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Quantum Probabilities for the Causal Ordering of Events

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Abstract: We develop a new formalism for constructing probabilities associated with the causal ordering of events in quantum theory, where an event is defined as the emergence of a measurement record on a detector. We start with constructing probabilities for the causal ordering events in classical physics, where events are defined in terms of worldline coincidences. Then, we show how these notions generalize to quantum systems, where there exists no fundamental notion of trajectory. The probabilities constructed here are experimentally accessible, at least in principle. Our analysis here clarifies that the existence of quantum orderings of events do not require quantum gravity effects: it is a consequence of the quantum dynamics of matter, and it appears in the presence of a fixed background spacetime.

Keywords: causal order; quantum dynamics; quantum superposition; quantum measurements

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1. Introduction

A bet on any type of race (with humans, horses, chariots, or cars) is equivalent to the assignment of probabilities to a causal ordering of events. The relevant events are the crossings of the finish line by the runners, and the causal ordering of such events is the results of the race. In this sense, assigning probabilities to causal orderings is both one of the oldest applications of probabilistic thinking, dating at least to the ancient Olympics, and one of the most common uses of probability theory today. In this paper, we describe causal ordering of events (COoE) for quantum systems, where an event is defined as the emergence of a macroscopic measurement record that is localized in space and in time [1,2]. We construct the probabilities for such causal orderings, and we suggest physical set-ups where such probabilities can be measured, at least in principle.

1.1. Motivation

This work is partially motivated by the recent studies of indefinite causal ordering of events in quantum computing [3,4]. In that context, the word "event" is used to denote an operation on a quantum system, for example, a step in an algorithm. An indefinite sequence of operations can arguably lead to significant advances in quantum computation and other technologies [5–8]. The most common set-up to witness such phenomena involves the quantum switch, that is, a quantum operation in which two or more quantum channels act on a quantum system with the order of application determined by the state of another quantum system. Systems that manifest indefinite causal order in this sense have been realized in the laboratory [9–11].

This quantum informational notion of causal ordering differs and may even conflict with relativistic causality—for a detailed analysis of this issue, see [12,13]. The meaning of the term "event" is crucial in this context. In this paper, we employ a notion of event that is similar to the crossing of the finishing line by a racer. This notion closely reflects the notion of an event in relativity, where physical events are invariantly defined in terms of worldline coincidences. Einstein emphasized this perspective in his first review of General

Relativity [14], see also [15,16]. For example, a particle-detection event is defined as the intersection of the particle's and the detector's worldlines. With this definition, we use the lightcone structure of spacetime, in order to define causal relations between events. Since events are defined in terms of trajectories, the dynamical behavior of trajectories directly influences the properties of causal ordering. If the trajectories are stochastic, then the ordering of events is also a stochastic variable.

Quantum theory does not admit trajectories as physical observables, so in quantum systems, we have to define events and their causal ordering differently. We define an event as the occurrence of a measurement outcome, that is, the emergence of a measurement record on a detector. This is the most conservative definition of a quantum event. It is also the most natural one in the Copenhagen interpretation [17], and the standard use of the term in particle physics. If the quantum events can be embedded in spacetime, i.e., if we can associate a spacetime point or region to the emergence of a measurement record, then we can define quantum probabilities for the COoE that are natural analogues of the classical ones.

Our definition of events as measurement records is certainly different from the definition in terms of operations, so no results that are based on the latter approach are relevant here. We have to use identical terms, but the meaning is different. For example, when we talk about an indefinite COoE, we mean the existence of superposition states of the form $\psi = \sum_a c_a \psi_a$, where each ψ_a corresponds to a unique causal order of detection events. Such states are generic by virtue of the superposition principle, and, as we will show, they can be readily identified. This means that experiments to identify the existence of indefinite causal order are relatively straightforward in our approach. The simplest set-up requires time-of-arrival measurements in multi-particles systems; if any of the particles is in a superposition state for position or momentum, then the experiment will record an indefinite order of events.

Our notion of events may be more appropriate when discussing gravitational effects on quantum systems. For example, it can readily be employed in order to identify a generalization of the equivalence principle for quantum systems [18–21], which also accounts for effects such as superposition or entanglement. In absence of the notion of a quantum trajectory, a formulation in terms of measurement records appears as the simplest option. Furthermore, we expect that our description of COoE will enable us to analyze the temporal and causal properties of multi-partite quantum systems interacting through gravity [22]. The analysis of such systems has gained increased prominence in recent years, since some of their quantum properties may be experimentally accessible in the medium term [23,24].

We must note that throughout this paper, we use the word "causal" exclusively in its temporal sense, namely, it denotes a temporal relation between two events, where one is earlier than the other. It does not have the connotation of one event being a cause of another, a concept that is ill defined in standard quantum theory.

1.2. Our Results

Our analysis of the causal order of quantum events requires the treatment of the time associated with an event as a quantum observable. It is an old result by Pauli [25] that time cannot be treated as a self-adjoint operator. The only way to have time as an observable is to represent it by a Positive-Operator-Valued measure (POVM). To this end, we use the Quantum Temporal Probabilities (QTP) approach that has been developed for constructing probabilities for temporal observables [1,2,26,27].

The key idea in QTP is to distinguish between the time parameter of Schrödinger's equation and the time variable associated with particle detection [28,29]. The latter is then treated as a macroscopic quasi-classical variable associated with the detector degrees of freedom. A quasi-classical variable is a coarse-grained quantum variable that satisfies appropriate decoherence conditions, so that its time evolution can be well approximated by classical equations [30–32]. Hence, the detector admits a dual description: in microscopic scales, it is described by quantum theory, but its macroscopic records are expressed in terms of classical spacetime coordinates. The key point here is that the spacetime coordinates of a quantum event are random variables, and they can be used to define quantum probabilities for the causal order of events.

The treatment of time as a random variable is the crucial difference of the QTP method from the traditional description of quantum measurements that originates from von Neumann [33]. In von Neumann's theory, the time t of a measurement (or the time interval) is fixed a priori. Hence, t is a parameter to the system, and not a random variable. Most measurement schemes share this feature with von Neumann's measurements, for example, particle detection models [34], continuous-time measurements [35], or measurements for stroboscopic tomography [36].

Since the time of a measurement is a random variable, we can straightforwardly compare the detection times for different detectors, in order to define probabilities for the COoE in multi-partite quantum systems. The COoEs are also treated as random variables, described by POVMs, i.e., they are generalized quantum observables. We also show that probabilities for quantum COoEs can also be defined even if there is no macroscopic record about the time at which the events occur. To this end, we construct a simple detection model, in which different orderings of events correspond to different measurement records. Hence, the probabilities of such records coincide with probabilities for different causal orders.

