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Lie Symmetries of the Wave Equation on the Sphere Using Geometry

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Abstract: A semilinear quadratic equation of the form $A_{ij}(x)u^{ij} = B_i(x,u)u^i + F(x,u)$ defines a metric A_{ij}; therefore, it is possible to relate the Lie point symmetries of the equation with the symmetries of this metric. The Lie symmetry conditions break into two sets: one set containing the Lie derivative of the metric wrt the Lie symmetry generator, and the other set containing the quantities $B_i(x,u), F(x,u)$. From the first set, it follows that the generators of Lie point symmetries are elements of the conformal algebra of the metric A_{ij}, while the second set serves as constraint equations, which select elements from the conformal algebra of A_{ij}. Therefore, it is possible to determine the Lie point symmetries using a geometric approach based on the computation of the conformal Killing vectors of the metric A_{ij}. In the present article, the nonlinear Poisson equation $\Delta_g u - f(u) = 0$ is studied. The metric defined by this equation is 1 + 2 decomposable along the gradient Killing vector ∂_t. It is a conformally flat metric, which admits 10 conformal Killing vectors. We determine the conformal Killing vectors of this metric using a general geometric method, which computes the conformal Killing vectors of a general $1+(n-1)$ decomposable metric in a systematic way. It is found that the nonlinear Poisson equation $\Delta_g u - f(u) = 0$ admits Lie point symmetries only when $f(u) = ku$, and in this case, only the Killing vectors are admitted. It is shown that the Noether point symmetries coincide with the Lie point symmetries. This approach/method can be used to study the Lie point symmetries of more complex equations and with more degrees of freedom.

Keywords: Lie point symmetries; wave equation; sphere; conformal Killing vectors; $1+(n-1)$ decomposable metric

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1. Introduction

The homogenous wave equation on the sphere is given by the equation

$$u_{tt} = u_{xx} + (\cot x)u_x + \frac{1}{\sin^2 x}u_{yy}. \quad (1)$$

H. Azad and M. Mustafa [1], using the package MathLie [2], determined the Lie algebra of the Lie point symmetries of (1), and classified its subalgebras up to conjugancy. Subsequently, they performed similarity reduction for each subalgebra, and in some cases of the two-dimensional subalgebra, they provided the invariant solution.

In [3], Freire observed that Equation (1) is a particular case of the Poisson equation $\Delta_g u = 0$ for the metric

$$ds^2_{2+1} = -dt^2 + dx^2 + \sin^2 x dy^2. \quad (2)$$

The Lie point and the Noether point symmetries of the nonlinear Poisson equation of the form

$$\Delta_g u - f(u) = 0 \tag{3}$$

have been studied in [4] for certain forms of the function $f(u)$ and $n > 2$, where n is the dimension of the space. Using the results of [4], Freire studied the Lie point symmetries of (1), which is a particular case of (3) for $f(u) = 0$, $n = 3$ and metric (2).

However, it should be pointed out that the situation here is different than the one considered in [4]. Indeed, having the Lie symmetry conditions for a generic metric and a general function $f(u)$, one has two possibilities: either to leave the metric unspecified and consider various types of functions $f(u)$, or to specify the metric and determine the functions $f(u)$ for which the symmetry conditions are satisfied.

In [4], the authors considered the first scenario, that is, they assumed a generic metric and considered certain forms of $f(u)$. In the present work, the situation is different because the metric is specified. Therefore, one has to use the second approach, that is, to fix the metric (2) and use the Lie symmetry conditions to find for which functions $f(u)$ Lie point symmetries are admitted. This is what it is carried out in the present work.

In the second approach, one needs the conformal Killing vectors (CKVs) of the metric (2). To our knowledge, there is no a package available to perform that, at least for a complex metric and/or a significant number of independent variables, and one has to use available results from Riemannian geometry. This case, although more demanding, has the advantage that it is applicable to higher dimensions and more complex metrics due to the existence of the plethora of general relevant geometric results. In the Appendix A, we use a systematic method, which determines the CKVs in a $1 + (n - 1)$ decomposable Riemannian space to compute the CKVs of (2).

The structure of the paper is as follows. In Sections 3–6, we present the geometric results, which shall be used in the "solution" of the system of Lie conditions. In Section 6, we use these results to compute the CKVs of the metric defined by (2). In Section 7, we write the Lie point symmetry conditions and find that the only homogeneous wave equations on the sphere that admit Lie point symmetries are the ones for which $f(u) = ku$ where k is a constant. Furthermore, the Lie symmetry generators are the Killing vectors (KVs) of the metric (2) and the vector $b(x)\partial_u$, where $b(x)$ is a solution of the wave equation. In Section 8, it is shown that the Lie symmetries are also Noether symmetries, and it is demonstrated how the conserved Noether currents are computed. Finally, in Section 10, we draw our conclusions.

2. Symmetries of a Metric

Consider a Riemannian space with metric g_{ab}, a vector field X, and the Lie derivative

$$L_X g_{ab} = g_{ab,c} X^c + X_{a,b} + X_{b,a}. \tag{4}$$

A *symmetry* of the metric is defined by the relation

$$L_X g_{ab} = K_{ab} \tag{5}$$

where K_{ab} is a symmetric tensor. The basic symmetries of g_{ab} are defined by the tensors $K_{ab} = 0, cg_{ab}, \psi(x)g_{ab}$, and in each of these cases, the vector field X is called the *Killing vector* (KV), *Homothetic vector* (HV), and *conformal Killing vector* (CKV), respectively. Each type of these vectors forms a Lie algebra. In order to compute the vector X, one solves the equation

$$g_{ab,c} X^c + X_{a,b} + X_{b,a} = K_{ab} \tag{6}$$

Obviously, the general solution cannot be performed for all metrics, and we have to consider special simplified classes of metrics. One special class of metrics are the maximally symmetric metrics, or metrics of constant curvature.

