

A New Class of Separable Lagrangian Systems Generalizing Sawada–Kotera System

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Abstract: Some characteristics of stationary flows of the Sawada–Kotera system lend themselves to generalization, producing a large class of separable Lagrangian systems with two degrees of freedom. All of these systems come in couples that have the same equations of motion, although they are not related by a gauge transform. Some nonpolynomial examples are provided.

Keywords: integrable systems; separation of coordinates; generalized Hénon–Heiles systems

1. Introduction

The authors have long been interested in constants of motion for Lagrangian systems with finite degrees of freedom, and, as a special case, in integrable systems. We have recently come across some such system, which had been introduced in the field of solitons and of a chaotic dynamical system. As a result of our investigation into their integrability, we found that it can be generalized to a wider class, which may be of some interest for possible future applications in PDE settings.

Our most direct motivation was the study of the Lagrangian

$$L_{SK} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \left(\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 + \frac{1}{3}q_2^3 \right), \quad (1)$$

which we call the Sawada–Kotera system. It is integrable, by means of the energy and the supplementary first integral

$$K_{SK} = \dot{q}_1 \dot{q}_2 + q_1 q_2 + \frac{1}{3}q_1^3 + q_1 q_2^2. \quad (2)$$

This case is also separable in the coordinates $(q_1 + q_2, q_1 - q_2)$, and its solution can be expressed through elliptic functions. This result was obtained by Aizawa and Saito [1], but their names did not stick, and the system is found associated with Sawada and Kotera, and is connected with soliton theory [2–4].

Modern theories of integrable systems introduce the fundamental concept of the *soliton* to study completely integrable systems having an infinite number of degrees of freedom [5]. Examples of soliton equations are the Korteweg–de Vries (KdV) equation [6], the nonlinear Schrödinger (NLS) equation [7], and the Sine–Gordon (SG) equation [8]. Generally, soliton solutions represent stable solutions of nonlinear PDEs, where nonlinearity and dispersion are balanced [9]. With reference to the scope of our work, we report two relevant results. A method developed by Blaszk [10] constructs Lagrangian and Hamiltonian functions for stationary flows of some well-known soliton equations, including Sawada–Kotera hierarchies. In [11], a new parametrization for higher-order nonlinear equations is applied to stationary flows of soliton solutions. Interestingly, the dynamics of such systems comes out in the form of Newton equations of motion.



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The Sawada–Kotera Lagrangian system is a special case of the following family of Lagrangian functions, depending on the parameter $b \in \mathbb{R}$:

$$L_b = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \left(\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{b}{3} q_2^3 \right). \tag{3}$$

See, for instance, the larger family of Hamiltonian functions marked as (1) in [12] (p. 277), which we restricted with the choice $\omega_1 = \omega_2 = a = 1$. Since these Lagrangians are autonomous, the associated Lagrange equations have the first integral of energy

$$E_b = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \left(\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{b}{3} q_2^3 \right). \tag{4}$$

The special case of $b = 1$ is the influential Hénon–Heiles system that was introduced in [13] in order to model a Newtonian axially symmetric galactic system. Such a model has no analytic first integral independent of energy as proved by Ito [14], and its chaotic dynamic behavior has been extensively studied [15–17]. Note that Wojciechowski calls after Hénon–Heiles the whole family in his formula (1) quoted above.

By contrast, two other members of the L_b family are known to be integrable. One is the case $b = -6$ studied by Dorizzi, Grammaticos, and Ramani [18], and independently integrated by Wojciechowski [12].

The case $b = -1$ is precisely the Sawada–Kotera system, which is integrable too as we mentioned above. In the same spirit of solitons, the case $b = -6$ is called Korteweg–de Vries.

Let us mention a recent paper by Sottocornola [19], which deals with the separation of variables for seven integrable systems related to Hénon–Heiles, and presents some open questions.

2. Main Result

Theorem 1. *The two smooth Lagrangians in two degrees of freedom*

$$L = \frac{1}{2}(q_1^2 + q_2^2) - U(q_1, q_2), \quad \tilde{L} = \dot{q}_1 \dot{q}_2 - \tilde{U}(q_1, q_2) \tag{5}$$

have the same Lagrange equations and, hence, share the two energy first integrals

$$E = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + U(q_1, q_2), \quad \tilde{E} = \dot{q}_1 \dot{q}_2 + \tilde{U}(q_1, q_2), \tag{6}$$

if and only if there exist two smooth functions f, g of one variable such that

$$U(q_1, q_2) = f(q_1 + q_2) + g(q_1 - q_2), \quad \tilde{U}(q_1, q_2) = f(q_1 + q_2) - g(q_1 - q_2). \tag{7}$$

If this happens, the change in variables $(x, y) = (q_1 + q_2, q_1 - q_2)$ separates the Lagrange equations into

$$\ddot{x} = -2f'(x), \quad \ddot{y} = -2g'(y). \tag{8}$$

In Equation (8), each of the two separated differential equations has its own energy as a conserved quantity, which helps when solving with numerical methods [20].

