foundations

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On the Semi-Local Convergence of a Jarratt-Type Family Schemes for Solving Equations

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Abstract: We study semi-local convergence of two-step Jarratt-type method for solving nonlinear equations under the classical Lipschitz conditions for first-order derivatives. To develop a convergence analysis we use the approach of restricted convergence regions in combination to majorizing scalar sequences and our technique of recurrent functions. Finally, the numerical example is given.

Keywords: nonlinear equation; Banach space; semi-local convergence; Jarratt-type scheme

Citation: Argyros, C.I.; Argyros, I.K.; Shakhno, S.; Yarmola, H. On the Semi-Local Convergence of a Jarratt-Type Family Schemes for Solving Equations. *Foundations* **2022**, *2*, 234–241. https://doi.org/10.3390/foundations2010017

Academic Editor: Sotiris K. Ntouyas

Received: 24 January 2022
Accepted: 15 February 2022
Published: 17 February 2022

1. Introduction

Let us consider an equation

$$F(x) = 0, \tag{1}$$

where $F : D \subset X \to Y$ is a nonlinear Fréchet-differentiable operator, X and Y are Banach spaces, D is an open convex subset of X. To find the approximate solution $x^* \in D$ of (1) very often the Newton method is used [1,2]:

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k),\ k = 0, 1, \ldots, \tag{2}$$

The method (2) has quadratic order of convergence. To increase the convergence order multi-step schemes had been developed [3–12]. Some of them are based on Newton schemes. Such algorithms require more evaluations of function and its derivatives per iteration. For example, Jarratt [11] examined a fourth order algorithm which required to compute one function and two derivative per iteration. Sharma and Arora [11] studied forth and six order Jarratt-like methods, which use one and two function, respectively, two derivatives and one matrix inversion per iteration. Jarratt-, King-, Ostrowski-type methods contain real parameters. The order of convergence depends on the values of these parameters.

In this article, we consider Jarratt-type scheme

$$\begin{aligned} y_k &= x_k - \alpha F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \gamma A_k^{-1}B_k(y_k - x_k),\ k = 0, 1, \ldots, \end{aligned} \tag{3}$$

where α, β, γ are nonzero scalar parameters, $B_k = F'(x_k)^{-1}(F'(y_k) - F'(x_k))$ and $A_k = I + \beta B_k$. Similar scheme was proposed in [11] and local convergence was studied. In this article, we develop a semi-local convergence analysis of method (3) under classical Lipschitz conditions only for first-order derivatives. The results can certainly be extended further along the same lines if instead of the Lipschitz condition we use the Hölderian one.

2. Majorizing Sequence

Let L_0, L and μ be positive parameters. We shall show in Section 3 that scalar sequence $\{t_k\}$ defined for $k = 0, 1, 2, \ldots$ by

$$
\begin{aligned}
t_0 &= 0,\ s_0 = \mu, \\
t_{k+1} &= s_k + \frac{|\gamma| L (s_k - t_k)^2}{1 - (L_0 t_k + |\beta| L (s_k - t_k))}, \\
s_{k+1} &= t_{k+1} + \frac{|\alpha| \left[L(t_{k+1} - t_k)^2 + 2(1 + L_0 t_k)(t_{k+1} - s_k) + 2\left|1 - \frac{1}{\alpha}\right|(1 + L_0 t_k)(s_k - t_k) \right]}{2(1 - L_0 t_{k+1})}
\end{aligned} \tag{4}
$$

where is majorizing for scheme (3).

Next, we provide two results for the convergence of sequence (4).

Lemma 1. *Assume that for each $k = 0, 1, 2, \ldots$*

$$L_0 t_k + |\beta| L (s_k - t_k) < 1. \tag{5}$$

Then, sequence $\{t_k\}$ generated by (4) is strictly increasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound t^*.

Proof. By the definition of sequence $\{t_k\}$ and (4) the conclusions immediately follow. □