It is important to emphasize that there is no relation between the COoE considered here and a quantum causal structure of spacetime, as commonly postulated in quantum gravity research. The quantum behavior of the COoE, considered here, is due to the quantum nature of matter, and it coexists peacefully with a fixed background spacetime. In fact, the background spacetime structure is essential for defining quantum probabilities for the COoE. This is not particular to our definition of events: quantum gravity is not necessary for an indefinite causal order even when events are defined in terms of operations [37].

The structure of this paper is the following. In Section 2, we provide a general definition of the notion of an event, and explain how we can construct probabilities for the causal order of events. In Section 3, we apply these definitions to classical physics, including Hamiltonian mechanics and stochastic processes. In Section 4, we define probabilities for the causal ordering of quantum events, using temporal observables. In Section 5, we present a simple model for the quantum order of events in absence of records about temporal observables. In Section 6, we summarize our results.

2. Main Concepts

In this section, we present a general characterization of events, and we identify the mathematical properties that are satisfied by a causal order of events.

By "event", we mean a uniquely identifiable occurrence with definite characteristics. In classical mechanics, events are typically defined as the intersection of two worldlines. For example, one worldline may correspond to a particle and the other to an observer with a particle detector; their coincidence is a particle-detection event. We can improve on this description, by defining an event as the first intersection of a worldline with a specific time-like surface. We can consider, for example, the worldline of a runner crossing the world tube of the finish line in a marathon.

However, in quantum theory, definite characteristics are attributed only to measurement outcomes; trajectories are not observables. For this reason, we will define events in terms of measurement records. For example, a detection event is the "click" of a particle detector. Hence, we identify event with changes in a macroscopic apparatus that denote the occurrence of a measurement.

Let E be a set of events in the physical system under study. Events are discrete occurrences, so E is a discrete set. We will denote events by Greek letters, α, β, and so on.

The events in the set E may be ordered causally. We say that $\alpha \prec \beta$, if event α occurs prior to an event β. A *causal order* on E is the consistent assignment of the order relation $\prec$ to pairs of elements of E. A causal order satisfies the properties of a partial-order relation, namely:

1. Irreflexivity: It is never true that $\alpha \prec \alpha$.
2. Asymmetry: If $\alpha \prec \beta$, then $\beta \prec \alpha$ is false.
3. Transitivity: If $\alpha \prec \beta$ and $\beta \prec \gamma$, then $\alpha \prec \gamma$.

In a partial order, it is not necessary that all pairs of elements are related with the order relation. Physically, we can distinguish two cases. Some elements may be *simultaneous*, in which case we write $\alpha \sim \beta$. Or they may be *uncomparable*, in which case we write $\alpha|\beta$. Therefore, we define a causal order as a partial order that also include the distinction between simultaneous and incomparable pairs of elements.

We will denote the set of all possible causal orders on E by $CO(E)$. We will denote elements of $CO(E)$ by capital Greek letters. For example, in a set E that consists of two distinct elements α and β, there are four possible causal orders:

- $M_1 = \{\alpha \prec \beta\}$;
- $M_2 = \{\beta \prec \alpha, \}$;
- $M_3 = \{\alpha|\beta\}$;
- $M_4 = \{\alpha \sim \beta\}$.

We say that a causal order defines a *time order* on E, if there exist no pair $\alpha, \beta \in E$ such that $\alpha|\beta$. We will denote the set of all time orders on E by $TO(E)$. Clearly, $TO(E) \subset CO(E)$.

Ever since Newton, we define the causal ordering of physical events in terms of the spacetime causal structure. That is, we consider a four-dimensional manifold M with points (x^0, x^1, x^2, x^3) that is equipped with a partial ordering relation $<$ that defines the causal structure of spacetime.

- In non-relativistic physics, $x < y$ if $x^0 < y^0$, and $x \sim y$ if $x^0 = y^0$. There are no incomparable elements.
- In relativistic physics, $x < y$, if y lies in the future lightcone of x, and $x|y$ if x is spacelike separated from y. Spacelike separated events are incomparable; there are no simultaneous events.

Since all physical events occur in spacetime, we consider embeddings X of the set of events E into spacetime, that is, onto maps $X : E \to M$. Then, the pullback of the spacetime causal structure with respect to X defines a causal order on E, that is:

$$\alpha \prec \beta, \;\text{ if }\; X(\alpha) < X(\beta) \tag{1}$$

Hence, the physical COoE reflects the causal structure on spacetime. Here, we associated events with spacetime points—a more general analysis should associate events with spacetime regions, but this will not be needed in this paper.

It is imperative to distinguish COoEs from the causal structure of spacetime. As long as we ignore gravitational interactions, the latter is fixed and unchanging. It is defined by the lightcone structure of Minkowski spacetime, or of any other background spacetime. However, COoEs are not fixed: they can be stochastic or quantum variables. This is because they depend on the embeddings X, which are themselves stochastic or quantum variables. The quantum behavior of the COoEs does not require a quantum behavior of spacetime, as postulated in quantum gravity theories. As a matter of fact, in quantum gravity proper, we expect to have no external spacetime causal structure, hence, the definitions of the COoEs given here do not work. The difficulties that arise from this fact are known as the *problem of time* in quantum gravity [38–40].

3. COoE in Classical Physics

In this section, we construct probabilities for the COoEs for classical systems, namely, for Hamiltonian systems and for systems described by stochastic processes.

3.1. Classical Mechanics

Let Γ be the state space of a classical system; we will denote the elements of Γ by ξ. By Hamilton's equation, a system found at ξ at time $t = 0$, will evolve to a point $\sigma_t(\xi)$ at time t. The map σ_t is a diffeomorphism.

An event in classical mechanics corresponds to the first intersection of a state space trajectory with a surface on Γ. Surfaces of codimension s are locally determined by the vanishing of s functions on Γ, hence, we can represent an event α by a set of s functions F^i_α, where $i = 1, 2, \ldots s$. A set E of n events consists n such families. For simplicity, we will consider only surfaces of codimension one in this paper, so that one event corresponds to a single function on Γ.

The causal ordering in the set of events is defined through the parameter of time evolution t, which is assumed to coincide with the Newtonian absolute time. For any event α, we define the null set N_α of α, as the set of all $\xi \in \Gamma$, such that the equation $F_\alpha[\sigma_t(\xi)] = 0$ has no solution for all $t \geq 0$. Then, we define the time T_α of the event α, as a function $T_\alpha : \Gamma - N_\alpha \to \mathbb{R}^+$, such that $T_\alpha(\xi)$ is the smallest positive value of t that solves the equation $F_\alpha[\sigma_t(\xi)] = 0$. This means that $T_\alpha(\xi)$ is the time it takes a trajectory that starts at ξ to cross the surface $F_\alpha = 0$ for the first time.

