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Dynamics and Bifurcation Analysis of an Eco-Epidemiological Model in a Crowley–Martin Functional Response with the Impact of Fear [†]

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Abstract: This article describes a three-species food web model that was developed by considering the interaction between susceptible prey, infected prey, and predator species. It is assumed that susceptible prey species grow logistically in the absence of predators. It is assumed that predators consume susceptible and infected prey and that infected prey consumes susceptible prey. We consider the effect of fear on susceptible prey due to the predator species. Furthermore, the predator consumes its prey in the form of Holling-type and Crowley–Martin-type interactions. Also, infected prey consume susceptible prey in the form of a Holling-type interaction. The conditions of all biologically feasible equilibrium points were examined. The local stability of the systems around these equilibrium points was investigated. Furthermore, the occurrence of Hopf-bifurcation concerning fear ϱ in the system was investigated. Finally, we demonstrate some numerical simulation results to illustrate our main analytical findings.

Keywords: infected prey; fear effect; Crowley–Martin; equilibrium point; stability; bifurcation

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1. Introduction

Eco-epidemiological systems are used to investigate the dynamic connection between predator and prey in one population or in a population of susceptible and infected animals. Mathematical models have become significant instruments in examining the flow of disease and how to enable prevention. Since Kermack and Mckendrick's pioneering work on SIRS [1], epidemiological models have attracted a lot of interest from researchers. Ecology and epidemiology are two distinct essential and significant areas of research. The Lotka [2] and Volterra [3] models, important advances in current mathematical ecology, can be examined using a system of dynamical equations. This is referred to as the study of infection spread between interacting organisms. A biological representation in terms of the mathematical modelling of communications among the population density of predators and the population density of prey is called a "functional response". In the modelling of biological systems, there are numerous types of functional responses, namely, the Holling type [4,5], Beddington–DeAngelis-type responses, and Crowley–Martin-type responses; Arditi and Ginzburg's [6] approach, which provides much more information on predator–prey systems in terms of Crowley–Martin functional responses, has become available in recent decades. In the recent era, some renowned authors, e.g., [7,8], have used functional responses, such as the Crowley–Martin-type functional response, to make the model system more realistic and controllable in representing the eco-system [9–11]. To the best of our knowledge, no one has examined a three-species food web eco-epidemiological model with Holling-type I, II, and Crowley–Martin functional responses, along with the impact of fear and disease on prey populations. Motivated by this, we explore a three-species food web eco-epidemiological model with Holling type I, Holling type II, and Crowley–Martin functional responses to fear in susceptible prey populations due to predator species. The occurrence of Hopf-bifurcation analysis for the proposed model in relation to the existence of the fear effect is also considered. The rest of the paper is structured as follows: In Section 2, we present the mathematical analysis that is investigated. Section 3 deals with the points of equilibrium in the boundary and their stability. In Section 4, we determine the existence of the interior points of equilibria $E^*(s^*, i^*, p^*)$ and investigate their local stability. The occurrence of Hopf-bifurcation is discussed in Section 5. Numerical simulations are examined in relation to the proposed model in Section 6. Section 7 concludes the paper.

2. Model Formation

The framework models the relationship between the population density of prey and infection, which leads to a corresponding set of non-linear differential equations. The suggested framework was applied to examine the non-linear population density of susceptible prey, infected prey, and predators using the following biological model:

$$\left.\begin{array}{rcl} \frac{d\mathcal{S}}{dT} &=& r_1\mathcal{S}(1-\frac{\mathcal{S}+\mathcal{I}}{\mathcal{K}}) - \lambda\mathcal{I}\mathcal{S} - \frac{\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}, \\ \frac{d\mathcal{I}}{dT} &=& \lambda\mathcal{I}\mathcal{S} - d_1\mathcal{I} - \frac{b_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}, \\ \frac{d\mathcal{P}}{dT} &=& -d_2\mathcal{P} + \frac{cb_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}} + \frac{c\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}. \end{array}\right\} \tag{1}$$