The next result uses stronger convergence conditions but easier to verify. Let us first define polynomials on the interval $[0, 1)$ by

$$p_1(t) = (L_0 + |\beta| L) t^2 + (|\gamma| - |\beta|) L t - |\gamma| L$$

and

$$p_2(t) = |\alpha| L (1 + t)^2 t - |\alpha| L (1 + t)^2 + 2 L_0 t^3.$$

By these definitions $p_1(0) = -|\gamma| L$, $p_1(1) = L_0$, $p_2(0) = -|\alpha| L$, $p_2(1) = 2L_0$. It follows that p_1 and p_2 have roots in $(0, 1)$ by the intermediate value theorem. Denote smallest such roots by δ_1 and δ_2, respectively. Let

$$a = \frac{|\gamma| L s_0}{1 - |\beta| L s_0}, \quad b = \frac{|\alpha| \left[L t_1^2 + 2(t_1 - s_0) + 2\left|1 - \frac{1}{\alpha}\right| s_0 \right]}{2(1 - L_0 t_1)},$$

$$c = \max\{a, b\}, \quad \delta_3 = \min\{\delta_1, \delta_2\},$$

and

$$\delta = \max\{\delta_1, \delta_2\}.$$

It is worth noticing that all these parameters depend on the minimal data L_0, L and μ. Then, we can show the second result on majorizing sequence for scheme (3).

Lemma 2. *Assume*

$$0 \le c \le \delta_3 \le \delta \le 1 - L_0 \mu \tag{6}$$

and

$$f(t) \le 0 \ at \ t = \delta \tag{7}$$

or

$$g(t) \le 0 \; at \; t = \delta, \tag{8}$$

where $f(t) = t|\alpha| + |\alpha - 1| + \frac{L_0 \mu t}{1-t} - t$ *and* $g(t) = |\alpha| L\mu(1+t)^2 t + 2t|\alpha| + 2|\alpha - 1| + 2L_0\mu t(1 + t + t^2) - 2t$. *Then, the sequence* $\{t_k\}$ *converges to* t^*.

Proof. Induction shall be used to show
$(H_k^{(1)})$:

$$0 \le \frac{|\gamma| L(s_k - t_k)}{1 - (L_0 t_k + |\beta| L(s_k - t_k))} \le \delta;$$

$(H_k^{(2)})$:

$$0 \le \frac{|\alpha| \left[L(t_{k+1} - t_k)^2 + 2(1 + L_0 t_k)(t_{k+1} - s_k) + 2\left|1 - \frac{1}{\alpha}\right|(1 + L_0 t_k)(s_k - t_k) \right]}{2(1 - L_0 t_{k+1})} \le \delta(s_k - t_k);$$

$(H_k^{(3)})$:

$$t_k \le s_k \le t_{k+1}.$$

These items are true for $k = 0$ by the definition of sequence $\{t_k\}$ and (6). Then, it also follows

$$0 \le s_0 - t_0 \le \delta^0 \mu, \quad 0 \le s_1 - t_1 \le \delta\mu, t_1 - s_0 \le \delta\mu \; and \; t_1 \le \frac{(1 - \delta^2)}{1 - \delta}\mu.$$

Assume

$$0 \le s_k - t_k \le \delta^k \mu, \quad and \; t_k \le \frac{(1 - \delta^{k+1})}{1 - \delta}\mu.$$

Evidently, $(H_k^{(1)})$ holds provided

$$|\gamma| L\delta^k \mu + L_0 \delta \frac{(1 - \delta^{k+1})\mu}{1 - \delta} + |\beta| L\delta^{k+1}\mu - \delta \le 0.$$

This estimate (7) holds if

$$h_k^{(1)}(t) \le 0 \; at \; t = \delta_1. \tag{9}$$

where recurrent functions are defined on $[0, 1)$ by

$$h_k^{(1)}(t) = |\gamma| L t^{k-1}\mu + L_0(1 + t + \ldots + t^k)\mu + |\beta| L t^k \mu - 1. \tag{10}$$

By this definition the following relationship between two consecutive polynomials can be found:

$$\begin{aligned} h_{k+1}^{(1)}(t) &= h_{k+1}^{(1)}(t) - h_k^{(1)}(t) + h_k^{(1)}(t) \\ &= |\gamma| L t^k \mu + L_0(1 + t + \ldots + t^{k+1})\mu + |\beta| L t^{k+1}\mu - 1 + h_k^{(1)}(t) \\ &\quad -|\gamma| L t^{k-1}\mu - L_0(1 + t + \ldots + t^k)\mu - |\beta| L t^k \mu + 1 \\ &= h_k^{(1)}(t) + [|\gamma| L t^k \mu - |\gamma| L t^{k-1}\mu + L_0 t^{k+1}\mu + |\beta| L t^{k+1}\mu - |\beta| L t^k \mu] \\ &= h_k^{(1)}(t) + p_1(t) t^{k-1}\mu. \end{aligned} \tag{11}$$

In particular, (11) gives

$$h_{k+1}^{(1)}(\delta_1) = h_k^{(1)}(\delta_1). \tag{12}$$

Define function on the interval $[0,1)$ by

$$h_{\infty}^{(1)}(t) = \lim_{k\to\infty} h_k^{(1)}(t). \tag{13}$$

It follows from (10) and (13) that

$$h_{\infty}^{(1)}(t) = \frac{L_0\mu}{1-t} - 1.$$

So, (9) holds if

$$\frac{L_0\mu}{1-t} - 1 \le 0 \; at \; t = \delta_1, \tag{14}$$

which is true by (6).