Suppose that the initial state of the system corresponds to a probability distribution $\rho(\xi)$. Then, we can construct joint probability distributions for the times of events:

$$p(t_1, t_2, \ldots, t_n) = \int d\xi \rho(\xi) \delta[T_1(\xi) - t_1] \delta[T_2(\xi) - t_2] \ldots \delta[T_n(\xi) - t_n]. \tag{2}$$

These probability densities are not normalized to unity. For proper normalization, we have to include the probability densities for no events, which corresponds to the null sets N_α. For example, for $n = 2$, we have the probability densities $p(t_1, t_2)$ as above, together with the probability densities:

$$p(N_1, t_2) = \int d\xi \chi_{N_1}(\xi) \delta[T_2(\xi) - t_2] \tag{3}$$

$$p(t_1, N_2) = \int d\xi \chi_{N_2}(\xi) \delta[T_1(\xi) - t_1] \tag{4}$$

$$p(N_1, N_2) = \int d\xi \chi_{N_1}(\xi) \chi_{N_2}(\xi). \tag{5}$$

where χ_C is the characteristic function of a set C.

We can define the following four causal orders for the two events:

- $M_1 = \{1 \prec 2\}$ corresponds to $t_1 < t_2$, or N_2 together with finite t_1.
- $M_2 = \{2 \prec 1\}$ corresponds to $t_2 < t_1$, or N_1 together with finite t_2.
- $M_3 = \{1|2\}$ corresponds to N_1 and N_2.
- $M_4 = \{1 \sim 2\}$ corresponds to $t_1 = t_2$.

Then, we obtain the associated probabilities:

$$\begin{array}{rcl} p(M_1) & = & \int_0^\infty dt_1 \int_0^{t_1} dt_2 p(t_1, t_2) + \int_0^\infty dt_1 p(t_1, N_2) \\ p(M_2) & = & \int_0^\infty dt_2 \int_0^{t_2} dt_1 p(t_1, t_2) + \int_0^\infty dt_2 p(N_1, t_2) \\ p(M_3) & = & p(N_1, N_2) \\ p(M_4) & = & \int_0^\infty dt p(t, t). \end{array} \tag{6}$$

This procedure is straightforwardly generalized to n events.

As an illustration, consider a system of two free particles of mass m in one dimension, with state space $\Gamma = \{x_1, x_2, p_1, p_2\}$. We restrict to $x_1 \leq 0$ and $x_2 \leq 0$, and we consider the pair of events that correspond to either of the two particles crossing the line $x = 0$. Hence, the two functions that define events are $F_1 = x_1$ and $F_2 = x_2$. The equations of motion are $x_1(t) = x_1 + p_1 t/m$ and $x_2(t) = x_2 + p_2 t/m$. We straightforwardly find that

$N_1 = \{(x_1, x_2, p_1, p_2)|p_1 \leq 0\}$ and $N_2 = \{(x_1, x_2, p_1, p_2)|p_2 \leq 0\}$, that is, the particles never cross the line $x = 0$ if they have non-positive momentum. Similarly, we compute the time functions $T_1 = -mx_1/p_1$ and $T_2 = -mx_2/p_2$.

It is simple to identify the subsets of Γ that correspond to the different causal orders:

$$\begin{aligned} M_1 &= \{(x_1, x_2, p_1, p_2)|p_1 > 0, p_2 > 0, x_1 p_2 > x_2 p_1\} \cup \{(x_1, x_2, p_1, p_2)|p_1 > 0, p_2 \leq 0\}, \\ M_2 &= \{(x_1, x_2, p_1, p_2)|p_1 > 0, p_2 > 0, x_2 p_1 > x_1 p_2\} \cup \{(x_1, x_2, p_1, p_2)|p_1 \leq 0, p_1 > 0\}, \\ M_3 &= \{(x_1, x_2, p_1, p_2)|p_1 \leq 0, p_2 \leq 0\}, \\ M_4 &= \{(x_1, x_2, p_1, p_2)|p_1 > 0, p_2 > 0, x_2 p_1 = x_1 p_2\}. \end{aligned} \tag{7}$$

The associated probabilities are simply $p(M_i) = \int d\xi \chi_{M_i}(\xi)\rho(\xi)$. Note that M_4 is a set of measure zero, so, in general, the associated probability vanishes.

Suppose, for example, that both particles start from $x_0 < 0$, and that they have the same momentum distribution $f(p)$, so that:

$$\rho(x_1, x_2, p_1, p_2) = \delta(x_1 - x_0)\delta(x_2 - x_0) f(p_1) f(p_2). \tag{8}$$

Then, we compute, $p(M_1) = p(M_2) = w_+ - \frac{1}{2}w_+^2$, and $p(M_3) = (1 - w_+)^2$, where $w_+ = \int_0^\infty dp f(p)$ is the fraction of particles with positive momentum.

3.2. Stochastic Processes

The analysis of Section 3.1 passes with little change to classical stochastic systems. Consider a system characterized by a state space Γ with elements ξ. Let us denote by $P(\Gamma)$ the space of paths on Γ, that is, of continuous maps from the time interval $[0, T]$ to Γ. Here, we are restricting to paths between an initial time $t = 0$, and a final time $t = T$. A stochastic system is described by a probability measure μ on $P(\Gamma)$, such that the expectation of any function A of $P(\Gamma)$ is given by:

$$\langle A \rangle = \int d\mu[\xi(\cdot)] A[\xi(\cdot)] \tag{9}$$

Again, an event α is defined by the first intersection of a path with a surface, and it can be represented by a function F_α on Γ. We can still define a null space N_α, and a time function T_α; however, in absence of a deterministic evolution law, these objects are defined on the space of paths $P(\Gamma)$, and not on Γ. In particular, we define by N_α the subset of $P(\Gamma)$ that consists of paths $\xi(\cdot)$ for which the equation $F_\alpha(\xi(t)) = 0$ has no solution for any $t \in [0, T]$; we will denote the complement of N_α by $\bar{N}_\alpha$. We also define the time function T_α for any path $\xi(\cdot) \in \bar{N}_\alpha$ by setting the value $T_\alpha[\xi(\cdot)]$ on a path $\xi(\cdot)$ equal to the smallest value of t such that $F_\alpha(\xi(t)) = 0$.