3. Two Lagrangians for the Sawada–Kotera Equations

Consider again the Sawada–Kotera cubic Lagrangian

$$L_{SK} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \left(\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 + \frac{1}{3} q_2^3 \right). \tag{9}$$

The associated Sawada–Kotera Lagrange equations are

$$\ddot{q}_1 = -q_1(1 + 2q_2), \quad \ddot{q}_2 = -q_2 - q_1^2 - q_2^2. \tag{10}$$

The energy first integral is

$$E_{\text{SK}} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \left(\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 + \frac{1}{3}q_2^3 \right). \quad (11)$$

and there is one more first integral, which is quadratic in the velocities

$$K_{\text{SK}} = \dot{q}_1 \dot{q}_2 + \left(q_1 q_2 + \frac{1}{3}q_1^3 + q_1 q_2^2 \right). \quad (12)$$

Let us change a sign in the formula for K_{SK} and make it into a second Lagrangian function

$$\tilde{L} = \dot{q}_1 \dot{q}_2 - \left(q_1 q_2 + \frac{1}{3}q_1^3 + q_1 q_2^2 \right). \quad (13)$$

It happens that

$$\partial_{q_1} L_{\text{SK}} - \frac{d}{dt} \partial_{\dot{q}_1} L_{\text{SK}} = \partial_{q_2} \tilde{L} - \frac{d}{dt} \partial_{\dot{q}_2} \tilde{L}, \quad (14)$$

$$\partial_{q_2} L_{\text{SK}} - \frac{d}{dt} \partial_{\dot{q}_2} L_{\text{SK}} = \partial_{q_1} \tilde{L} - \frac{d}{dt} \partial_{\dot{q}_1} \tilde{L}, \quad (15)$$

so that the Euler–Lagrange equation for the two system are exactly the same:

$$\ddot{q}_2 = -q_2 - q_1^2 - q_2^2, \quad \ddot{q}_1 = -q_1(1 + 2q_2). \quad (16)$$

The first integral of energy for \tilde{L} happens to coincide with K_{SK} of (12) above

$$\tilde{E} = \dot{q} \cdot \frac{\partial \tilde{L}}{\partial \dot{q}} - \tilde{L} = \dot{q}_1 \dot{q}_2 + q_1 q_2 + \frac{1}{3}q_1^3 + q_1 q_2^2 = K_{\text{SK}} \quad (17)$$

(the central dot is the scalar product). The relationship between L_{SK} and \tilde{L} is very close.

4. Proof of the Main Theorem

Suppose we have a Lagrangian of the form

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - U(q_1, q_2), \quad (18)$$

where the potential U is smooth but it is not necessarily a polynomial function. The associated Euler–Lagrange equations are

$$\ddot{q}_1 = -\frac{\partial U}{\partial q_1}(q_1, q_2), \quad \ddot{q}_2 = -\frac{\partial U}{\partial q_2}(q_1, q_2). \quad (19)$$

Inspired by the Sawada–Kotera results of the previous Section, let us consider a second Lagrangian of the form

$$\tilde{L} = \dot{q}_1 \dot{q}_2 - \tilde{U}(q_1, q_2), \quad (20)$$

whose equations of motions are

$$\ddot{q}_2 = -\frac{\partial \tilde{U}}{\partial q_1}(q_1, q_2), \quad \ddot{q}_1 = -\frac{\partial \tilde{U}}{\partial q_2}(q_1, q_2). \quad (21)$$

Let us impose that the equations (19) coincide with those in (21):

$$\frac{\partial U}{\partial q_1} \equiv \frac{\partial \tilde{U}}{\partial q_2}, \quad \frac{\partial U}{\partial q_2} \equiv \frac{\partial \tilde{U}}{\partial q_1}. \quad (22)$$

We claim that this occurs if and only if

$$U(q_1, q_2) = f(q_1 + q_2) + g(q_1 - q_2), \quad (23)$$

$$\tilde{U}(q_1, q_2) = f(q_1 + q_2) - g(q_1 - q_2), \quad (24)$$

for some smooth functions f, g .