In the above biological systems, the susceptible prey population fears the predator population. The reproduction rate of the susceptible prey population will decrease due to fear of the predator. Here, the conditions are $\mathcal{S}(0) \geq 0, \mathcal{I}(0) \geq 0$ and $\mathcal{P}(0) \geq 0$. The condition for the fear effect is

$$\mathcal{F}_1(\varrho, p) = \frac{1}{1+\varrho p} \tag{2}$$

This describes the level of fear in susceptible prey as a consequence of the predator. Here, ϱ represents the quantity of fear. Given the epidemiological meaning of ϱ, the following condition applies:

$$\varrho(0,p) = \mathcal{F}_1(\varrho,0) = 1, lim\varrho \to \infty\mathcal{F}_1(\varrho,p) = 0 = limp \to \infty\mathcal{F}_1(\varrho,p)$$
$$\frac{\partial\mathcal{F}_1(\varrho,p)}{\partial\varrho} < 0, \frac{\partial\mathcal{F}_1(\varrho,p)}{\partial p} < 0.$$

In this work, we incorporate the prey and the fear effect ϱ. Then, the system is as follows:

$$\left.\begin{array}{rcl} \frac{d\mathcal{S}}{dT} &=& \frac{r_1\mathcal{S}}{1+\mathcal{F}\mathcal{P}}(1-\frac{\mathcal{S}+\mathcal{I}}{\mathcal{K}}) - \lambda\mathcal{I}\mathcal{S} - \frac{\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}, \\ \frac{d\mathcal{I}}{dT} &=& \lambda\mathcal{I}\mathcal{S} - d_1\mathcal{I} - \frac{b_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}, \\ \frac{d\mathcal{P}}{dT} &=& -d_2\mathcal{P} + \frac{cb_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}} + \frac{c\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}. \end{array}\right\} \tag{3}$$

Table 1 displays the specific biological meanings of the parameters.

In the system, (3) has many parameters with different units. It is inconvenient to solve the system (3), so for convenience, we reduce the system to non-dimensional equations using the following transformations: Here, $s = \frac{\mathcal{S}}{\mathcal{K}}, i = \frac{\mathcal{I}}{\mathcal{K}}, p = \frac{\mathcal{P}}{\mathcal{K}}$, with non-dimensional time $t = \lambda\mathcal{K}T$. Now, (3) becomes,

$$\left.\begin{array}{rcl} \frac{ds}{dt} &=& \frac{rs}{1+\varrho p}(1-s-i) - is - \frac{s\alpha p}{(1+\zeta s)(1+\eta p)} \\ \frac{di}{dt} &=& is - di - \frac{\theta ip}{a+i} \\ \frac{dp}{dt} &=& -\delta p + \frac{c\theta ip}{a+i} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)}. \end{array}\right\} \tag{4}$$

Here, the conditions are, $r = \frac{r_1}{\lambda \mathcal{K}}, \alpha = \frac{\alpha_1}{\lambda \mathcal{K}}, d = \frac{d_1}{\lambda \mathcal{K}}, \theta = \frac{b_1}{\lambda \mathcal{K}}, \delta = \frac{d_2}{\lambda \mathcal{K}}, \varrho = \frac{\mathcal{F}}{\mathcal{K}}$. According to the preliminary criteria, $\{s(0), i(0), p(0)\} \geq 0$. The conditions are described in $\mathbb{R}^3_+$.

Table 1. Specific biological meanings of the parameters (3).

Parameters	Units	Physiological Representation
$\mathcal{S}$	Components per unit area (tons)	Population density of susceptible prey
$\mathcal{I}$	Components per unit area (tons)	Population density of prey with infection
$\mathcal{P}$	Components per unit area (tons)	Population density of predator
r_1	Per day (T^{-1})	Prey population densities growth rate
$\mathcal{K}$	Components per unit area (tons)	The carrying capacity
λ	Per day (T^{-1})	Infection rate
a	Per day (T^{-1})	Constant of half-saturation
α_1	Per day (T^{-1})	Susceptible prey to predator consumption
b_1	Per day (T^{-1})	Capture rate by predator
c	Per day (T^{-1})	Conversion rate of prey to predator
d_1, d_2	Per day (T^{-1})	Diseased prey and predator death rate
$\mathcal{F}$	Components per unit area (tons)	Impact of fear
ζ, η	Per day(T^{-1})	Constant of feeding rate

3. The Existence Point of the Equilibrium

The system (4) has three points of equilibrium and one endemic point of equilibrium.