The definition of joint probabilities for the times of events proceeds in a similar way to Section 2. For example, the joint probability distribution for n events is:

$$p(t_1, t_2, \ldots, t_n) = \int d\mu[\xi(\cdot)]\delta(t_1 - T_1(\xi(\cdot))\delta(t_2 - T_2(\xi(\cdot)) \ldots \delta(t_n - T_n(\xi(\cdot)). \tag{10}$$

The space of paths $P(\Gamma)$ is split into mutually exclusive and exhaustive subsets, each corresponding to an element of $CO(E)$. For two events, we have four elements of $CO(E)$, which correspond to the following subsets:

$$\begin{aligned} M_1 &= \{\gamma \in P(\Gamma)|T_1[\gamma] < T_2[\gamma]\} \cup (N_2 \cap \bar{N}_1), \\ M_2 &= \{\gamma \in P(\Gamma)|T_2[\gamma] < T_1[\gamma]\} \cup (N_1 \cap \bar{N}_2), \\ M_3 &= N_1 \cap N_2, \\ M_4 &= \{\gamma \in P(\Gamma)|T_1[\gamma] = T_2[\gamma]\}. \end{aligned} \tag{11}$$

As an example, consider the case of a Wiener process. We have two particles undergoing Brownian motion, that is, each particle is described by the evolution of a single-time probability density ρ on $\mathbb{R}$, by:

$$\frac{\partial \rho}{\partial t} = \frac{D}{2}\frac{\partial^2 \rho}{\partial x^2}, \tag{12}$$

where D is the diffusion constant. For a particle that starts at $x_0 = -L$, the probability density of crossing the line $x = 0$ is:

$$f(t) = \frac{1}{\sqrt{2\pi D t}}\frac{L}{2t}e^{-\frac{L^2}{2Dt}}, \tag{13}$$

with the probability of not crossing $x = 0$ for any $t \in [0, \infty)$ equal to $\frac{1}{2}$.

We assume that the two particles move independently, and that the associated diffusion constants are different, D_1 and D_2 (this is possible, for example, if the particle masses are different). The joint probability density that the first crosses $x = 0$ at time t_1 and the second at time t_2 is simply $f_1(t_1)f_2(t_2)$, where f_i is the probability density (13), with diffusion constant D_i. Then, we evaluate:

$$\begin{array}{rcl} p(M_1) & = & \frac{1}{2\pi}\arctan\left(\sqrt{D_1/D_2}\right) + \frac{1}{4}, \\ p(M_2) & = & \frac{1}{2\pi}\arctan\left(\sqrt{D_2/D_1}\right) + \frac{1}{4}, \\ p(M_3) & = & \frac{1}{4}, \end{array} \tag{14}$$

where we ignored M_4, as it is of measure zero.

4. COoE in Quantum Systems

In this section, we define probabilities for the COoE in quantum systems.

4.1. Probability Assignment

For quantum systems, the definition of events in terms of paths crossing a surface does not work, because paths are not physical observables in quantum theory. The only meaningful observables are measurement outcomes. In the most common measurement scheme, namely, von Neumann measurements, the timing of the measurement events is fixed a priori. Hence, the causal order of events is also fixed.

We need a measurement scheme that treats the time of an event as a random variable, if we are to treat the causal order of events as a random variable quantum mechanically. This is achieved by the QTP approach that was described in the introduction.

Suppose that we have a particle detector located at a fixed region in space. Then, via QTP, we can construct a set of positive $\hat{\Pi}(t)$, such that the probability density of detection at time $t > 0$ is $p(t) = Tr(\hat{\rho}_0\hat{\Pi}(t))$, where $\hat{\rho}_0$ is the initial state if the particle. Together with the positive operator $\hat{\Pi}(N)$ of no detection, the operators $\hat{\Pi}_t$ define a POVM.

For example, we can construct a POVM for the time of arrival of a particle of mass m. We assume that the particle moves at a line and that the particle detector is located at $x = L$. In the momentum basis:

$$\langle k|\hat{\Pi}(t)|k'\rangle = \int \frac{dkdk'}{2\pi} S(k,k')\sqrt{v_k v_{k'}}e^{i(k-k')L - i(\epsilon_k - \epsilon_{k'})t}, \tag{15}$$

where $\epsilon_k = \sqrt{m^2 + k^2}$ is the particle's energy, v_k is the particle's velocity, and $S(k,k')$ is the *localization operator*, which is an operator that determines the irreducible spread of the detection record.

The sharpest localization is achieved for $S(k,k') = 1$. The operators $\hat{\Pi}(t)$ are not normalized to unity for $t \in [0, \infty)$. However, if we restrict to quantum states with strictly positive momentum content, the contribution to the total probability from negative values of t is negligible, and we can consider the normalization of $\hat{\Pi}(t)$ in the full real line. In this case:

$$\int_{-\infty}^{\infty} dt \hat{\Pi}(t) = \hat{I}. \tag{16}$$

The derivation of Equation (15) from a detailed modelling of system-apparatus coupling is provided in Ref. [27]. Equation (15) for $S(k,k') = 1$ was earlier derived in Ref. [41], as a generalization of Kijowski's POVM for the time of arrival of non-relativistic particles [42].

For n independent detectors, each detecting a different particle, we can identify a POVM $\hat{\Pi}(t_1, t_2, \ldots, t_n) = \hat{\Pi}_1(t_1) \otimes \hat{\Pi}_2(t_2) \otimes \ldots \otimes \hat{\Pi}_n(t_n)$, where $\hat{\Pi}_i(t_i)$ corresponds to the POVM for the i-th detector. In general, the detectors are different, so the localization operators $S_i(k,k')$ are different for each detector. Thus, we can define probability densities $p(t_1, t_2, \ldots, t_n)$ for the n measurement events, and we can follow the same procedure as in Section 3, in order to obtain probabilities for different causal orders of n measurement events.

For example, for two events, 1 and 2, we have the three COoEs $M_1 = \{1 \prec 2\}$, $M_2 = \{2 \prec 1\}$, and $M_3 = \{1||2\}$. The following positive operators define a POVM for the causal orders:

$$\begin{aligned} \hat{E}(M_1) &= \int_0^\infty dt_2 \int_0^{t_2} dt_1 \hat{\Pi}_1(t_1) \otimes \hat{\Pi}_2(t_2) + [\hat{I} - \hat{\Pi}_1(N)] \otimes \hat{\Pi}_2(N), \\ \hat{E}(M_2) &= \int_0^\infty dt_1 \int_0^{t_1} dt_2 \hat{\Pi}_1(t_1) \otimes \hat{\Pi}_2(t_2) + \hat{\Pi}_1(N)[\hat{I} - \hat{\Pi}_2(N)], \\ \hat{E}(M_3) &= \hat{\Pi}_1(N) \otimes \hat{\Pi}_2(N), \end{aligned} \tag{17}$$

Assume that the two events correspond to the detection of identical particles with the two detectors located at $x_1 = L_1$ and $x_2 = L_2$ from the source—see Figure 1. We use the POVM (15) for both $\hat{\Pi}_1(t)$ and $\hat{\Pi}_2(t)$. Taking $-\infty$ for the lower bound in the time integral, we find:

$$\hat{E}(M_1) = \frac{1}{2}\hat{I} + \hat{B} \tag{18}$$

$$\hat{E}(M_2) = \frac{1}{2}\hat{I} - \hat{B} \tag{19}$$

$$\hat{E}(M_3) = 0 \tag{20}$$

where the operator $\hat{B}$ is defined as:

$$\hat{B} = \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} [\theta(t_1 - t_2) - \theta(t_2 - t_1)] \hat{\Pi}_1(t_1) \hat{\Pi}_2(t_2). \tag{21}$$

Any initial state that does not lie in an eigenspace of the operator $\hat{B}$ defines an indefinite COoE.