Similarly, $(H_k^{(2)})$ holds if we show instead

$$0 \le \frac{|\alpha|L(1+\delta)^2\delta^k\mu + 2\delta|\alpha| + 2|\alpha-1|}{2\Big(1 - L_0\Big(\frac{(1-\delta^{k+2})\mu}{1-\delta}\Big)\Big)} \le \delta, \tag{15}$$

where we also used

$$0 \le t_{k+1} - t_k = (t_{k+1} - s_k) + (s_k - t_k) \le (1+\delta)(s_k - t_k)$$

or if

$$h_k^{(2)}(t) \le 0 \; at \; t = \delta_2, \tag{16}$$

where polynomials $h_k^{(2)}$ are defined on the interval $[0,1)$ by

$$h_k^{(2)}(t) = |\alpha|L(1+t)^2 t^k\mu + 2t|\alpha| + 2|\alpha-1| + 2L_0t(1+t+\ldots+t^{k+1})\mu - 2t. \tag{17}$$

By this definition one obtains

$$\begin{aligned} h_{k+1}^{(2)}(t) &= h_{k+1}^{(2)}(t) + h_k^{(2)}(t) - h_k^{(2)}(t) \\ &= |\alpha|L(1+t)^2t^{k+1}\mu + 2t|\alpha| + 2|\alpha-1| + 2L_0t(1+t+\ldots+t^{k+2})\mu - 2t \\ &\quad + h_k^{(2)}(t) - |\alpha|L(1+t)^2t^k\mu - 2t|\alpha| - 2|\alpha-1| - 2L_0t(1+t+\ldots+t^{k+1})\mu + 2t \\ &= h_k^{(2)}(t) + p_2(t)t^k\mu, \end{aligned} \tag{18}$$

so,

$$h_{k+1}^{(2)}(\delta_2) = h_k^{(2)}(\delta_2). \tag{19}$$

Define function

$$h_{\infty}^{(2)}(t) = \lim_{k\to\infty} h_k^{(2)}(t),$$

so by this definition and (17)

$$h_{\infty}^{(2)}(t) = 2t|\alpha| + 2|\alpha-1| + \frac{2L_0t\mu}{1-t} - 2t,$$

so (19) holds if

$$h_{\infty}^{(2)}(\delta) \le 0, \tag{20}$$

which is true by (7).

If (8) holds instead of (7) then (16) holds if

$$h_1^{(2)}(t) = g(t) \leq 0.$$

It follows that sequence $\{t_k\}$ is increasing an bounded from above by $\frac{\mu}{1-\delta}$ and such it converge to t^*. □

3. Semi-Local Convergence

The conditions (A) shall be used. Assume:

(A1) There exist $x_0 \in D$ and $\mu_0 > 0$ such that $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq \mu_0$, $\mu = \mu_0|\alpha|$.
(A2) $\|F'(x_0)^{-1}(F'(z) - F'(x_0))\| \leq L_0\|z - x_0\|$ for each $z \in D$ and some $L_0 > 0$. Let $D_0 = U(x_0, \frac{1}{L_0}) \cap D$.
(A3) $\|F'(x_0)^{-1}(F'(u) - F'(v))\| \leq L\|u - v\|$ for each $u, v \in D_0$ and some $L > 0$.
(A4) Conditions of Lemma 1 or Lemma 2 hold.
(A5) $U[x_0, t^*] \subset D$.

The semi-local convergence analysis is based on conditions (A).

Theorem 1. *Assume conditions (A) hold. Then, sequence $\{x_k\}$ generated by scheme (3) exists in $U(x_0, t^*)$, stays in $U(x_0, t^*)$ for each $k = 0, 1, \ldots$ and converges to a solution $x^* \in U[x_0, t^*]$ of equation $F(x) = 0$.*

Proof. Items

$$\|y_m - x_m\| \leq s_m - t_m \tag{21}$$

and

$$\|x_{m+1} - y_m\| \leq t_{m+1} - s_m \tag{22}$$

shall be proved using induction.