3. Maximally Symmetric Metrics

Maximally symmetric metrics are the units of “symmetry”, and have a direct relation with the standard Euclidean concepts of symmetry, e.g., spherical symmetry.

Definition 1. *A maximally symmetric metric g_{ab} is a metric whose curvature tensor R_{abcd} satisfies the relation*

$$R_{abcd} = K g_{abcd}. \tag{7}$$

where $K = \frac{R}{n(n-1)}$ is a constant, R is the curvature scalar, and the tensor

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{8}$$

A flat metric is a maximally symmetric metric for which $K = 0$. The covariant definition of a flat metric is $R_{abcd} = 0$.

A maximally symmetric metric is characterized by the number of admitted KVs as follows (see pp. 238–239 in [5]):

Theorem 1. *A non-degenerate metric g_{ab} is a metric of constant curvature iff it is a maximally symmetric metric iff it admits $\frac{1}{2}n(n+1)$ KVs iff the equations of geodesics admit $\frac{1}{2}n(n+1)$ linearly independent linear first integrals ($I = \xi_i \dot{x}^i$ where $\xi_{(i;j)} = 0$).*

All maximally symmetric metrics are conformally flat and have the global property that they can be written in the form

$$ds^2 = \frac{1}{(1 + \frac{K}{4}x^r x_r)^2} dx^i dx_i \tag{9}$$

where $i = 0, 1, \ldots, n$ and

$$K = \frac{R}{n(n-1)} \tag{10}$$

where R is the scalar curvature of the space and $K = -1, +1, 0$ for a negative, positive, and zero curvature, respectively.

4. The Conformal Algebra of Maximally Symmetric Metrics

Definition 2. *Two metrics, g_{ab}, η_{ab}, are conformally related if they satisfy the condition*

$$g_{ab} = \psi(x)\eta_{ab} \tag{11}$$

where $\psi(x)$ is the conformal factor.

Two conformally related metrics share the same CKVs. From (9), it follows that a maximally symmetric metric is conformally related to the flat metric with the conformal factor

$$U = \frac{1}{1 + \frac{K}{4}\mathbf{x}^2}. \tag{12}$$

Therefore, the conformal algebra of the metric g_{ab} is the same with conformal algebra of the flat metric.

It is well known [5–8] that a flat metric of dimension n admits $\frac{n(n+1)}{2}$ KVs, 1 gradient HV and $\frac{(n+1)(n+2)}{2}$ Special CKVs (a CKV is called special if the conformal factor ψ satisfies the condition $g^{ij}\psi_{;ij} = 0$).

Specifically, the conformal algebra of the flat space

$$ds^2 = \varepsilon dt^2 + \delta_{AB} dy^A dy^B \tag{13}$$

where $\varepsilon = \pm 1$ and $A, B = 1, 2, \ldots, n-1$ consists of the following vectors.

$n-$ gradient KVs: ∂_t, ∂_A.
$\frac{n(n-1)}{2}$ non-gradient KVs: $X_R^{1A} = y^A\partial_t - \varepsilon t\partial_A, X_R^{AB} = y^B\partial_A - y^A\partial_B$.
One HV: $H = t\partial_t + \sum_A y^A\partial_A$.
$\frac{(n+1)(n+2)}{2}$ Special CKVs: $X_C^1 = \frac{1}{2}(t^2 - \varepsilon\sum_A (y^A)^2)\partial_t + t\sum_A y^A\partial_A, X_C^A = ty^A\partial_t + \frac{1}{2}(-\varepsilon t^2 + (y^A)^2 - \sum_{B\neq A}(y^B)^2)\partial_A + y^A\sum_{B\neq A} y^B\partial_B$ where $A, B = 1, 2, \ldots, n-1$ with conformal factor $\psi_C^1 = t$ and $\psi_C^A = y^A$.

From these, by taking into consideration the conformal factor $U(x)$, we have that the conformal algebra of a maximally symmetric metric g_{ab} consists of the vectors of Table 1.

Table 1. The conformal algebra of the metric of constant curvature (9).

Param	**Type**	**Symbol**	**Number**	$\hat{\psi}$	$\hat{F}_{ij}$
a_i	CKV	$\mathbf{P}_a$	n	$-\frac{KU}{2}x_I$	$KU^3 x_{[a}\eta_{b]I}$
a_{ab}	KV	$\mathbf{r}_{AB}$	$\frac{n(n-1)}{2}$	0	$U^2\eta_{ABab} + KU^3\eta_{\dot{A}\dot{B}\cdot[b}^{\phantom{\dot{A}\dot{B}\cdot}c} x_{a]}x_c$
b	CKV	$\mathbf{H}$	1	$1 - \frac{KU}{2}\mathbf{x}^2$	0
b_a	SCKV	$\mathbf{K}_a$	n	$2Ux_I$	$4U^3 x_{[a}\eta_{b]I}$

The metric g_{ab} being maximally symmetric must admit $n(n+1)/2$ KVs. In Table 1, we have only the $n(n-1)/2$ KVs $\mathbf{r}_{ab}$. Therefore, there must exist another n KVs. Furthermore, we note that the HV is a gradient CKV. Because gradient vector fields are always convenient, we are looking for combinations of these CKVs, which produce KVs and gradient CKVs. We find that the vectors

$$\begin{aligned}\mathbf{I}_I &= \mathbf{P}_I + \frac{K}{4}\mathbf{K}_I = \partial_I + \frac{K}{4}\left[2x^b x_I - (x_c x^c)\delta_I^b\right]\partial_b \\ &= \left[1 - \frac{K\mathbf{x}^2}{4}\right]\partial_I + \frac{K}{2}x^b x_I\partial_b\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{C}_I &= \mathbf{P}_I - \frac{K}{4}\mathbf{K}_I = \partial_a - \frac{K}{4}\left[2x^b x_I - (x_c x^c)\delta_I^b\right]\partial_b \\ &= \left[1 + \frac{K\mathbf{x}^2}{4}\right]\partial_I - \frac{K}{2}x^b x_I\partial_b\end{aligned} \tag{15}$$

are the required ones. Indeed, $\mathbf{I}_I$ is a KV and $\mathbf{C}_I$ is a proper gradient CKV with the conformal functor

$$\psi(\mathbf{C}_I) = -KUx_I. \tag{16}$$

We collect these results in Table 2. The non-tensorial indices $A, B, I = 1, \ldots, n$ count vector fields. F_{ab} is the antisymmetric part $X_{[ab]}$ of the non-gradient CKV, called the bivector of X.