- The $E_0(0,0,0)$ is the point of equilibrium, which is trivial.
- $E_1(\frac{r-1}{r}, 0, 0)$ is the free of infection and free of predator point of equilibrium which exists for $r > 1$.
- The absence of predator point of equilibrium is $E_2(\hat{s}, \hat{i}, 0)$, where, $\hat{s} = d + 2$, $\hat{i} = \frac{r(1-d-2)-1}{r+1}$, which exists for $r(1-2-d) > 1$
- The endemic equilibrium is $E^*(s^*, i^*, p^*)$, where $i^* = \frac{a(a\delta+(\delta-c\alpha)s^*)}{(c\alpha s^*+(c\theta-\delta)(1+\zeta s^*)(1+\eta p^*))}$, $p^* = \frac{ac(s^*-d)(1+\zeta s^*)}{(c\alpha s^*+(c\theta-\delta)(1+\zeta s^*))}$, and the $s^*$is the quadratic equation's one and only positive root, $\mathcal{A}S^2 + \mathcal{B}S + \mathcal{C} = 0$, where,

$$\mathcal{A} = r(\alpha c + \theta c - \delta), \mathcal{B} = (\theta c - \delta)(ar - r) + \alpha c((1 + \varrho p) - r) + a(\delta(1 + \varrho p) + (\delta - c\alpha)r),$$
$$\mathcal{C} = -a(r(1 + \varrho p))(c\theta - \delta) + (c\alpha(1 + \varrho p)(d) - a\delta((1 + \varrho p) + r))).$$

Where an endemic equilibrium exists for $\delta > \alpha c$, $r > 1$, $s^* - d > \frac{(1+r)a\delta}{a\alpha}$, and $a\delta + s^*(\delta - \alpha c)$.

4. Local Stability Analysis

I. We begin by determining the system's (4) Jacobian matrix, $J(E) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$,

where,

$$n_{11} = \frac{r}{1+\varrho p}(1-2s) - i\left(\frac{r}{1+\varrho p} + 1\right) - \frac{\alpha p}{(1+\zeta s)^2(1+\eta p)}, n_{12} = -s(\frac{r}{1+\varrho p} + 1),$$
$$n_{13} = \frac{prs}{(1+\varrho p)^2}(1-s-i) - \frac{\alpha s}{(1+\zeta s)(1+\eta p)^2}, n_{21} = i, n_{22} = s - d - \frac{a\theta p}{(a+i)^2},$$
$$n_{23} = -\frac{\theta i}{(a+i)}, n_{31} = \frac{c\alpha p}{(1+\zeta s)^2(1+\eta p)}, n_{32} = \frac{ac\theta p}{(a+i)^2}, n_{33} = -\delta + \frac{c\theta i}{a+i} + \frac{\alpha c s}{(1+\zeta s)(1+\eta p)^2}.$$

Theorem 1.

- $E_0(0,0,0)$*, the trivial equilibrium point, is locally stable if* $r < 1$*, otherwise it is unstable.*
- $E_1(\frac{r-1}{r},0,0)$ *is an infection-free and predator-free equilibrium point, which is locally stable if* $c\alpha < \delta$ *and* $1 > r(1-d-2)$*, otherwise it is unstable.*

Proof. The characteristic equation of the point of equilibrium E_0 is $(\lambda_{01} - (r-1))(\lambda_{02} - (-d-2))(\lambda_{03} + \delta) = 0$, $\lambda_{01} = r-1, \lambda_{02} = -d-2, \lambda_{03} = -\delta$. Here, $\lambda_{02} < 0, \lambda_{03} < 0$ $E_0(0,0,0)$ is the trivial equilibrium point. It is locally stable if $r < 1$; otherwise it is unstable.