We employ Equation (15), to compute the matrix elements of $\hat{B}$:

$$\begin{aligned} \langle k_1, k_2 | \hat{B} | k_1', k_2' \rangle &= i S_1(k_1, k_1') S_2(k_2, k_2') \sqrt{v_{k_1} v_{k_2} v_{k_1'} v_{k_2'}} \\ &\times \; e^{i(k_1 - k_1')L_1 + i(k_2 - k_2')L_2} \delta(\epsilon_{k_1} + \epsilon_{k_2} - \epsilon_{k_1'} - \epsilon_{k_2'}) \, \mathrm{PV} \frac{1}{\epsilon_{k_2} - \epsilon_{k_2'}}. \end{aligned} \tag{22}$$

where PV stands for the Cauchy principal value.

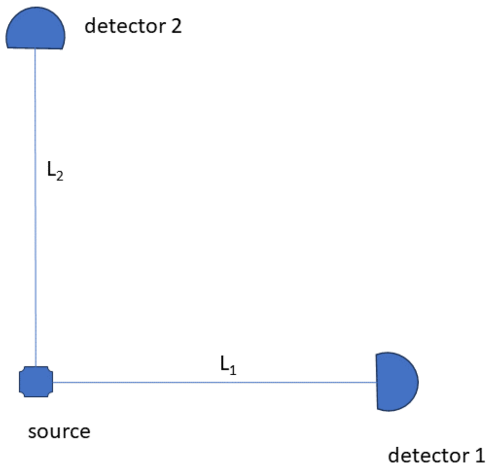

Figure 1. A set-up by which to measure the causal order of two events that correspond to detections at detectors 1 and 2.

Consider a general initial state of the form $|\psi\rangle = \sum_i c_i |\psi_{1i}\rangle \otimes |\psi_{2i}\rangle$. The probability densities associated with the three orders are:

$$p(M_1) = \frac{1}{2} + w, \;\; p(M_2) = \frac{1}{2} - w, \;\; p(M_3) = 0, \tag{23}$$

where $w = \langle\psi|\hat{B}|\psi\rangle$.

By expressing the delta function in Equation (22) as an integral $(2\pi)^{-1}\int_{-\infty}^{\infty} ds \exp[-i(\epsilon_{k_1} + \epsilon_{k_2} - \epsilon_{k_1'} - \epsilon_{k_2'})s]$, we can bring the asymmetry w into a form that is convenient for calculations:

$$w = \sum_{ij} c_i c_j^* \int_{-\infty}^{\infty} ds \, [\alpha_{ij}^{(1)}(s)\dot{\alpha}_{ij}^{(2)}(s) - \dot{\alpha}_{ij}^{(1)}(s)\alpha_{ij}^{(2)}(s)]. \tag{24}$$

Here, we wrote:

$$\alpha_{ij}^{(a)}(s) = \int \frac{dkdk'}{2\pi} \psi_{ai}(k)\psi_{aj}^*(k')S_a(k,k')\sqrt{v_k v_{k'}} e^{i(k-k')L_a - i(\epsilon_k - \epsilon_{k'})s} \, \mathrm{PV}\frac{1}{\epsilon_k - \epsilon_{k'}}, \tag{25}$$

where $a = 1, 2$.

4.2. Examples

We analyze the case of massless particles, $m = 0$, and ideal detector, $S(k,k') = 1$. For two particles prepared in the same state $\psi_0(k)$, that is centered around $x = 0$. However, the distances traveled by the two particles may be different, $L_1 \neq L_2$. We take for ψ_0 a Gaussian centered around k_0:

$$\psi_0(k) = (2\pi\sigma^2)^{-1/4} \exp\left[(k - k_0)^2/(4\sigma^2)\right]. \tag{26}$$

Then, we find that w in Equation (24) equals $Q_1[\sigma(L_1 - L_2)]$, where:

$$Q_1(\delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-2(x-\delta)^2} \mathrm{erf}\left(\sqrt{2}x\right). \tag{27}$$

The dependence of the function Q on δ is plotted in Figure 2. As expected, w vanishes for $L_1 = L_2$ and tends to $\pm\frac{1}{2}$ for large differences between L_1 and L_2, in which case the ordering of the events is almost certain.

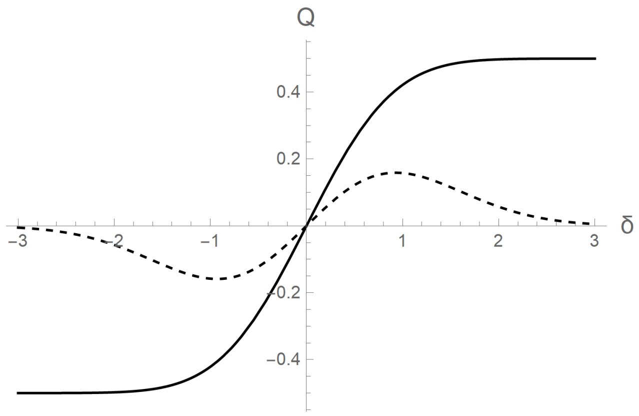

Figure 2. The function Q_1 of Equation (27) (solid) and the function Q_2 of Equation (30) (dashed) as a function of δ.

The probabilities in the example above could have been derived from a classical theory. To see quantum behavior, we consider an superposition state for the first particle:

$$\psi(k_1, k_2) = \frac{1}{\sqrt{2(1+\nu)}} \psi_0(k_1) \left[1 + e^{ik_1\ell}\right] \psi_0(k_2), \tag{28}$$

where ℓ is a path difference for the first particle in one component of the superposition and $\nu = \int dk_1 dk_2 |\psi_0(k)|^2 \cos(k\ell)$. For the Gaussian (26), $\nu = e^{-\sigma^2\ell^2/2}\cos(k_0\ell)$. For this initial state:

$$w = \frac{Q_1(\delta) + 2Q_2(\delta)\cos\left(\frac{k_0}{\sigma}\delta\right)}{2\left[1 + e^{-\delta^2/2}\cos\left(\frac{k_0}{\sigma}\delta\right)\right]}, \tag{29}$$

where $\delta = \sigma\ell$, and the following function is plotted in Figure 2:

$$Q_2(\delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-x^2-(x-\delta)^2} \mathrm{erf}\left(\sqrt{2}x\right), \tag{30}$$

In Figure 3, we plot w as a function of δ and of k_0/σ. The quantum nature of the system is manifested in the oscillatory behavior of the probabilities.