Table 2. The $(n+1)(n+2)/2$ CKVs of (9).

#	**Type**	**Components**	$\hat{F}_{ab}$	$\hat{\psi}$	$\hat{\psi}^{;a}$
$\mathbf{I}_I$	KV (non-grad)	$\frac{1}{U}\left[(2U-1)\delta_I^\alpha + \frac{1}{2}KUx_I x^\alpha\right]$	$2KU^3 x_{[\alpha}\eta_{\beta]I}$	0	0
$\mathbf{r}_{AB}$	KV (non-grad)	$\delta_{[A}^c\delta_{B]}^d x_c$	$U^2\eta_{ABab} + KU^3\eta_{AB.[b}^{c} x_{a]}x_c$	0	0
$\mathbf{H}$	CKV (grad)	$x^\alpha\partial_\alpha$	0	$1 - \frac{KU\cdot\mathbf{x}^2}{2}$	$-KU^2 x_a = -K\widehat{H}_\alpha$
$\mathbf{C}_I$	CKV (grad)	$\frac{1}{U}\left[\delta_I^b - \frac{KU}{2}x_I x^b\right]$	0	$-KUx_I$	$-K\widehat{C}_{(I)\alpha}$

We note that the new basis consists of the $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ KVs $\{\mathbf{I}_a, \mathbf{M}_{ab}\}$, no HV and $n+1$ proper CKVs $\{\mathbf{H}, \mathbf{C}_a\}$. These vectors are in total $\frac{(n+1)(n+2)}{2}$ as it is necessary for a metric of constant curvature. Furthermore all KVs of this basis are not gradient and all proper CKVs are gradient. This observation is very useful in applications.

We note that the vectors $\{\mathbf{H}, \mathbf{C}_a\}$ *are not* SCKVs except in the case $K = 0$. Indeed from the last column we compute

$$\widehat{\psi}_{;ab}(\mathbf{H}) = -K\widehat{\mathbf{H}}_{(;ab)} = -K\widehat{\psi}(\mathbf{H})g_{ab} = -p\widehat{\psi}(\mathbf{H})g_{ab} \tag{17}$$

where we have set

$$p = K. \tag{18}$$

A similar calculation applies to $\widehat{\psi}_{;ab}(\mathbf{C}_I)$. Relations (17) and (18) show two more facts:

1. The gradient vectors $\widehat{\psi}_{,a}(\mathbf{H})$ and $\widehat{\psi}_{,a}(\mathbf{C}_I)$ are gradient CKVs of the metric g_{ab} (not the flat metric!);
2. For non-flat metrics ($K \neq 0$) the gradient vector $\psi_{,a}$ is non-null.

5. The Conformal Algebra of a $1 + (n - 1)$ Decomposable Metrics

We continue with the determination of the conformal algebra of $1 + (n - 1)$ decomposable metric S_{ab}.

Definition 3. *A metric S_{ab} is called $1 + (n - 1)$ globally decomposable if it admits a non-null covariantly constant vector field k^a.*

The vector field k^a is a gradient KV. The metric of a $1 + (n - 1)$ can be written in the form

$$ds^2 = \epsilon(k)(dx^1)^2 + g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \tag{19}$$

where $\mathbf{k} = \partial x^1$, $\epsilon(\mathbf{k}) = \pm 1$ is the sign of k^a and the quantities $g_{\alpha\beta}(x^\gamma)$ are the components of the metric tensor S_{ab} on the $(n - 1)$ space x^1 =constant.

The CKVs of the $1 + (n - 1)$ decomposable metric are computed from the CKVs of the $n - 1$ metric $g_{\mu\nu}$ as indicated in the following Theorem [9].

Theorem 2. *Consider the $1 + (n - 1)$, $(n \geqq 3)$ decomposable Riemannian metric S_{ab}, which is as follows:*

$$ds_n^2 = S_{ab}dx^a dx^b = S_{11}(dx^1)^2 + g_{\mu\nu}(x^\rho)dx^m dx^\nu$$

where the reduced $n - 1$ metric $g_{\mu\nu}$ ($\mu\nu \neq 1$) is non-decomposable and the decomposing vector is the $M^a = \delta^a_1$ with sign $\varepsilon = M^a M_a$. The conformal algebra of S_{ab} has as follows:

1. The KVs of S_{ab} are the ones of $g_{\mu\nu}$ (if any) plus the Killing vector M^a, which decomposes the metric. All these vectors are non-gradient.

2. S_{ab} admits the gradient HV:

$$X_a = C_1 x^1 M_a + k_\mu \delta^\mu_a$$

provided the reduced $n - 1$ metric $g_{\mu\nu}$ admits the gradient HV k^μ with conformal factor C_1.

3. The metric S_{ab} admits a non-gradient special CKV X^a:

$$X^a = \left[\frac{1}{2}a(x^1)^2 - a\varepsilon \int k_\mu dx^\mu\right] M^a + ax^1\delta^a_\mu k^\mu$$

with conformal factor $\psi = ax^1$, provided the reduced $n - 1$ metric $g_{\mu\nu}$ admits the proper gradient HV k^μ with homothetic factor a.