The characteristic equation of the point of equilibrium E_1 is, $(\lambda_{11} - ((1-r)))$ $(\lambda_{12} - (1-d-2-\frac{1}{r}))(\lambda_{13} - (\frac{-\alpha(r-1)}{ra+(r-1)} - \delta)) = 0, \lambda_{11} = 1-r, \lambda_{12} = 1-d-2-\frac{1}{r}, \lambda_{13} = \frac{-c\alpha(r-1)}{ra+(r-1)} - \delta$. Here, $E_1(\frac{r-1}{r},0,0)$, the infection-free and the predator-free equilibrium point, is locally stable if $c\alpha < \delta$ and $1 > r(1-d-2)$, otherwise it is unstable. □

Theorem 2. *The equilibrium* $E_2(\hat{s},\hat{i},0)$*, which indicates absence of the predator, is asymptotically stable if* $\delta > c(\theta+\alpha)$*.*

Proof. The matrix in the form of the Jacobian at E_2 is $J(E_2) = \begin{pmatrix} o_{11} & o_{12} & o_{13} \\ o_{21} & o_{22} & o_{23} \\ o_{31} & o_{32} & o_{33} \end{pmatrix}$, where,

$$o_{11} = r(1-2\hat{s}) + i(r+1), o_{12} = (-1-r)\hat{s}, o_{13} = -\frac{\alpha\hat{s}}{(1+\zeta\hat{s})}, o_{21} = \hat{i}, o_{22} = s-d-2,$$
$$o_{23} = -\frac{\theta\hat{i}}{a+\hat{i}}, o_{31} = 0, o_{32} = 0, o_{33} = \frac{c\alpha\hat{s}}{1+\zeta\hat{s}} - \delta + \frac{c\theta\hat{i}}{a+\hat{i}}.$$

The E_2 characteristic equation is, $\lambda^3 + \mathcal{T}\lambda^2 + \mathcal{U}\lambda + \mathcal{V} = 0$. Here,

$$\mathcal{T} = -o_{11} - o_{33}, \; \mathcal{U} = -o_{21}o_{12} + o_{33}o_{11}, \; \mathcal{V} = o_{12}o_{21}o_{33}.$$

According to the Routh–Hurwitz criterion, if, and only if, $\mathcal{T}, \mathcal{V}$ and $\mathcal{T}\mathcal{U} - \mathcal{V}$ are non-negative, then the real parts are non-positive roots of the above characteristic equation. Now, $\mathcal{T}\mathcal{U} - \mathcal{V} = -o_{11}(-o_{12}o_{21} + o_{33}(o_{33} + o_{11}))$. Now, the necessary criterion for o_{33} to be non-positive is $\delta > c(\alpha+\theta)$. If the above condition in the Theorem is satisfied, the E_2 is locally asymptotically stable. □

Theorem 3. *The endemic or positive point of equilibrium* E^* *is asymptotically stable.*

Proof. The matrix in the form of the Jacobian at E^* is $J(E^*) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, where,

$$r_{11} = -\frac{s^*(1-r+ar+(1+r)i^*+2rs^*)}{(1+\zeta s^*)^2(1+\eta p^*)}, r_{12} = -s^*(\frac{r}{1+\varrho p^*}+1),$$
$$r_{13} = \frac{p^* r s^*}{(1+\varrho p^*)^2}(1-s^*-i^*) - \frac{\alpha s^*}{(1+\zeta s^*)(1+\eta p^*)}, r_{21} = i^*,$$
$$r_{22} = \frac{a\theta p^* i^*}{(a+i^*)^2}, r_{23} = \frac{\theta i^*}{(a+i^*)}, r_{31} = \frac{c\alpha p^*}{((1+\zeta s^*)^2(1+\eta p^*))}, r_{32} = \frac{ac\theta p^*}{(a+i^*)^2}, r_{33} = 0.$$

The E^* characteristic equation is

$$\lambda^3 + \mathcal{F}\lambda^2 + \mathcal{G}\lambda + \mathcal{H} = 0, \tag{5}$$

where, $\mathcal{F} = -r_{11} - r_{33}, \mathcal{G} = -r_{21}r_{12} + r_{22}r_{11} - r_{13}r_{31} + r_{23}r_{32}, \mathcal{H} = r_{13}(-r_{22}r_{31} + r_{21}r_{32}) + r_{23}(r_{12}r_{31} - r_{11}r_{32})$. If $\mathcal{F} > 0, \mathcal{H} > 0, \mathcal{F}\mathcal{G} - \mathcal{H} > 0$. According to the Routh–Hurwitz criterion, if, and only if, $\mathcal{F}, \mathcal{H}, \mathcal{F}\mathcal{G} - \mathcal{H}$ are non-negative, then the real parts are non-positive roots of the above characteristic equation. The E^* is locally asymptotically stable. □

5. Hopf-Bifurcation Analysis

The periodic solutions arise or depart due to changes in the system parameters, which is called Hopf-bifurcation. The eigenvalues of the Jacobian matrix have a negative real part with a complex conjugate, which means that bifurication can occur.