By (A1) one has

$$\|y_0 - x_0\| = |\alpha|\|F'(x_0)^{-1}F(x_0)\| \leq |\alpha|\mu_0 = \mu = s_0 - t_0 = s_0 < t^*,$$

so (21) holds for $m = 0$, and $y_0 \in U(x_0, t^*)$. Suppose it holds for all values of m smaller or equal to $k - 1$ Let $v \in U(x_0, t^*)$. Then, in view of (A1) and (A2) one obtains

$$\|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq L_0\|v - x_0\| \leq L_0 t^* < 1,$$

leading to $F'(v)^{-1} \in L(Y, X)$ and

$$\|F'(v)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|v - x_0\|} \tag{23}$$

by the Lemma on invertible linear operators due to Banach [2,13]. Then, one has

$$\|B_m\| \leq \|F'(x_m)^{-1}F'(x_0)\|\|F'(x_0)^{-1}(F'(y_m) - F'(x_m))\| \leq \frac{L\|y_m - x_m\|}{1 - L_0\|x_m - x_0\|}. \tag{24}$$

and

$$\|\beta B_m\| \leq \frac{|\beta|L\|y_m - x_m\|}{1 - L_0\|x_m - x_0\|} \leq \frac{|\beta|L(s_m - t_m)}{1 - L_0 t_m} < 1,$$

so

$$\|A_m\| \le \frac{1}{1-\frac{|\beta|L(s_m-t_m)}{1-L_0t_m}},$$

so

$$\begin{aligned}\|x_{m+1}-y_m\| &\le |\gamma|\|A_m^{-1}\|\|B_m\|\|y_m-x_m\| \\ &\le |\gamma|\frac{1-L_0t_m}{1-L_0t_m-|\beta|L(s_m-t_m)}\frac{L(s_m-t_m)^2}{1-L_0t_m}=t_{m+1}-s_m. \end{aligned} \tag{25}$$

By scheme (3), one can write

$$\begin{aligned}F(x_{m+1}) &= F(x_{m+1})-F(x_m)+F(x_m) \\ &= F(x_{m+1})-F(x_m)-F'(x_m)(x_{m+1}-x_m)+F'(x_m)(x_{m+1}-x_m) \\ &\quad -\frac{1}{\alpha}F'(x_m)(y_m-x_m)+F'(x_m)(y_m-x_m)-F'(x_m)(y_m-x_m) \\ &= \int_0^1\left[F'(x_m+\theta(x_{m+1}-x_m))-F'(x_m)\right]d\theta(x_{m+1}-x_m) \\ &\quad +F'(x_m)(x_{m+1}-y_m)-F'(x_0)(x_{m+1}-y_m)+F'(x_0)(x_{m+1}-y_m) \\ &\quad +\left(1-\frac{1}{\alpha}\right)(F'(x_m)-F'(x_0)+F'(x_0))(y_m-x_m),\end{aligned}$$

so

$$\begin{aligned}|\alpha|\|F'(x_0)^{-1}F(x_{m+1})\| &= |\alpha|\left[\frac{L}{2}\|x_{m+1}-x_m\|+(1+L_0\|x_m-x_0\|)\|x_{m+1}-y_m\|\right. \\ &\quad \left.+\left|1-\frac{1}{\alpha}\right|(1+L_0\|x_m-x_0\|)\|y_m-x_m\|\right] \\ &\le |\alpha|\left[\frac{L}{2}(t_{m+1}-t_m)^2+(1+L_0t_m)(t_{m+1}-s_m)\right. \\ &\quad \left.+\left|1-\frac{1}{\alpha}\right|(1+L_0t_m)(s_m-t_m)\right].\end{aligned} \tag{26}$$

It them follows from (3), (4), (23) for $v=x_{m+1}$, and (26) that

$$\|y_{m+1}-x_{m+1}\| \le \|F'(x_{m+1})^{-1}F'(x_0)\||\alpha|\|F'(x_0)^{-1}F(x_{m+1})\| \le s_{m+1}-t_{m+1}, \tag{27}$$

where we also used. The following have also be used

$$\|x_{m+1}-x_0\| \le \|x_{m+1}-y_m\|+\|y_m-x_0\| \le t_{m+1}-s_m+s_m-t_0=t_{m+1}<t_*$$

so $x_{m+1}\in U(x_0,t^*)$. Notice also that

$$\|y_{m+1}-x_0\| \le \|y_{m+1}-x_{m+1}\|+\|x_{m+1}-x_0\| \le s_{m+1}-t_{m+1}+t_{m+1}-t_0=s_{m+1}<t_*,$$

so $y_{m+1}\in U(x_0,t^*)$.