4. All proper CKVs of the metric S_{ab} are non-gradient and given by the formula

$$X_a = \left[\int g(x^1)dx^1\lambda(x^\rho) + C\right] M_a - \frac{1}{p}g(x^1)\lambda(\mathbf{k})_{,\mu}\delta^\mu_a \tag{20}$$

where

a. The function $g(x^1)$

$$g(x^1) = \sin(\sqrt{p}x^1), \cos(\sqrt{p}x^1) \quad \textit{for } \epsilon p = 1$$
$$g(x^1) = \sinh(\sqrt{p}x^1), \cosh(\sqrt{p}x^1) \quad \textit{for } \epsilon p = -1$$

p is a non-zero constant.

b. $\lambda(\mathbf{k})_{,\mu}$ *is a gradient CKV of the reduced* $n-1$ *metric* $g_{\mu\nu}$ *with conformal factor* $p\lambda(x^\rho)$ $(p \neq 0)$. *The conformal factor of* X^a *equals* $\psi(x^a) = g(x^1)\lambda(x^\rho)$ *and its bivector is given by the formula* $F_{ab}(X) = \varepsilon \int g(x^1)dx^1 \lambda_{,\mu}(x^\rho)(\delta_a^1 \delta_b^\mu - \delta_b^1 \delta_a^\mu)$.

At this point, we have all the means to compute the functions $f(u)$ for Equation (3).

6. The Geometry of the Metric (2)

The metric

$$ds^2_{2+1} = -dt^2 + dx^2 + \sin^2 x dy^2. \tag{21}$$

defined by the homogenous wave Equation (3) is 1 + 2 decomposable along the gradient KV ∂_t. Furthermore, the 2-metric

$$ds^2_2 = dx^2 + \sin^2 x dy^2 \tag{22}$$

is the metric of a maximally symmetric space with positive Gaussian curvature $R = 2$ therefore admits $\frac{(2+1)(2+2)}{2} = 6$ CKVs. Three of these vectors are Killing vectors (KVs) and the remaining three are proper CKVs. The homothetic vector (HV) is not admitted. It is easily found (using any algebraic computing program) that the KVs ξ_1, ξ_2, ξ_3 (say) are

$$\xi_1 = \partial_y, \xi_2 = \sin y \partial_x + \cot x \cos y \partial_y, \xi_3 = \cos y \partial_x - \cot x \sin y \partial_y. \tag{23}$$

The 1 + 3 metric (2) is conformally flat (this can be shown easily by showing that the Cotton tensor vanishes), and therefore admits $\frac{(3+1)(3+2)}{2} = 10$ CKVs. These are the four KVs $\xi_1, \xi_2, \xi_3, \partial_t$ and six proper CKVs.

The computation of the six CKVs are computed using Theorem 2. It is found that there are two sets of proper CKVs, the non-gradient and the gradient ones. The non-gradient CKVs are

$$\begin{aligned}
\xi_5 &= -\cos t \sin y \sin x \partial_t - \sin t \sin y \cos x \partial_x - \sin t \cos y \sin x \partial_y \\
\xi_6 &= \sin t \sin y \sin x \partial_t - \cos t \sin y \cos x \partial_x - \cos t \frac{\cos y}{\sin x} \partial_y \\
\xi_7 &= \cos t \cos y \sin x + \sin t \cos y \cos x \partial_x - \sin t \frac{\sin y}{\sin x} \sin x \partial_y \\
\xi_8 &= -\sin t \cos y \sin x + \cos t \cos y \cos x \partial_x - \cos t \frac{\sin y}{\sin x} \sin x \partial_y
\end{aligned}$$

with conformal factors

$$\psi_5 = \sin t \sin y \sin x; \psi_6 = \cos t \sin y \sin x; \psi_7 = -\sin t \cos y \sin x; \psi_8 = -\cos t \cos y \sin x$$

and the gradient CKVs are

$$\begin{aligned}
\xi_9 &= -\cos t \cos x \partial_t + \sin t \sin x \partial_x \\
\xi_{10} &= \sin t \cos x \partial_t + \cos t \sin x \partial_x
\end{aligned}$$

with conformal factors

$$\psi_9 = \sin t \cos x; \psi_{10} = \cos t \cos x.$$

It is important to note that the conformal factors of the all proper CKVs satisfy the relation

$$\psi_{;ij} = -\psi g_{ij} \tag{24}$$

that is, the vectors $\psi_{,i}$ are gradient CKVs.

In a Riemannian space, which admits CKVs the following results are well known [7,8].

Lemma 1. *Assume that the vector ξ^i is a CKV of the metric g_{ij} with conformal factor $-(\lambda - a)$ i.e., $L_{\xi^i \partial_i} g_{ij} = -(\lambda - a) g_{ij}$. Then, the following relations hold:*

$$g^{jk} L_\xi \Gamma^i_{.jk} = g^{jk} \xi^i_{.,jk} + \Gamma^i_{,l} \xi^l - \xi^i_{.,l} \Gamma^l + (a - \lambda) \Gamma^i_{.} \tag{25}$$

$$g^{jk} L_\xi \Gamma^i_{.jk} = \frac{2-n}{2} (a - \lambda)^{,i} \tag{26}$$

where $n = g^{jk} g_{kj}$ is the dimension of the space.

Proof. The proof of (25) is straightforward by using the definition of $L_\xi \Gamma^i_{.jk}$ and contracting with g^{jk}. To prove (26), one uses the identity

$$L_\xi \Gamma^i_{.jk} = \frac{1}{2} g^{ir} \left[\nabla_k L_\xi g_{jr} + \nabla_j L_\xi g_{kr} - \nabla_r L_\xi g_{kj} \right] \tag{27}$$

and replaces $L_\xi g_{ij} = (a - \lambda) g_{ij}$ to find

$$\begin{aligned} L_\xi \Gamma^i_{.jk} &= \frac{1}{2} g^{ir} \left[(a-\lambda)_{,k} g_{jr} + (a-\lambda)_{,j} g_{kr} - (a-\lambda)_{,r} g_{kj} \right] \\ &= \frac{1}{2} \left[(a-\lambda)_{,k} \delta^i_j + (a-\lambda)_{,j} \delta^i_k - g^{ir} (a-\lambda)_{,r} g_{kj} \right]. \end{aligned}$$

□

Contracting with g^{jk} the result follows.