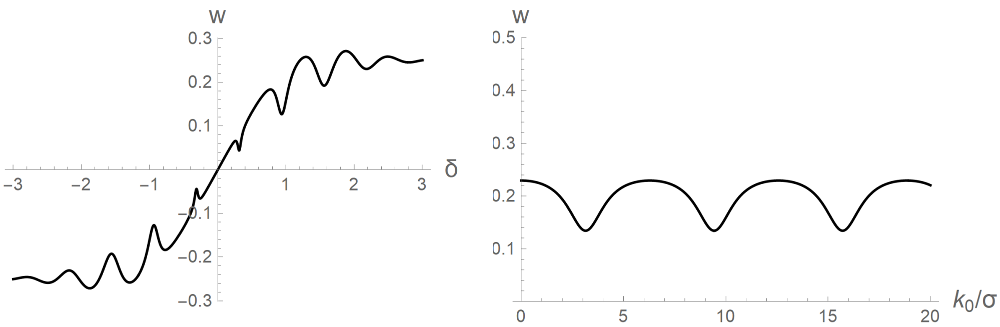

Figure 3. The asymmetry w of Equation (29) as a function of δ for constant $k_0/\sigma = 10$ (**left**) and as a function of k_0/σ for constant $\delta = 1$ (**right**).

4.3. Causal Orders for Several Events

As a demonstration, we show how the analysis above works for three events, say, 1, 2, and 3. We have the following causal orders:

- Six total orders that correspond to $t_1 < t_2 < t_3$ and all permutations of the indices.
- Six orders of the form $t_1|t_2$, $t_1 < t_3, t_2 < t_3$ and $t_1|t_2$, $t_3 < t_1, t_3 < t_2$, with all permutations of the indices.

- Six orders for the form $t_1|t_2, t_2|t_3, t_1 < t_3$ and $t_1|t_2, t_2||t_3, t_3 < t_1$, modulo all permutations of the indices.
- A single order for $t_1|t_2, t_2|t_3, t_1|t_3$.

Here, we ignored the case of simultaneity, because in the particle detection model, the corresponding events are of measure zero.

A system of three distinguishable particles is described by a quantum state $\psi(k_1, k_2, k_3)$. Suppose that the particles are detected at distances L_1, L_2, and L_3 from the source, respectively. We assume that the momentum content of the states is positive for all particles. Then, there is zero probability of no-detection events, and only the probabilities for the six total orders are non-vanishing. The associated positive operators are as follows:

$$\begin{aligned} \langle k_1, k_2, k_3|\hat{\Pi}(t_a < t_b < t_c)|k_1', k_2', k_3'\rangle &= -S_1(k_1, k_1')S_2(k_2, k_2')S_3(k_3, k_3')\sqrt{v_{k_1}v_{k_2}v_{k_3}v_{k_1'}v_{k_2'}v_{k_3'}} \\ \times \quad & e^{i(k_1-k_1')L_1+i(k_2-k_2')L_2+i(k_3-k_3')L_3}\delta(\epsilon_{k_1}+\epsilon_{k_2}+\epsilon_{k_3}-\epsilon_{k_1'}-\epsilon_{k_2'}-\epsilon_{k_3'})\mathrm{PV}\left[\frac{1}{(\epsilon_{k_a}-\epsilon_{k_a'})(\epsilon_{k_c}-\epsilon_{k_c'})}\right], \end{aligned}$$

where (a, b, c) is a permutation of $(1, 2, 3)$. Note that the positive operator for a causal order and its inverse coincide, $\hat{\Pi}(t_1 < t_2 < t_3) = \hat{\Pi}(t_3 < t_2 < t_1)$. The generalization for an arbitrary number of events is straightforward.

5. Probabilities for the COoE via a Detection Model

In the examples of Section 4, the measurements of the causal ordering of events are coarse-grained. This means that the measuring apparatuses record the time of detection events, and the probabilities for the causal order of events are obtained by integrating over detection times. The construction is formally similar to that of classical physics, even if there are no paths at the fundamental level. In either context, coarse-graining means that we have ignored significant information, pertaining to the detection time of each individual particle.

However, in quantum theory, it may be possible to define probabilities for causal ordering of events as fine-grained observables. This means that we can define the COoE even if the apparatus makes no record of the times of individual events. In this section, we will present a simple model that provides such probabilities for the case of two potential events, and which can be straightforwardly generalized for n events.

5.1. The Model

The key idea is to direct a pair of particles towards a detector that can record either of them, but not both. As an example, we consider a three-level system (3LS), with states $|0\rangle$, $|1\rangle$, and $|2\rangle$. Suppose that particle 1 can excite only the transition $0 \to 1$, and particle 2 only the transition $0 \to 2$. If after the interaction of the particles with the three-level system, we find the system in state $|1\rangle$, we can surmise that particle 1 was detected first and particle 2 was not detected, and vice versa.

To conform with our definition of an event with a measurement record, we must place an identical 3LS after the first, in which the particle not absorbed by the first can be detected. However, this is superfluous for identifying the COoE in this system. If particle 1 is recorded in the first 3LS, particle 2 will either be recorded in the second 3LS, or it will not be recorded at all. Both cases correspond to the same order of events: the record of 1 is prior to the record of 2. Hence, as far as the COoE is concerned, the presence of the second 3LS makes no difference. Hence, for two events, a single 3LS suffices.

This set-up is straightforwardly generalized for determining the probabilities for n events. We require n particles that can be sharply distinguished by their energies and $n-1$ systems with $n+1$ energy levels, so that the detection of each particle can be associated with a single transition.

To implement our model, we assume that the incoming particles are described by a free scalar field $\hat{\phi}(x)$ with mass m. The two particles are distinguished by their initial energies; we can assume that they are prepared from different sources. The particles interact with one 3LS, which we take to be located at $\mathbf{x} = 0$.

The total Hamiltonian is a sum of three terms $\hat{H}_\phi + \hat{H}_{3LS} + \hat{H}_{int}$, where $\hat{H}_\phi$ is the field Hamiltonian, expressed in terms of field creation and annihilation operators:

$$\hat{H}_\phi = \int d\mathbf{k}\epsilon_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}, \tag{31}$$

where $d\mathbf{k}$ stands for $d^3k/(2\pi)^3$;

$$\hat{H}_{3LS} = \Omega_1|1\rangle\langle 1| + \Omega_2|2\rangle\langle 2| \tag{32}$$

is the 3LS Hamiltonian, and the following interaction Hamiltonian:

$$\hat{H}_{int} = \sum_{a=1}^{2}\lambda_a\int\frac{dk}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}}\hat{u}_{a+} + \hat{a}^\dagger_{\mathbf{k}}\hat{u}_{a-}), \tag{33}$$

describes a dipole coupling between the field and the 3LS. Here, λ_a is the coupling constants associated with the transition $0 \to a$, $\hat{u}_{a+} = |a\rangle\langle 0|$ and $\hat{u}_{a-} = |0\rangle\langle a|$; $a = 1,2$. This Hamiltonian is a variation of Lee's Hamiltonian that is commonly employed in the study of spontaneous decay [32].

