foundations

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On the Local Convergence of a $(p+1)$-Step Method of Order $2p+1$ for Solving Equations

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Abstract: The local convergence of a generalized $(p+1)$-step iterative method of order $2p+1$ is established in order to estimate the locally unique solutions of nonlinear equations in the Banach spaces. In earlier studies, convergence analysis for the given iterative method was carried out while assuming the existence of certain higher-order derivatives. In contrast to this approach, the convergence analysis is carried out in the present study by considering the hypothesis only on the first-order Fréchet derivatives. This study further provides an estimate of convergence radius and bounds of the error for the considered method. Such estimates were not provided in earlier studies. In view of this, the applicability of the given method clearly seems to be extended over the wider class of functions or problems. Moreover, the numerical applications are presented to verify the theoretical deductions.

Keywords: convergence analysis; iterative methods; Fréchet derivative; Banach spaces

Citation: Sharma, J.R.; Argyros, I.K.; Singh, H.; Argyros, M.I. On the Local Convergence of a $(p+1)$-Step Method of Order $2p+1$ for Solving Equations. *Foundations* **2022**, *2*, 242–250. https://doi.org/10.3390/foundations2010018

Academic Editor: Sotiris K. Ntouyas

Received: 29 January 2022
Accepted: 15 February 2022
Published: 20 February 2022

1. Introduction

Problems in applied mathematics are frequently formulated as the systems of nonlinear equations. Considering the Banach spaces, X and Y, the mathematical formulation of a given problem can be expressed in the form

$$F(x) = 0, \tag{1}$$

where $F : D \subset X \to Y$ is a Fréchet-differentiable [1] mapping, and D is an open convex set of X. The analytical solutions of the formulated nonlinear models are rather complicated to obtain, but on the contrary, the iterative methods (see [2,3]) provide the numerical solution up to the desired accuracy. Numerous iterative methods have been presented (see, for example, [2,4–6], and references therein) over the years for this purpose.

One of the crucial components for the development of iterative methods is the analysis of their convergence behavior. The most common approach to estimate the order of convergence of an iterative method includes the Taylor's series expansions, which inherently involve higher-order derivatives ($F^{(i)},\ i = 1,2,\ldots$), and some assumptions on $F^{(i)}$. However, such assumptions limit the applicability of techniques, since most require the computation of the first-order derivative only. Consider a real valued function [7], $F : D \subseteq \mathbb{R} \to \mathbb{R},\ D = \left[-\frac{1}{2}, \frac{5}{2}\right]$, which is defined as

$$F(x) = \begin{cases} x^3 \ln(x^2) + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Consequently, we have

$$F'(x) = 3x^2 \ln(x^2) + 5x^4 - 4x^3 + 2x^2,$$
$$F''(x) = 6x \ln(x^2) + 20x^3 - 12x^2 + 10x,$$

and

$$F'''(x) = 6 \ln(x^2) + 60x^2 - 24x + 22.$$

Apparently, $F'''(x)$ is unbounded in the given interval. Therefore, the Taylor's series expansion might not be a suitable approach to study the convergence behavior of an iterative technique.

Additionally, the convergence behavior of an iterative technique is significantly affected by the selection of initial approximation in the neighborhood of the solution. It is worth noticing that the set assumptions on $F^{(i)}$ $(i = 1, 2, \ldots)$ further reduce the convergence region to a significant extent. Therefore, it is essential to enlarge the convergence domain by avoiding these additional hypotheses. In this sense, the convergence analysis of iterative techniques should include a measure of closeness of the initial estimate to the solution. In fact, many authors (see [6–16]) have adopted an appropriate methodology to establish the local or semilocal convergence analysis by considering the hypothesis of Lipschitz continuity on first-order derivatives only. Furthermore, the bounds of the error estimates and the convergence radius can be computed by defining some real functions and parameters.

In view of the above facts, we shall study the local convergence analysis of a generalized $(p + 1)$-step iterative method of order $2p + 1$, developed in [4], and which is expressed as follows:

$$\begin{aligned}
y_n^{(0)} &= x_n - aF'(x_n)^{-1}F(x_n),\\
y_n^{(1)} &= y_n^{(0)} - B_n F'(x_n)^{-1}F(x_n),\\
y_n^{(2)} &= y_n^{(1)} - g(x_n, y_n)F'(x_n)^{-1}F(y_n^{(1)}),\\
&\cdots\\
y_n^{(p-1)} &= y_n^{(p-2)} - g(x_n, y_n)F'(x_n)^{-1}F(y_n^{(p-2)}),\\
x_{n+1} = y_n^{(p)} &= y_n^{(p-1)} - g(x_n, y_n)F'(x_n)^{-1}F(y_n^{(p-1)}),
\end{aligned} \tag{2}$$

where 'a' is parameter, $y_n^{(0)} = y_n$, $B_n = \frac{1}{12}(13I - 9A_n)$, $g(x_n, y_n) = \frac{1}{2}(5I - 3A_n)$, and $A_n = F'(x_n)^{-1}F'(y_n)$. Let us note that for any $x \in D \subseteq \mathbb{R}^m$, $F'(x) : D \to \mathbb{R}^m$ is the first Fréchet derivative [1]. Clearly, the above-given technique requires the computation of derivatives of an order not more than one, but the order of convergence was proved in [4] using the assumptions of the derivatives up to order $2p + 1$. Our prime objective here is to weaken the conditions of [4], and further, to estimate the upper bounds of the convergence radius, which will definitely expand the applicability of the considered technique.