We note that for $n = 2$ the $g^{jk} L_\xi \Gamma^i_{.jk} = 0$, that is, the connection coefficients are disassociated from the conformal factor. This is a singular case, and it is the reason that the case $n = 2$ was not considered in [4].

7. The Lie Point Symmetries

As it is done in [3], we consider the Equation (3) for an arbitrary function $f(u)$ and the metric (2), that is, we consider the equation $\Delta_g u + f(u) = 0$, or equivalently,

$$H(x^i, u, u_{,i}, u_{,ij}) \equiv g^{ij} u_{,ij} - \Gamma^i u_i + f(u) = 0 \tag{28}$$

where $x^i = \{t, x, y\}$, $g^{ij} = diag(-1, 1, \sin^2 x)$ and $\Gamma^k = g^{ij} \Gamma^i_{jk}$.

The Lie point symmetry vector is

$$X = \xi^i(x^j, u) \partial_i + \eta(x, u) \partial_u \tag{29}$$

and the Lie symmetry condition is

$$X^{[2]}(H) = \lambda H$$

where $\lambda(x^i, u, u_i)$ is a function to be determined. $X^{[2]}$ is the second prolongation of X given by the expression

$$X^{[2]} = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta^{(1)}_i \frac{\partial}{\partial u_i} + \eta^{(2)}_{ij} \frac{\partial}{\partial u_{ij}}. \tag{30}$$

where

$$\eta_i^{(1)} = \frac{D\eta}{Dx^i} - u_j \frac{D\xi^j}{Dx^i} = \eta_{,i} + u_i\eta_u - \xi^j_{,i}u_j - u_i u_j \xi^j_{,u} \quad (31)$$

$$\eta_{ij}^{(2)} = \frac{D\eta_i^{(1)}}{Dx^j} - u_{jk}\frac{D\xi^k}{Dx^j} = \eta_{ij} + (\eta_{ui}u_j + \eta_{uj}u_i) - \xi^k_{,ij}u_k + \eta_{uu}u_i u_j - (\xi^k_{.,ui}u_j + \xi^k_{.,uj}u_i)u_k$$
$$+\eta_u u_{ij} - (\xi^k_{.,i}u_{jk} + \xi^k_{.,j}u_{ik}) - \left(u_{ij}u_k + u_i u_{jk} + u_{ik}u_j\right)\xi^k_{.,u} - u_i u_j u_k \xi^k_{uu}.$$

The introduction of the function $\lambda(x^i, u, u_i)$ in the defining equation allows one to consider the variables x^i, u, u_i to be independent [10].

When the symmetry condition $X^{[2]}(H) = \lambda H$ is applied to (28), the following system of equations results [10–18] in

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^i + auf_{,u} + bf_{,u} - \lambda f = 0 \quad (32)$$

$$g^{ij}\xi^k_{,ij} - 2g^{ik}a_i - \xi^k_{,i}\Gamma^i + \xi^i\Gamma^k_{,i} + (a-\lambda)\Gamma^k = 0 \quad (33)$$

$$L_{\xi^i\partial_i}g^{ij} = (\lambda - a)g^{ij} \quad (34)$$

$$\eta = a(x^i)u + b(x^i) \quad (35)$$

$$\xi^k_{,u} = 0 \Leftrightarrow \xi^k(x^i). \quad (36)$$

These relations coincide with those given in [10] (p. 115) if we consider the following correspondence:

This Notation	**Ibragimov Notation**
$a(x)$	$\sigma(x)$
$f(x,u)$	$-\psi(x,u)$
B^i	$-b^i$
$B^i_{,u}$	0

The strategy to solve these conditions is the following.

Because we know the metric, it is possible (in principle!) to solve condition (34) and determine the CKVs ξ^i. Using these vectors in (33), one determines the value of λ. Having these results and using (32), one determines the possible functions $f(u)$, and consequently, the equations of the form (28), which admit Lie point symmetries. Having the admitted Lie symmetries, one may use similarity reduction and possibly determine invariant solutions of (28). This latter part has been carried out for the case $f(u) = 0$ in [1]. Finally, using the fact that Equation (28) follows from the Lagrangian [4]

$$L = \frac{1}{2}\sin x\left(-u_t^2 + u_x^2 + \frac{1}{\sin^2 x}u_y + 2f(u)\right)$$

one determines the Noether point symmetries and finds the corresponding Noether currents, which reduce this equation possibly to one that can be solved with quadratures. This has been considered in [3] for $f(u) = ku$, where k is a constant.

In the present case, it is possible to compute the CKVs of the $1 + (3-1)$ decomposable metric; therefore, we follow the above algorithm.

We start with (34). This states that ξ^i is a CKV of (2). These vectors have been determined in Section 6.

Next, we consider condition (33).

Using property (25) of Lemma 1, condition (33) becomes

$$g^{jk}L_\xi\Gamma^i_{.jk} = 2g^{ik}a_i. \quad (37)$$

Again, using (26) of Lemma 1 to replace $g^{jk}L_{\xi}\Gamma^{i}_{.jk}$, we find that eventually, condition (33) gives

$$\frac{2-n}{2}(a-\lambda)^{,i} = 2a^{,i}$$

where n is the dimension of the space. In our case, $n=3$; therefore,

$$(a-\lambda)^{,i} = -4a^{,i}$$

or

$$a - \lambda = -4a - 2C \tag{38}$$

where C is a constant, which counts for the KVs and the HV. We conclude that condition (33) expresses the conformal factor in terms of a, and furthermore provides the value of λ:

$$\lambda = 5a + 2C. \tag{39}$$

Due to (38), condition (34) becomes

$$L_{\xi^i\partial_i}g_{ij} = 2(2a+C)g_{ij} \tag{40}$$

Still, we have to consider the remaining Lie symmetry condition (32):

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^{i} + auf_{,u} + bf_{,u} - \lambda f = 0$$

from which follows

$$g^{ij}a_{ij} - a_{,i}\Gamma^{i} + af_{,u} = 0 \Leftrightarrow \Delta_g a = -af_{,u} \tag{41}$$

$$g^{ij}b_{ij} - b_{,i}\Gamma^{i} + bf_{,u} - \lambda f = 0 \Leftrightarrow \Delta_g b = \lambda f - bf_{,u}. \tag{42}$$

The obvious implication of (41) is (because $a(x^i)$) $f_{,u} = 0$; therefore, the function $f(u) = ku$, where k is a constant.

