$ and $X_{j\,t-1} = k_j$ then,

$$\begin{gathered}
k_1 - \alpha_{1\,1}\;k_1 - \alpha_{1\,2}\,k_1 - \cdots - \alpha_{1i}k_1 - \\
\alpha_{2\,1}k_2 - \alpha_{2\,2}k_2 - \cdots + \alpha_{2\,i}\,k_2 - \cdots - \\
\alpha_{j\,1}k_j - \alpha_{j\,2}k_j - \cdots - \alpha_{j\,i}\,k_j = C_1
\end{gathered}$$

$$k_1 - \beta_{1\,1}\ \ k_1 - \beta_{1\,2}k_1 - \cdots - \beta_{1i}k_1 -$$
$$\beta_{2\,1}k_2 - \beta_{2\,2}k_2 - \cdots - \beta_{2\,i}\,k_2 - \cdots -$$
$$\beta_{j\,1}k_j - \beta_{j\,2}k_j - \cdots - \beta_{j\,i}\,k_j = C_2$$
$$\vdots$$
$$k_1 - \gamma_{1\,1}\ \ k_1 - \gamma_{1\,2}k_1 - \cdots - \gamma_{1i}k_1 -$$
$$\gamma_{2\,1}k_2 - \gamma_{2\,2}k_2 - \cdots - \gamma_{2\,i}\,k_2 - \cdots -$$
$$\gamma_{j\,1}k_j - \gamma_{j\,2}k_j - \cdots - \gamma_{j\,i}\,k_j = C_j$$

Factoring the constants yields the following:

$$\begin{aligned}
&\left(1 - \textstyle\sum_i \alpha_{1i}\right)k_1 - \left(\textstyle\sum_i \alpha_{2\,i}\right)k_2 - \quad \cdots \quad - \left(\textstyle\sum_i \alpha_{j\,i}\right)k_j = C_1 \\
&-\left(\textstyle\sum_i \beta_{1i}\right)k_1 + \left(1 - \textstyle\sum_i \beta_{2\,i}\right)k_2 - \quad \cdots \quad - \left(\textstyle\sum_i \beta_{j\,i}\right)k_j = C_2 \\
&\vdots \\
&-\left(\textstyle\sum_i \gamma_{1i}\right)k_1 - \left(\textstyle\sum_i \gamma_{2\,i}\right)k_2 - \quad \cdots \quad + \left(1 - \textstyle\sum_i \gamma_{j\,i}\right)k_j = C_j
\end{aligned}$$

In matrix form, this is represented as

$$\begin{bmatrix} (1 - \sum_i \alpha_{1i}) & -\sum_i \alpha_{2\,i} & - \cdots - & -\sum_i \alpha_{j\,i} \\ -\sum_i \beta_{1i} & (1 - \sum_i \beta_{2\,i}) & - \cdots - & -\sum_i \beta_{j\,i} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_i \gamma_{1i} & -\sum_i \gamma_{2\,i} & - \cdots + & (1 - \sum_i \gamma_{j\,i}) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_j \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_j \end{bmatrix}$$

where $Ak = c, k = A^{-1}c$.

This results in a system of simultaneous linear equations with the number of variables equal to the number of equations, i.e., it will always be a square matrix.

The solution of the system constitutes the vector k of the long-term cointegration equation.

To achieve the above, that is, the general solutions converging to their particular solutions, the complementary solutions would have to be dynamically stable, or they would converge to nullity.

$X_{i\,t} = A_{ii}b_i{}^t, i = 1, 2, \ldots n$ represent the number of variables.

Since all the solutions of the characteristic equation are real numbers, $b_j \in R$.

To fulfill the above, it is required that $|b_i| < 1$, and the particular solutions will be as follows:

$$\lim_{t\to\infty} X_{i\,t} = \lim_{t\to\infty} A_{ii}\ \ b_i{}^t = 0 \tag{26}$$

On the other hand, to achieve $|b_i| < 1$, no doubt the necessary and sufficient condition is that the series involved in the system of difference equations has to be stationary, which is consistent with a single time series.

3. Concluding Remarks

Finally, it is possible to observe the way in which the solution of the VAR cointegration models was generalized for *i* variables and *j* lags where the estimators obtained are very useful in observing the structural components of the phenomenon to be analyzed, so that the idea of obtaining BLUEs is not a necessary condition. Even so, in the case of heterodasticity and autocorrelation in the errors, it is accepted that the estimator loses efficiency; that is, it is a good estimator, although it is not the best.

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