To prove this, let us make a change in the dependent variables through a 45° rotation in the (q_1, q_2) plane, introducing new variables r_1, r_2, V, \tilde{V} :

$$V(r_1, r_2) := U\left(\frac{r_1 + r_2}{\sqrt{2}}, \frac{r_2 - r_1}{\sqrt{2}}\right), \quad (25)$$

$$\tilde{V}(r_1, r_2) := \tilde{U}\left(\frac{r_1 + r_2}{\sqrt{2}}, \frac{r_2 - r_1}{\sqrt{2}}\right). \quad (26)$$

that is to say,

$$U(q_1, q_2) = V\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}\right), \quad (27)$$

$$\tilde{U}(q_1, q_2) = \tilde{V}\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}\right), \quad (28)$$

In terms of the new variables, the differential Equation (22) becomes

$$\frac{\partial(\tilde{V} + V)}{\partial r_2} = 0, \quad \frac{\partial(\tilde{V} - V)}{\partial r_1} = 0, \quad (29)$$

which means that there exist one-variable functions φ, ψ such that

$$\tilde{V} + V = 2\varphi(r_1), \quad \tilde{V} - V = -2\psi(r_2), \quad (30)$$

that is,

$$\tilde{V}(r_1, r_2) = \varphi(r_1) - \psi(r_2), \quad V(r_1, r_2) = \varphi(r_1) + \psi(r_2), \quad (31)$$

which are equivalent to (23).

The Euler–Lagrange Equation (19) has the first integral of energy

$$E = \dot{q}_1 \partial_{\dot{q}_1} L + \dot{q}_2 \partial_{\dot{q}_2} L - L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + U(q_1, q_2), \quad (32)$$

while the first integral of energy for (21) is

$$\tilde{E} = \dot{q}_1 \partial_{\dot{q}_1} \tilde{L} + \dot{q}_2 \partial_{\dot{q}_2} \tilde{L} - \tilde{L} = \dot{q}_1 \dot{q}_2 + \tilde{U}(q_1, q_2). \quad (33)$$

Whenever conditions (22) hold, the Euler–Lagrange equations coincide, and E and \tilde{E} are the first integrals for both. In this case, in terms of the new variables $(x, y) = (q_1 + q_2, q_1 - q_2)$ (just a little simpler than r_1, r_2 above), the Lagrangian function L becomes

$$L = \frac{1}{4}\dot{x}^2 - f(x) + \frac{1}{4}\dot{y}^2 - g(y), \quad (34)$$

for which the Euler–Lagrange equations are separated:

$$\ddot{x} = -2f'(x), \quad \ddot{y} = -2g'(y). \quad (35)$$

Similarly, the Lagrangian \tilde{L} becomes

$$\tilde{L} = \frac{1}{4}\dot{x}^2 - f(x) - \left(\frac{1}{4}\dot{y}^2 - g(y)\right). \quad (36)$$

Given a Lagrangian function $L(t, q, \dot{q})$, $q, \dot{q} \in \mathbb{R}^n$, it is easy to build other Lagrangians which have the same motions: simply take an arbitrary smooth scalar function $G(t, q)$, and define

$$\mathcal{L}(t, q, \dot{q}) := L(t, q, \dot{q}) + \frac{\partial G}{\partial t}(t, q) + \frac{\partial G}{\partial q}(t, q) \cdot \dot{q}. \tag{37}$$

Then we say that L and \mathcal{L} are related by a *gauge transform*. Our Lagrangian functions L in (18) and \tilde{L} in (20) are *not* related by a gauge transform, because their difference is quadratic in the velocities.

5. Examples

Here are some some sample systems that are covered by our results.

Example 1. (Recovering Sawada–Kotera). The functions

$$f(x) = \frac{1}{4}x^2 + \frac{1}{6}x^3, \quad g(y) = \frac{1}{4}y^2 - \frac{1}{6}y^3, \tag{38}$$

give the Sawada–Kotera U, \tilde{U} :

$$U(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + q_1^2q_2 + \frac{1}{3}q_2^3 = f(q_1 + q_2) + g(q_1 - q_2), \tag{39}$$

$$\tilde{U}(q_1, q_2) = q_1q_2 + \frac{1}{3}q_1^3 + q_1q_2^2 = f(q_1 + q_2) - g(q_1 - q_2). \tag{40}$$

Hence, from (19), the separated Euler–Lagrange equations for $x = q_1 + q_2, y = q_1 - q_2$ are

$$\dot{x} = -x - x^2, \quad \dot{y} = -y + y^2. \tag{41}$$

Example 2. The class of system to which our result applies is not limited to the polynomial potential. Let us consider the following Lagrangian, which is a variation on Calogero’s potential [21] for n -bodies constrained on a line with inversely quadratic pair potentials:

$$L(q_1, q_2) = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \left(\frac{\alpha_f}{(q_2 + q_1)^2} + \frac{\alpha_g}{(q_2 - q_1)^2} \right). \tag{42}$$