In the last section, it has been shown that the conformal factors of the CKVs of the metric (2) satisfy the relation

$$\psi_{;ij} = -\psi g_{ij} \tag{43}$$

from which follows

$$\Delta_g\psi = -3\psi. \tag{44}$$

From (34) we have $2\psi = -(a-\lambda)$, and replacing λ from from (39), we find

$$\psi = 2a + 2C \tag{45}$$

Combining (44) and (45), we find

$$2\Delta_g a + 3(2a+C) = 0. \tag{46}$$

Then, condition (41) becomes

$$3(2a+C) = 2ak.$$

or

$$2(k-3)a = 3C. \tag{47}$$

We consider two cases.

Case a. $k \neq 3$

Then, a is a constant, which means that $a=0$ (homothetic vector is not admitted), and subsequently, $C = \lambda = 0$. In this case, only the KVs ξ_1, ξ_2, ξ_3 survive. Condition (42) gives $\Delta_g b = -bk$.

We conclude that for metric (2), the wave equations that admit Lie point symmetries are of the form

$$\Delta_g u = ku \tag{48}$$

where $k \neq 3$. The Lie symmetry vectors are

$$X_{1,2,3,4,5} = \xi_{1,2,3}(x^i), \partial_t, b(x^i)\frac{\partial}{\partial u} \tag{49}$$

where $\xi_{1,2,3}$ are the KVs (23) and $b(x^i)$ is a solution of Equation (48). These vectors coincide with the ones found in [1,3], except the Lie symmetry $u\frac{\partial}{\partial u}$. This is reasonable, because $a = 0$.

Note: In [3], the author refers to [4]'s case $f(u) = u^p$, where $p \neq \frac{n+2}{n-2}$. In our case, $p = \frac{3+2}{3-2} = 5 \neq 1$, which also gives $a = \lambda = 0$, and therefore agrees with our result.

Case b: $k = 3$

In this case, we have $C = 0$, and (47) is trivially satisfied; therefore a stays unspecified. Now, we use (42), from which follows

$$\begin{aligned} \lambda &= 0 \Rightarrow a = 0 && (50)\\ \Delta_g b &= -bk && (51) \end{aligned}$$

We conclude that the Lie symmetries are the same as in the case $k \neq 3$.

Therefore the result of $k \neq 3$ applies to all values of k.

8. Noether Point Symmetries

The Noether point symmetries are special Lie point symmetries, which in addition satisfy Noether's condition:

$$X^{[1]}L + LD_i\xi^i = D_i\phi^i \tag{52}$$

where $\phi^j(x, u, u_i, u_{ij,\ldots})$ is the Noether function,

$$D_i\phi^j = \frac{\partial\phi^j}{\partial x^i} + u_k\frac{\partial\phi^j}{\partial u} + u_{ks}\frac{\partial\phi^j}{\partial u^s} + \ldots$$

and $X^{[1]}$ is the first prolongation of the vector field X.

The nonlinear Poisson equation in a general space with metric g_{ij} follows from the Lagrangian [4]

$$L = \sqrt{g}\left(\frac{1}{2}g^{jk}u_ju_k - F(u)\right)$$

where

$$F_u = \frac{dF}{du} = f(u). \tag{53}$$

Using the relations

$$\begin{aligned} (\sqrt{g}g^{jk})_{,k} &= \sqrt{g}g^{rs}\Gamma^i_{rs} && (54)\\ (\sqrt{g})_{,i} &= \sqrt{g}\Gamma^k_{ik} && (55)\\ (\sqrt{g}g^{sk})_{,i} &= -\sqrt{g}(g^{sl}\Gamma^k_{li} + g^{kl}\Gamma^s_{li}) + \sqrt{g}g^{sk}\Gamma^l_{il} && (56) \end{aligned}$$

and the Lie symmetry condition $\eta = a(x)u + b(x)$ (see (35)), one computes

$$\begin{aligned} X^{[1]}L &= \frac{1}{2}\xi^i\left(-g^{jl}\Gamma^k_{li} - g^{kl}\Gamma^j_{li} + g^{jk}\Gamma^l_{li}\right)u_ju_k + \sqrt{g}(a\delta^k_i - \xi^k_{,i})g^{ij}u_ju_k \\ &\quad + \sqrt{g}(b_k + ua_k)u^k - \sqrt{g}(au+b)F_{,u} - \xi^i(\sqrt{g})_{,i}F(u) \end{aligned}$$

while
$$LD_i\xi^i = \sqrt{g}\left(\frac{1}{2}g^{jk}u_j u_k - F(u)\right)\xi^i_{,i}$$

Replacing the lhs of the Noether condition (52) and collecting terms, it follows the general result:

$$X^{[1]}L + LD_i\xi^i = \frac{1}{2}\sqrt{g}\left[-L_\xi g^{jk} + (2a + \xi^i_{;i})g^{jk}\right]u_j u_k + \sqrt{g}(b_k + ua_k)u^k - \sqrt{g}(au+b)f(u) - \sqrt{g}\xi^i{}_{;i}F(u). \tag{57}$$