The induction for (21) and (22) is completed. It follows that sequence $\{x_k\}$ is Cauchy in a Banach space X and, as such, it converges to some $x^*\in U[x_0,t^*]$ (since $U[x_0,t^*]$ is a closed set).

By letting $m\to\infty$ in (26) and using the continuity of F we conclude $F(x^*)=0$. □

The parameters $\frac{1}{L_0}$ or $\frac{\mu}{1-\delta}$ given in closed form can replace t^* in Theorem 1.

A uniqueness of the solution result follows next.

Proposition 1. *Assume:*

(i) *The point x^* is a solution of equation $F(x) = 0$ in D;*
(ii) *Condition (A2) holds.*

Then, the only solution of Equation (1) in the region D_0 is x^*.

Proof. Let $y \in D$ with $F(y) = 0$. Set $M = \int_0^1 F'(y + \theta(x^* - y))d\theta$. Then, in view of (A2) one obtains

$$\|F'(x_0)^{-1}(M - F'(x_0))\| \leq \frac{L_0}{2}[\|x^* - x_0\| + \|y - x_0\|] < \frac{L_0}{2}\frac{2}{L_0} = 1,$$

so $y = x^*$ is obtained from the invertibility of M and $0 = F(y) - F(x^*) = M(y - x^*)$. □

4. Numerical Example

Let us consider the nonlinear equation

$$F(x) = x^3 - q = 0,$$

where a function F is defined on $D = U(x_0, 1 - q)$ and $q \in (0, 1)$. Let $x_0 = 1$. Then, we obtain

$$\mu_0 = \frac{1 - q}{3},\ L_0 = 3 - q,\ L = 2\min\left\{2 - q,\ 1 + \frac{1}{L_0}\right\}.$$

If we choose $q = 0.85$ then

$$x^* = \sqrt[3]{q} \approx 0.947268237185910,\ D = D_0 = (0.85,\ 1.15),\ \mu_0 = 0.05,\ L_0 = 2.15,\ L = 2.3.$$

Now, verify conditions of Lemma 1 and Theorem 1 for $\alpha = 1, \beta = \gamma = 0.5$. Majorizing sequences

$$\begin{aligned} t_k &= \{0,\ 0.05305039787798409,\ 0.06021265690968219,\ 0.06036488186993019, \\ &\quad 0.06036495221495131,\ 0.06036495221496634\}, \\ s_k &= \{0.05,\ 0.060146682761 79737,\ 0.06036485126494786,\ 0.06036495221494478, \\ &\quad 0.06036495221496634,\ 0.06036495221496634\} \end{aligned}$$

converge to $t_* < \dfrac{1}{L_0}$. Moreover, condition (5) holds for each k.

Table 1 gives error estimates (21) and (22). The solution x^* is obtained at three iterations for $\varepsilon = 10^{-10}$. Therefore, conditions of Theorem 1 are satisfied and $\{x_k\}$ converge to $x^* \in U[x_0, t^*]$.

Table 1. Error estimates.

k	x_{k+1}	$\lvert y_k - x_k\rvert$	$s_k - t_k$	$\lvert x_{k+1} - y_k\rvert$	$t_{k+1} - s_k$
0	0.947437582128778	5.0000×10^{-2}	5.0000×10^{-2}	2.5624×10^{-3}	3.0504×10^{-3}
1	0.947268237192221	1.6931×10^{-4}	7.0963×10^{-3}	3.0261×10^{-8}	6.5974×10^{-5}
2	0.947268237185910	6.3117×10^{-12}	1.5219×10^{-4}	0	3.0605×10^{-8}

5. Conclusions

Method (3) has been used extensively for solving equations. The local convergence analysis of method (3) has been given under various conditions. However, the semi-local which is more interesting has not been given. That is why we presented it in this study using majorizing sequences, Lipschitz conditions, and recurrent functions. The results can certainly be extended further along the same lines if instead of the Lipschitz condition we use the Hölderian one. Our technique is very general, so it can be used to provide results on the semi-local convergence of other higher-order convergent methods along the same lines. The theoretical results are also justified by examples.

Author Contributions: Conceptualization I.K.A.; Methodology I.K.A.; Investigation I.K.A., C.I.A., S.S., and H.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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