5.2. Time Evolution

To derive the time evolution law for this model, we work in the interaction picture. Then, the quantum state satisfies the equation:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \sum_{a=1}^{2}\lambda_a\int\frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}}\hat{u}_{a+}e^{-i(\epsilon_{\mathbf{k}}-\Omega_a)t} + \hat{a}^\dagger_{\mathbf{k}}\hat{u}_{a-}e^{i(\epsilon_{\mathbf{k}}-\Omega_a)t})|\psi(t)\rangle. \tag{34}$$

We assume an initial two-particle state for the field and the ground state for the 3LS. The Hamiltonian employed here causes transitions only to one-particles states with an excited state for the 3LS. Hence, the state is of the form

$$|\psi(t)\rangle = \int d\mathbf{k}d\mathbf{k}'c(\mathbf{k},\mathbf{k}';t)|\mathbf{k},\mathbf{k}',0\rangle + \sum_a\int d\mathbf{k}d_a(\mathbf{k};t)|k,a\rangle, \tag{35}$$

Substituting into Equation (34), we obtain:

$$i\dot{c}(\mathbf{k},\mathbf{k}';t) = \sum_a\lambda_a\left[\frac{d_a(\mathbf{k};t)}{\sqrt{2\epsilon_{\mathbf{k}'}}}e^{i(\epsilon_{\mathbf{k}}-\Omega_a)t} + \frac{d_a(\mathbf{k}';t)}{\sqrt{2\epsilon_{\mathbf{k}}}}e^{i(\epsilon_{\mathbf{k}'}-\Omega_a)t}\right] \tag{36}$$

$$i\dot{d}_a(\mathbf{k};t) = 2\lambda_a\int\frac{d\mathbf{k}'}{\sqrt{2\epsilon_{\mathbf{k}'}}}c(\mathbf{k},\mathbf{k}';t)e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)t} \tag{37}$$

These equations are to be solved subject to the initial conditions $d_a(\mathbf{k};0) = 0$ and $c(\mathbf{k},\mathbf{k}';0) = c_0(\mathbf{k},\mathbf{k}')$, where $c_0(\mathbf{k},\mathbf{k}')$ is the initial state of the two particles. We integrate both sides of Equation (36) and substitute $c(\mathbf{k},\mathbf{k}';t)$ to Equation (37). We obtain:

$$\begin{aligned}\dot{d}_a(\mathbf{k};t) &= 2\lambda_a\int\frac{d\mathbf{k}'}{\sqrt{2\epsilon_{\mathbf{k}'}}}c_0(\mathbf{k},\mathbf{k}')e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)t} \\ &- 2\lambda_a\sum_b\lambda_b\int\frac{d\mathbf{k}'}{2\epsilon_{\mathbf{k}'}}e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)t}\int_0^t ds d_a(\mathbf{k};s)e^{i(\epsilon_{\mathbf{k}}-\Omega_b)s} \\ &- 2\frac{\lambda_a}{\sqrt{\epsilon_{\mathbf{k}}}}\sum_b\lambda_b\int\frac{d\mathbf{k}'}{2\sqrt{\epsilon_{\mathbf{k}'}}}e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)t}\int_0^t ds d_a(\mathbf{k}';s)e^{i(\epsilon_{\mathbf{k}'}-\Omega_b)s}.\end{aligned} \tag{38}$$

Equation (38) is exact. The term in the second line is proportional to the vacuum Wightman function $W(t) = \int\frac{d\mathbf{k}'}{2\epsilon_{\mathbf{k}'}}e^{-i\epsilon_{\mathbf{k}'}t}$, which drops at least with e^{-mt} for $m \neq 0$ and as t^{-2} for $m = 0$. Assuming that the particle starts sufficiently far from the detector, $d_a(\mathbf{k};t)$ becomes appreciable different from zero at times such that the term proportional to $W(t)$ is strongly suppressed.

The third-line term in Equation (38) is of a structure that commonly appears in elementary treatments of spontaneous decay [32,43]. It can be calculated by invoking a version of the Wigner–Weisskopf approximation. For $\Omega_a t >> 1$, this expression is strongly dominated by the term with $b = a$. By carrying out the integration over $\mathbf{k}'$, we obtain:

$$-\frac{\lambda_a^2}{2\pi^2\sqrt{\epsilon_{\mathbf{k}}}}\int_m^\infty d\epsilon \frac{(\epsilon^2 - m^2)^{3/2}}{\sqrt{\epsilon}}\int_0^t ds e^{-i(\epsilon-\Omega_a)(t-s)}d_a(\mathbf{k}, s). \tag{39}$$

The time integral is negligible except for values of ϵ around Ω_a. Hence, we are justified in substituting $(\epsilon^2 - m^2)^{3/2}/\sqrt{\epsilon}$ with $(\Omega_a^2 - m^2)^{3/2}/\sqrt{\Omega_a}$, and then, to extend integration over ϵ to $(-\infty, \infty)$. Then, the term (39) simplifies to $-\frac{1}{2}\eta_a\epsilon_{\mathbf{k}}^{-1/2}d_a(\mathbf{k}, t)$, where:

$$\eta_a = -\frac{\lambda_a^2(\Omega_a^2 - m^2)^{3/2}}{\pi\sqrt{\Omega_a}}. \tag{40}$$

Equation (38) becomes:

$$\dot{d}_a(\mathbf{k}; t) + \frac{1}{2}\eta_a\epsilon_{\mathbf{k}}^{-1/2}d_a(\mathbf{k}; t) = 2\lambda_a\int \frac{d\mathbf{k}'}{\sqrt{2\epsilon_{\mathbf{k}'}}}c_0(\mathbf{k}, \mathbf{k}')e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)t}. \tag{41}$$