In Section 7, it has been shown that the Lie symmetry conditions for the homogenous wave equation on the sphere require that $a = \lambda = 0$ and ξ^i is a KV; therefore $\xi^i{}_{;i} = 0$. Replacing these in (57) and using $f(u) = ku$, we find

$$X^{[1]}L + LD_i\xi^i = \sqrt{g}(b_k u^k - bku). \tag{58}$$

Then, the Noether symmetry condition (52) becomes

$$\sqrt{g}(b_k u^k - bku) \equiv \frac{\partial\phi^i}{\partial x^i} + u_k\frac{\partial\phi^i}{\partial u} + u_{ks}\frac{\partial\phi^i}{\partial u^s} + \dots$$

from which follows

$$\phi^i(x,u) \tag{59}$$
$$\frac{\partial\phi^j}{\partial x^i} = -\sqrt{g}bku \tag{60}$$
$$\frac{\partial\phi^j}{\partial u} = \sqrt{g}g^{ki}b_k. \tag{61}$$

Condition (61) implies
$$\phi^i = \sqrt{g}g^{ki}b_k u + R(u) \tag{62}$$

from which follows
$$\frac{\partial\phi^j}{\partial x^i} = \sqrt{g}g^{ki}ub_{;ki} = \sqrt{g}u\Delta_g b$$

Comparing with (60), we find

$$\Delta_g b + ku = 0 \tag{63}$$

which means that $b(x^i)$ *is* a solution of Equation (3). Therefore, for the KVs, the Noether condition is trivially satisfied, which means that for the homogenous wave equation on the sphere, the Lie symmetries are also Noether symmetries.

9. Conserved Noether Currents

The conserved vector corresponding to a Noether symmetry is given by the general expression [10]

$$C_i = L\xi_i + (n^a - \xi^k u^a_k)\left(\frac{\partial L}{\partial u^a_i} - D_j\frac{\partial L}{\partial u^a_{ij}}\right) + \frac{\partial L}{\partial u^a_{ij}}D_j(n^a - \xi^i u^a_j) \tag{64}$$

where L is the Lagrangian. In the present case, $a = 0$ and

$$L(x^i, u, u_i) = \frac{1}{2}\sin x u_t^2 - \frac{1}{2}\sin x u_x^2 - \frac{1}{2\sin x}u_y^2 + \frac{1}{2}k\sin x u^2$$

Therefore, (64) reduces to

$$C_i = L\xi_i - \xi^k u_k\frac{\partial L}{\partial u_i}. \tag{65}$$

We compute

$$\frac{\partial L}{\partial u} = k \sin x u; \frac{\partial L}{\partial u_t} = \sin x u_t; \frac{\partial L}{\partial u_x} = -\sin x u_x; \frac{\partial L}{\partial u_y} = -\frac{1}{\sin x} u_y \tag{66}$$

Using (64) and (66), one computes the conserved currents for the KVs of the metric (19).

We demonstrate the computation for the vector ∂_t and refer the reader to [3] for the remaining conserved vectors.

Let $C_t = (A^1, A^2, A^3)$ be the conserved vector for the KV ∂_t. Then, from (64) and (66), we compute

$$A_1 = -L$$

$$A_2 = -u_t(-\sin x u_x) = \sin x u_t u_x$$

$$A_3 = \frac{1}{\sin x} u_t u_y$$

Therefore,

$$C_i = -L\frac{\partial}{\partial t} + \sin x u_t u_x \frac{\partial}{\partial x} + \frac{1}{\sin x} u_t u_y \frac{\partial}{\partial y}. \tag{67}$$

10. Conclusions

We addressed the problem of finding the Lie point symmetries of the partial differential equation

$$H = A^{ij}(x) u_{ij} - F(x^r, u, u_i) = 0 \tag{68}$$

where

$$F(x^r, u, u_1) = B^k(x, u) u_k - f(x, u). \tag{69}$$

and $B^k(x,u), f(x,u)$ are arbitrary functions of their argument. In (68), there are two sets of unknown quantities, that is, the tensor A_{ij} and the functions $B^k(x,u), f(x,u)$. This means that in order for it to be possible for the Lie point symmetries to be determined, one of these sets must be specified. If the functions of the second set are assumed, one determines the CKVs of the metrics for which Lie point symmetries are admitted. Because the CKVs do not specify completely the metric, one finds essentially families of metrics. This approach has been taken in [4]. Here, we are given the metric, which we read from the Equation (3), which is a special case of (68), and we assume that $B^k(x,u) = 0, f(x,u) = f(u)$. Therefore, we compute the Lie point symmetries, and consequently, we determine the functions $f(x,u)$ for which Lie point symmetries are admitted. This completes the work of [3].

From the Lie symmetry condition (34), it follows that the Lie point symmetries are the CKVs of the metric $g^{ij} = diag(-1, 1, \sin^2 x)$. This is a $1 + (3-1)$ decomposable metric whose reduced 2D metric is a metric of maximal symmetry with curvature scalar $R = 2$. Using Theorem 2, we computed the CKVs of (34), and consequently, the Lie point symmetries. Using the rest of the Lie symmetry conditions, we found that $f(u) = ku$, where k is a constant. Finally, it has been shown that the Lie symmetries are also Noether symmetries, and it has been demonstrated how one computes the conserved Noether currents.

The geometric method we have considered in computing the Lie point symmetries and the functions $B^k(x,u), f(x,u)$ is general and can be applied to other known differential equations of the form (68), especially to the ones in which the metric defined by A^{ij} is more complex and of a higher dimension where the algebraic computing programmes might not answer.