The equations of motion are

$$\ddot{q}_1 = \frac{2\alpha_f}{(q_2 + q_1)^3} - \frac{2\alpha_g}{(q_2 - q_1)^3}, \quad \ddot{q}_2 = \frac{2\alpha_f}{(q_2 + q_1)^3} + \frac{2\alpha_g}{(q_2 - q_1)^3}. \tag{43}$$

The functions

$$f(x) = \frac{\alpha_f}{x^2}, \quad g(y) = \frac{\alpha_g}{y^2}, \tag{44}$$

give the potential in (42) and satisfy the hypotheses of Theorem 1. Hence, from (19), the separated Euler–Lagrange equations are

$$\ddot{x} = \frac{4\alpha_f}{x^3}, \quad \ddot{y} = \frac{4\alpha_g}{y^3}, \tag{45}$$

that have simple solutions.

Incidentally, to this system, we can apply the following result ([22] Section 3): for general Lagrangian functions $\frac{1}{2}m\|\dot{q}\|^2 - U(q)$, $q \in \mathbb{R}^n$, with U homogeneous of degree -2 , we have that for each motion, $\|q(t)\|^2 = \frac{2}{m}(Et^2 + It + J)$, where E is the energy and I, J are constants depending on the motion. In the present case, $m = 1$ and the relation becomes

$$q_1(t)^2 + q_2(t)^2 = 2(Et^2 + It + J), \tag{46}$$

$$x(t)^2 = 2(E_1t^2 + I_1t + J_1), \quad y(t)^2 = 2(E_2t^2 + I_2t + J_2). \tag{47}$$

Example 3. One more nonpolynomial system is this Lagrangian:

$$L(q_1, q_2) = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_2 + q_1)^2 + \cos(q_1 - q_2), \tag{48}$$

which has infinitely many potential wells along the line $q_1 + q_2 = 0$. The separated system with $x = q_1 + q_2, y = q_1 - q_2$ is

$$\dot{x} = -2x, \quad \dot{y} = -2 \sin y. \tag{49}$$

The motion with $x(0) = y(0) = q_1(0) = q_2(0) = 0, \dot{x}(0) = \dot{y}(0) = \dot{q}_1(0) = 2\sqrt{2} + 1/1000, \dot{q}_2(0) = 0$ visits all the potential wells in turn (Figure 1).

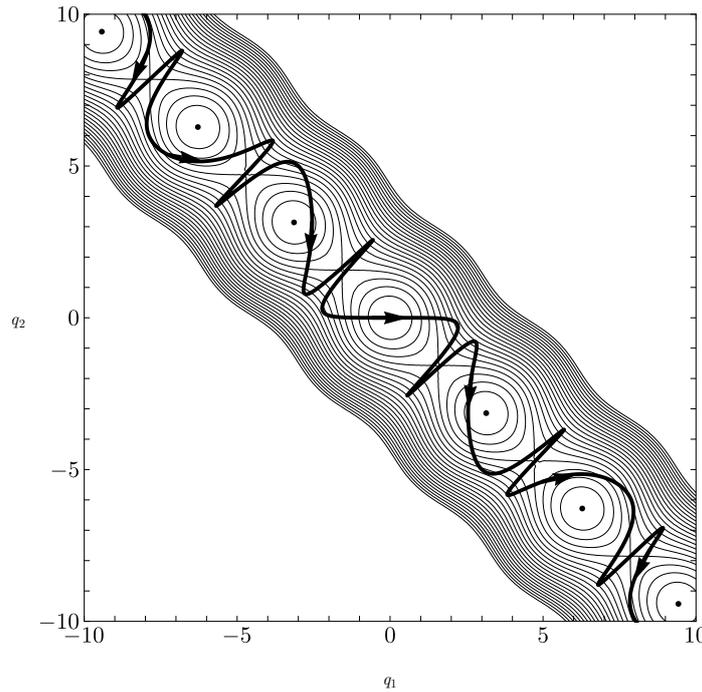


Figure 1. The trajectory of a motion of system (48), over the backdrop of the level curves and the minimum points of the potential. This kind of dynamics is not possible with polynomial potentials. The picture was made with Wolfram Mathematica.

6. Conclusions

A possible direction for future research could be a search for a new soliton equation whose stationary flows are separable in the sense described above. We would be very glad if our theory could lead to experimental and numerical analysis work too.

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Abbreviations

The following abbreviations are used in this manuscript:

PDE Partial Differential Equation

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