This is a linear inhomogenous equation of first order. The Green function for the corresponding homogeneous equation is simply $\theta(t - t')e^{-\frac{1}{2}\eta_a\epsilon_{\mathbf{k}}^{-1/2}(t-t')}$. Hence, we obtain:

$$d_a(\mathbf{k}; t) = 2\lambda_a\int \frac{d\mathbf{k}'}{\sqrt{2\epsilon_{\mathbf{k}'}}}c_0(\mathbf{k}, \mathbf{k}')\int_0^t ds e^{-\frac{1}{2}\eta_a\epsilon_{\mathbf{k}}^{-1/2}(t-s)}e^{-i(\epsilon_{\mathbf{k}'}-\Omega_a)s} = 2\lambda_a\int d\mathbf{k}' c_0(\mathbf{k}, \mathbf{k}')h_a(\epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k}'}; t),$$

where:

$$h_a(\epsilon, \epsilon'; t) = \frac{e^{-\frac{1}{2}\eta_a\epsilon^{-1/2}t} - e^{-i(\epsilon'-\Omega_a)t}}{\sqrt{2\epsilon'}\left[\frac{1}{2}\eta_a\epsilon^{-1/2} - i(\epsilon' - \Omega_a)\right]}. \tag{42}$$

The detection probability is non-negligible only if $\mathbf{k}$ is along the axis that connects the source to the detector. Hence, the problem is effectively one-dimensional. Therefore, we can substitute the initial state with $c_0(k, k')$, where $k, k' > 0$, and write $d_a(k; t) = 2\lambda_a\int \frac{dk'}{2\pi}c_0(k, k')h_a(\epsilon_k, \epsilon_{k'}; t)$.

5.3. An Example

Consider an initial state:

$$c_0(k, k') = \frac{1}{\sqrt{2}}\left[\psi_1(k)\psi_2(k') + \psi_1(k')\psi_2(k)\right], \tag{43}$$

where ψ_i, for $i = 1, 2$, is centered around momentum k_i, or, equivalently, on energy $\epsilon_i = \sqrt{k_i^2 + m^2}$. We assume that there is no overlap between ψ_1 and ψ_2. Then, we can approximate:

$$d_a(k; t) = \lambda_a[\psi_1(k)F_{2a}(t) + \psi_2(k)F_{1a}(t)], \tag{44}$$

where

$$F_{ia}(t) = \int \frac{dk}{2\pi}\psi_i(k)\frac{e^{-\Gamma_{ia}t} - e^{-i(\epsilon_k-\Omega_a)t}}{\sqrt{2\epsilon_k}[\Gamma_{ia} - i(\epsilon_k - \Omega_a)]}, \tag{45}$$

where $\Gamma_{1a} = \frac{1}{2}\eta_a\epsilon_2^{-1/2}$ and $\Gamma_{2a} = \frac{1}{2}\eta_a\epsilon_1^{-1/2}$. Then, the probability $p_a(t)$ that the 3LS is found in an excited state is given by:

$$p_a(t) = \int dk|d_a(k; t)|^2 = \lambda_a^2\left(|F_{1a}(t)|^2 + |F_{2a}(t)|^2\right). \tag{46}$$

Let the states $\psi_i(k)$ be well localized around $x = -L$, so that they can be written as $\chi(k - k_i)e^{ikL}$, where χ is a positive function peaked around $k = 0$, for example, a Gaussian.

Then, the typical behavior of $|F_{ia}(t)|^2$ is given in Figure 4. The function is negligible prior to the arrival time $t_a = mL/k_i$ of the particle to the locus of 3LS. Then, it jumps to a finite value, which then decays with a rate given by Γ_{ia}.

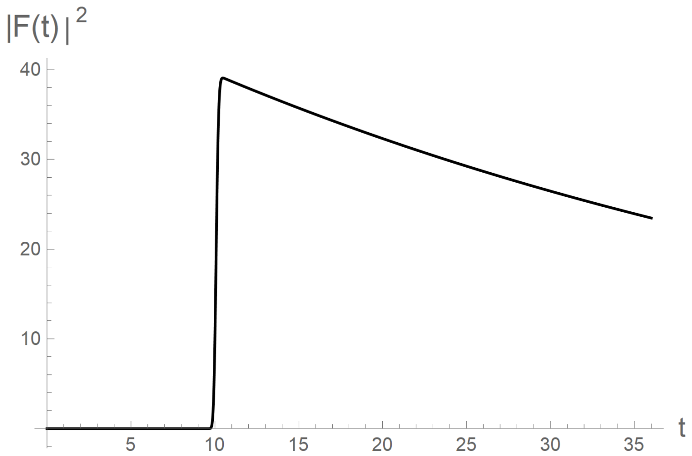

Figure 4. Typical plot of the functions $|F_{ia}(t)|^2$ as a function of time t, for a Gaussian function $\chi(k)$. The function jumps to a finite value when the particle arrives at the 3LS, and then it decays with a rate of Γ_{ia}.

The peak value of $|F_{ia}(t)|^2$ is approximately proportional to the Breit–Wigner term $[(\Gamma_{ia}^2 + (\epsilon_i - \Omega_a)^2]^{-1}$. Supposing that we choose $\epsilon_a \simeq \Omega_a$, and that $\Gamma_{ia} << |\Omega_1 - \Omega_2|$, for all $i, a = 1, 2$, and then, the terms $|F_{11}|^2$ and $|F_{22}|^2$ dominate in the probability assignment, and:

$$p_a(t) = \lambda_a^2 |F_{aa}(t)|^2. \tag{47}$$

The behavior of the probabilities is characteristic of resonant fluorescence. The 3LS absorbs one of the two particles, and after a time of order Γ_{aa}^{-1}, it re-emits the particle, albeit in a different direction. Hence, the energy of the fluorescent particle determines whether the ordering M_1 or M_2 was realized.

6. Conclusions

We provided a general definition of events in quantum theory, and showed how to construct probabilities associated with the causal ordering such events. Our notion of events is very different from that of Refs. [3,4], and it is naturally related to the relativistic notion of events. Our analysis clarifies that the existence of an indefinite quantum causal order of events has no relation to quantum gravity, as this causal order is a dynamical consequence of the quantum nature of the matter degrees of freedom. The COoE should not be conflated with the causal structure of spacetime, which we take to be fixed and unchanged in absence of gravity.

In Section 4, we showed that the quantum probabilities for the COoE have different behavior from the corresponding classical ones. We expect that it will be possible to prove an analogue of Bell's theorem, namely, that some probabilistic predictions for the causal orders cannot be reproduced by any classical probabilistic theory. However, the POVM does not factorize with respect to the subsystems, so any irreducibly quantum behavior of the COoEs in a multi-partite system will not be directly related to entanglement. In this sense, any Bell-type inequality for the COoE's would be similar to the Leggett–Garg inequalities for sequential measurements [44–46], rather than to the standard Bell inequalities.

The model systems considered in this paper are experimentally accessible. The set-ups considered in Section 5 are essentially quantum races, that is, the causal order of events coincides with the order that a number of distinguishable particles arrive in a specific finish line. The set-up of Section 6, when applied to photons, involves a variation of resonant fluorescence with specially engineered multi-level atoms that play the role of detectors for the causal ordering that is being realized.

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