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Appendix A

We write the metric (2) as

$$ds^2_{2+1} = -dt^2 + ds^2_2 \tag{A1}$$

where

$$ds^2_2 = dx^2 + \sin^2 x dy^2 \tag{A2}$$

is the metric of a maximally symmetric 2D Euclidian space with the curvature scalar $R = 2$. Therefore, ds^2_2 can be written in the form

$$ds^2_2 = \frac{1}{\left[1 + \frac{1}{4}(x_1^2 + y_1^2)\right]^2}(dx_1^2 + dy_1^2) \tag{A3}$$

for some coordinates x_1, y_1. ds^2_2 is given in the coordinates x, y. To bring ds^2_2 to the form (A3) and use the results of Section 5, we consider the coordinate transformation

$$x_1 = 2\tan\frac{x}{2}\cos y; y_1 = 2\tan\frac{x}{2}\sin y \tag{A4}$$

with inverse

$$\tan\frac{x}{2} = \frac{1}{2}\sqrt{x_1^2 + y_1^2}; y = \tan^{-1}\left(\frac{x_1}{y_1}\right). \tag{A5}$$

The six CKVs of the metric ds^2_2 in the coordinates x_1, y_1 are known (see Section 6). They are the three KVs:

$$\mathbf{I}_{y_1} = P_{y_1} + \frac{K}{4}K_{y_1} = \left[1 + \frac{1}{4}(y_1^2 - x_1^2)\right]\partial_{y_1} + \frac{1}{2}x_1 y_1 \partial_{x_1} \tag{A6}$$

$$\mathbf{I}_{x_1} = P_{x_1} + \frac{K}{4}K_{x_1} = \frac{1}{2}y_1 x_1 \partial_{y_1} + \left[1 + \frac{1}{4}(-y_1^2 + x_1^2)\right]\partial_{x_1} \tag{A7}$$

$$\mathbf{r}_{y_1 x_1} = 2\delta^c_{[y_1}\delta^d_{x_1]}x_{1c}\partial_d = (-y_1)\partial_{x_1} - x_1\partial_{y_1} = -(y_1\partial_{x_1} + x_1\partial_{y_1}) \tag{A8}$$

and the three proper gradient CKVs:

$$\mathbf{H} = y_1\partial_{y_1} + x_1\partial_{x_1} \tag{A9}$$

$$\mathbf{C}_{y_1} = P_{y_1} - \frac{K}{4}K_{y_1} = \left[1 - \frac{1}{4}(y_1^2 - x_1^2)\right]\partial_{y_1} - \frac{1}{2}y_1 x_1 \partial_{x_1} \tag{A10}$$

$$\mathbf{C}_{x_1} = P_{x_1} - \frac{K}{4}K_{x_1} = -\frac{1}{2}y_1 x_1 \partial_{y_1} + \left[1 - \frac{1}{4}(-y_1^2 + x_1^2)\right]\partial_{x_1} \tag{A11}$$

whose conformal factors are

$$\psi_H = \frac{1 - \frac{1}{4}(y_1^2 + x_1^2)}{1 + \frac{1}{4}(y_1^2 + x_1^2)}$$

$$\psi_{C_{y_1}} = \frac{-y_1}{1 + \frac{1}{4}(y_1^2 + x_1^2)}$$

$$\psi_{C_{x_1}} = \frac{-x_1}{1 + \frac{1}{4}(y_1^2 + x_1^2)}$$

and satisfy the condition

$$\psi_{C_I;\mu\nu} = -\psi g_{\mu\nu} \tag{A12}$$

where $g_{ab} = \frac{-1}{1+\frac{1}{4}(y_1^2+x_1^2)}\delta_{ab}$. These CKVs in the coordinates x, y are

$$\xi_4 = \sin x\partial_x, \xi_5 = -\sin y \cos x\partial_x - \frac{\cos y}{\sin x}\partial_y, \xi_6 = \cos y \cos x\partial_x - \frac{\sin y}{\sin x}\partial_y \tag{A13}$$

with conformal factors

$$\psi_4 = \cos x, \psi_5 = \sin y \sin x, \psi_6 = -\cos y \sin x. \tag{A14}$$

The conformal factors $\psi_{4,5,6}$ satisfy relation (A12).

Now that we know the gradient CKVs of the two-space ds_2^2 in the coordinates x, y, we are able to compute the CKVs of the decomposable metric ds_{1+2}^2

$$ds_{1+2}^2 = -dt^2 + \frac{1}{\left[1 + \frac{1}{4}(x_1^2 + y_1^2)\right]^2}(dx_1^2 + dy_1^2). \tag{A15}$$

using Theorem 2.

In order to apply Theorem 2, we read from (A12) $p = -1$. Also, the sign of $x^1 = t$ is $\varepsilon = -1$; therefore, $\varepsilon p = 1$ and $m(t) = \sin t, \cos t$. Because there is no HV, $b = 0$. Finally, we have that $f_{4,5,6}(t) = -(\sin t, \cos t)\psi_{4,5,6}$.

It is well known that if a $1 + (n-1)$ decomposable metric is such that the $n-1$ metric is maximally symmetric, then the $1 + (n-1)$ metric is conformally flat. This is the case here (equivalently one calculates the Cotton tensor of (2) and shows that vanishes). Therefore, the metric ds_{1+2}^2 has 10 CKVs, which can be computed using Theorem 2. Concerning the KVs, they are the ones of ds_2^2, that is, $\partial_t, \partial_y, I_{x^1}, I_{y^1}$.

Concerning the CKV, we find the following six CKVs:

$$\begin{aligned}
X_5 &= \sin t \cos x\partial_t + \cos t \sin x\partial_x \\
X_6 &= -\cos t \cos x\partial_t + \sin t \sin x\partial_x \\
X_7 &= \sin t \sin y \sin x\partial_t - \cos t \sin y \cos x\partial_x - \cos t\frac{\cos y}{\sin x}\partial_y \\
X_8 &= -\cos t \sin y \sin x\partial_t - \sin t \sin y \cos x\partial_x - \sin t\frac{\cos y}{\sin x}\partial_y \\
X_9 &= -\sin t \cos y \sin x + \cos t \cos y \cos x\partial_x - \cos t\frac{\sin y}{\sin x}\partial_y \\
X_{10} &= \cos t \cos y \sin x + \sin t \cos y \cos x\partial_x - \sin t\frac{\sin y}{\sin x}\partial_y
\end{aligned}$$

with conformal factors

$$\begin{aligned}
\phi_{5,7,9} &= \cos t\psi_{4,5,6} \\
\phi_{6.8.10} &= \sin t\psi_{4,5,6}.
\end{aligned}$$

The conformal factors $\phi_{5,\dots,10}$ satisfy condition (A12).

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