Theorem 4. *If the bifurcation parameter ϱ exceeds a critical point, the model (4) approaches Hopf-bifurcation. At $\varrho = \varrho^*$, the following Hopf-bifurcation conditions arise:*
1. $\mathcal{A}_1(\varrho^*)A(\varrho^*) - \mathcal{A}_3(\varrho^*) = 0$.
2. $\frac{d}{df}(Re(\lambda(\varrho)))|_{\varrho=\varrho^*} \neq 0$. *Here, λ is the root of the parametric solution correlated with the equilibrium interior point.*

Proof. For $\varrho = \varrho^*$, characteristic (5) is in the form

$$(\lambda^2(\varrho^*) + \mathcal{A}_2(\varrho^*))(\lambda(^*) + \mathcal{A}_1(\varrho^*)) = 0. \tag{6}$$

This indicates that the roots of the preceding equation are $\pm i\sqrt{\mathcal{A}_2(\varrho^*)}$ and $-\mathcal{A}_1(\varrho^*)$. To achieve the Hopf-bifurcation at $\varrho = \varrho^*$, the following transversality criterion must be fulfilled:

$$\tfrac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0.$$

For ϱ, the above Equation (6) has general roots

$$\lambda_1 = r(\varrho) + is(\varrho), \lambda_2 = r(\varrho) - is(\varrho), \lambda_3 = -\mathcal{A}_1(\varrho).$$

Weather check the criteria $\frac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0$. Let $\lambda_1 = r(\varrho) + is(\varrho)$ in (6), then we get $\mathcal{C}(\varrho) + i\mathcal{D}(\varrho) = 0$. Where,

$$\mathcal{C}(\varrho) = r^3(\varrho) + r^2(\varrho)\mathcal{A}_1(\varrho) - 3r(\varrho)s^2(\varrho) - s^2(\varrho)\mathcal{A}_1(\varrho) + \mathcal{A}_2(\varrho)r(\varrho) + \mathcal{A}_1(\varrho)\mathcal{A}_2(\varrho),$$
$$\mathcal{D}(\varrho) = \mathcal{A}_2(\varrho)s(\varrho) + 2r(\varrho)s(\varrho)\mathcal{A}_1(\varrho) + 3r^2(\varrho)s(\varrho) + s^3(\varrho).$$

In order to satisfy (6), we must have the variables $\mathcal{C}(\varrho) = 0$ and $\mathcal{D}(\varrho) = 0$, then calculating $\mathcal{C}$ and $\mathcal{D}$ with regard to ϱ,

$$\frac{d\mathcal{A}}{d\varrho} = \varsigma_1(\varrho)r^{'}(\varrho) - \varsigma_2(\varrho)s^{'}(\varrho) + \varsigma_3(\varrho) = 0, \tag{7}$$

$$\frac{d\mathcal{B}}{d\varrho} = \varsigma_2(\varrho)r^{'}(\varrho) + \varsigma_1(\varrho)s^{'}(\varrho) + \varsigma_4(\varrho) = 0, \tag{8}$$

where, $\varsigma_1 = 3r^2(\varrho) + 2r(\varrho)\mathcal{A}_1(\varrho) - 3s^2(\varrho) + \mathcal{A}_2(\varrho), \varsigma_2 = 6r(\varrho)s(\varrho) + 2s(\varrho)a_1(\varrho)$, $\varsigma_3 = r^2(\varrho)\mathcal{A}_1^{'}(\varrho) + s^2(\varrho)\mathcal{A}_1^{'}(\varrho) + \mathcal{A}_2^{'}(\varrho)r(\varrho), \varsigma_4 = \mathcal{A}_2^{'}(\varrho)s(\varrho) + 2r(\varrho)s(\varrho)\mathcal{A}_1^{'}(\varrho)$. On multiplying (7) by $\varsigma_1(\varrho)$ and (8) by $\varsigma_2(\varrho)$, respectively,

$$r(\varrho)^{'} = -\frac{\varsigma_1(\varrho)\varsigma_3(\varrho) + \varsigma_2(\varrho)\varsigma_4(\varrho)}{\varsigma_1{}^2(\varrho) + \varsigma_2{}^2(\varrho)}. \tag{9}$$

Substituting $r(\varrho) = 0$ and $s(\varrho) = \sqrt{\mathcal{A}_2(\varrho)}$ at $\varrho = \varrho^*$ on $\varsigma_1(\varrho), \varsigma_2(\varrho), \varsigma_3(\varrho)$, and $\varsigma_4(\varrho)$, we obtain $\varsigma_1(\varrho^*) = -2\mathcal{A}_2(2^*), \varsigma_2(\varrho^*) = 2\mathcal{A}_1(\varrho^*)\sqrt{\mathcal{A}_2(\varrho^*)},\ \varsigma_3(\varrho^*) = \mathcal{A}_3'(\varrho^*) - \mathcal{A}_2(\varrho^*)\mathcal{A}_1'(\varrho^*), \varsigma_4(\varrho^*) = \mathcal{A}_2'(\varrho^*)\sqrt{\mathcal{A}_2\varrho^*}$. Equation (9) implies

$$r'(\varrho^*) = \frac{\mathcal{A}_3'(\varrho^*) - (\mathcal{A}_1(\varrho^*\mathcal{A}_2(\varrho^*)))}{2(\mathcal{A}_2(\varrho^*) + \mathcal{A}_1^2(\varrho^*))}, \tag{10}$$

if $\mathcal{A}_3'(\varrho^*) - (\mathcal{A}_1(\varrho^*)\mathcal{A}_2(\varrho^*))' \neq 0$, which implies that $\frac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0$, and $\lambda_3(\varrho^*) = -\mathcal{A}_1(\varrho^*) \neq 0$. Therefore, the condition $\mathcal{A}_3'(\varrho^*) - (\mathcal{A}_1(\varrho^*)\mathcal{A}_2(\varrho^*))' \neq 0$. It has been demonstrated that the transversality criterion is satisfied; hence, the model (4) has attained the Hopf-bifurcation at $\varrho = \varrho^*$. □

6. Numerical Simulations

In this section, several numerical experiments on the system (4) are reported to verify the mathematical findings. The rate of fear ϱ is used as a control parameter. For the specified fixed parameter values, the numerical simulation is carried out using the MATLAB/MATHEMATICA software packages. With Runge–Kutta's numerical scheme, here $r = 0.2, \alpha = 0.3, \theta = 0.25, d = 0.1, \delta = 0.1, \zeta = 0.15, \eta = 0.15, \varrho = variable$.

Bifurcation of Fear ϱ

If $\varrho = 0.3$, then the model (4) is asymptotically stable about the positive point of equilibrium $E^*(0.52861, 0.0917829, 0.204774)$ and the other parameter values are the same. Now, we increase the value of the bifurcation parameter, $\varrho = 0.6$, and the model (4) loses its stability, producing a limit cycle at $E^*(0.4899, 0.0920924, 0.220149)$. The model (4) then meets the transversality criteria for $(Re(\lambda(\varrho)))|_{\varrho=\varrho^*} = 0.002185 \neq 0$. The Figure 1 shows the behavioural shifts of the system (4) from the impact of fear, $\varrho = 0.6$.

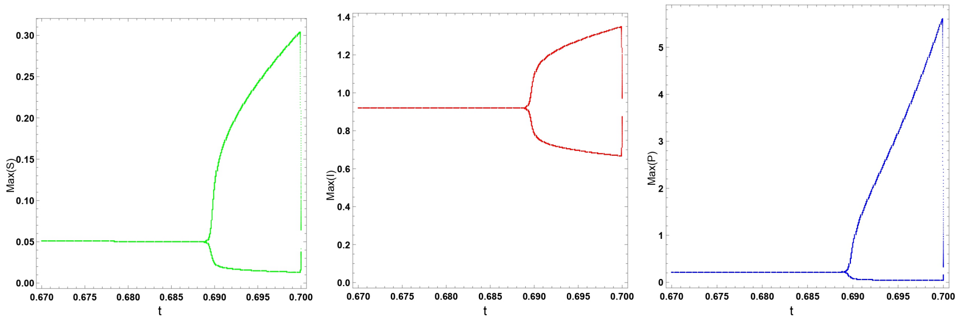

Figure 1. The dynamical change of the model (4) at $\varrho = 0.6$.

7. Conclusions

We investigated an eco-epidemiological system that included infection in the population density of prey, and fear in the susceptible prey population density, as a result of predator attacks on susceptible and diseased prey. In addition, each biologically possible point of equilibrium was represented (4). Furthermore, we investigated the suggested model's local stability (4) and observed the occurrence of Hopf-bifurcation. We determined that modifying the cost of fear ϱ has an instantaneous effect on the model's stability (4). As a result, Hopf-bifurcation constrained the developed analytical arguments around the E^* simulation findings. In the proposed models, we deduce that the existence of dread has a higher impact on the stability shifts via the Hopf-bifurcation.

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References

1. Kermack, W.O.; McKendrick, A.G. A contribution to the mathematical theory of epidemics. *Proc. R. Soc. Lond. Ser. Contain. Pap. Math. Phys. Character* **1927**, *115*, 700–721.
2. Lotka, A.J. *Elements of Physical Biology*; Williams & Wilkins: Philadelphia, PA, USA, 1925.
3. Volterra, V. *Variazioni e Fluttuazioni del Numero d'Individui in Specie Animali Conviventi*; Societá anonima tipografica "Leonardo da Vinci": Roma, Italy, 1927.
4. Ashwin, A.; Sivabalan, M.; Divya, A.; Siva Pradeep, M. Dynamics of Holling type II eco-epidemiological model with fear effect, prey refuge, and prey harvesting. In Proceedings of the 1st International Online Conference on Mathematics and Applications, online, 1–15 May 2023; MDPI: Basel, Switzerland, 2023.
5. Divya, A.; Sivabalan, M.; Ashwin, A.; Siva Pradeep, M. Dynamics of ratio dependent eco epidemiological model with prey refuge and prey harvesting. In Proceedings of the 1st International Online Conference on Mathematics and Applications, online, 1–15 May 2023; MDPI: Basel, Switzerland, 2023.
6. Arditi, R.; Ginzburg, L.R. Coupling in predator-prey dynamics: Ratio-dependence. *J. Theor. Biol.* **1989**, *139*, 311–326. [CrossRef]
7. Crowley, P.H.; Martin, E.K. Functional responses and interference within and between year classes of a dragonfly population. *J. N. Am. Benthol. Soc.* **1989**, *8*, 211–221. [CrossRef]
8. Panigoro, H.S.; Suryanto, A.; Kusumawinahyu, W.M.; Darti, I. Dynamics of an eco-epidemic predator–prey model involving fractional derivatives with power-law and Mittag–Leffler kernel. *Symmetry* **2021**, *13*, 785. [CrossRef]
9. Panigoro, H.S.; Anggriani, N.; Rahmi, E. Understanding the Role of Intraspecific Disease Transmission and Quarantine on the Dynamics of Eco-Epidemiological Fractional Order Model. *Fractal Fract.* **2023**, *7*, 610. [CrossRef]
10. Panigoro, H.S.; Suryanto, A.; Kusumawinahyu, W.M.; Darti, I. Dynamics of a fractional-order predator-prey model with infectious diseases in prey. *Commun. Biomath. Sci.* **2019**, *2*, 105–117. [CrossRef]
11. Rahmi, E.; Darti, I.; Suryanto, A.; Trisilowati, T. A Fractional-Order Eco-Epidemiological Leslie–Gower Model with Double Allee Effect and Disease in Predator. *Int. J. Differ. Equ.* **2023**, *2023*, 5030729. [CrossRef]