In what follows, the local convergence analysis is developed in Section 2, which includes the computation of the upper bounds of the convergence domain. Some numerical applications are given in Section 3. Section 4 contains the concluding remarks.

2. Convergence Analysis

To establish the local convergence analysis of the iterative technique (2), we define some real parameters and functions, and moreover, let the following suppositions (i–iii) hold.

(i) There exists a function $\phi_0 : [0, \infty) \to [0, \infty)$, continuous and non-decreasing, such that the equation:

$$\phi_0(t) - 1 = 0,$$

has the smallest solution $\rho \in (0, \infty)$.

(ii) There exist functions $\phi : [0, \rho) \to [0, \infty)$ and $\phi_1 : [0, \rho) \to [0, \infty)$, both continuous and non-decreasing, such that the equations:

$$\psi_0(t) - 1 = 0,$$

$$\psi_1(t) - 1 = 0,$$

have the smallest solutions $r_0, r_1 \in (0, \rho)$, respectively, where ψ_0 and ψ_1 are the functions defined on the interval $[0, \rho)$, and are expressed as

$$\psi_0(t) = \frac{\int_0^1 \phi((1-\theta)t)d\theta + |1-a| \int_0^1 \phi_1(\theta t)d\theta}{1 - \phi_0(t)},$$

$$\psi_1(t) = \frac{\int_0^1 \phi((1-\theta)t)d\theta + p(t) \int_0^1 \phi_1(\theta t)d\theta}{1 - \phi_0(t)}.$$

Here, $p : [0, \rho) \to [0, \infty)$ is defined as

$$p(t) = \frac{3(\phi_0(t) + \phi_0(\psi_0(t)t))}{4(1 - \phi_0(t))} + \frac{|3a - 2|}{3}.$$

(iii) Suppose that the equations:

$$\psi_k(t) - 1 = 0, \quad (k = 2, 3, \ldots, p),$$

have the smallest solutions $r_k \in (0, \rho)$, respectively, where ψ_k for each k is defined as

$$\psi_k(t) = q(t)\psi_{k-1}(t).$$

Here, $q : [0, \rho) \to [0, \infty)$, and further, $s : [0, \rho) \to [0, \infty)$ are defined as

$$q(t) = 1 + \frac{s(t) \int_0^1 \phi_1(\theta \psi_1(t)t)d\theta}{1 - \phi_0(t)},$$

$$s(t) = 1 + \frac{3(\phi_0(t) + \phi_0(\psi_0(t)t))}{2(1 - \phi_0(t))}.$$

Let us define

$$r = \min\{r_i\}, \quad i = 0, 1, 2, \ldots, p. \tag{3}$$

We shall show that r is the radius of convergence for the iterative method (2). Notice that, by definition of r, it follows that for all $t \in [0, r)$,

$$0 \le \phi_0(t) < 1, \tag{4}$$

$$0 \le \psi_i(t) < 1, \tag{5}$$

where $i = 0, 1, 2, \ldots, p$. Assume that $x^* \in D$. By taking x^* as center, we denote $U(x^*, r)$ as the open ball, and $U[x^*, r]$ as the closed ball, having a radius equal to 'r'. Before proceeding to the main result, it is required that the following conditions (A_1–A_4) hold:

(A_1) : The point x^* is the simple solution of Equation (1).

(A_2) : For each $x \in D$,

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le \phi_0(\|x - x^*\|).$$

Let $D_0 = D \cap U(x^*, \rho)$.

(A_3) : For each $z, w \in D_0$,

$$\|F'(x^*)^{-1}(F'(z) - F'(w))\| \le \phi(\|z - w\|),$$

and

$$\|F'(x^*)^{-1}F'(z)\| \le \phi_1(\|z - x^*\|).$$

(A_4) : $U[x^*, r] \subset D$.

Next, we present the convergence of method (2) using the conditions (A_1–A_4).

Theorem 1. *Under the conditions (A_1–A_4), and further choosing $x_0 \in U(x^*, r) - \{x^*\}$, the sequence $\{x_n\}$, generated by method (2), remains in $U(x^*, r)$ and converges to x^*.*

Proof. Let $v \in U(x^*, r)$. In view of the condition (A_2) and Equation (3), in turn we obtain

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \le \phi_0(\|v - x^*\|) < \phi_0(r) < 1. \tag{6}$$

The existence of invertible operators in Banach spaces (see [1]), together with (6), implies that $F'(v)^{-1} \in \mathcal{L}(Y, X)$, so that

$$||F'(v)^{-1}F'(x^*)|| \le \frac{1}{1 - \phi_0(||v - x^*||)}. \tag{7}$$

It follows from expression (7), for $v = x_0$, that $y_0^{(1)}, y_0^{(2)}, \ldots, y_0^{(p)}$ exist. Then, using the first sub-step of method (2) for $n = 0$,

$$y_0^{(0)} - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 - a)F'(x_0)^{-1}F(x_0). \tag{8}$$

Using (3), (A_2), (A_3), (7) (for $v = x_0$), and (8), in turn one finds that

$$\begin{aligned}\|y_0^{(0)} - x^*\| &\le \frac{\int_0^1 \phi((1-\theta)\|x_0 - x^*\|)d\theta + |1-a| \int_0^1 \phi_1(\theta(\|x_0 - x^*\|))d\theta}{1 - \phi_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \\ &\le \psi_0(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,\end{aligned} \tag{9}$$

which proves $y_0^{(0)} \in U(x^*, r)$. Furthermore, re-writing the second sub-step of (2) for $n = 0$, we have

$$\begin{aligned}y_0^{(1)} - x^* &= y_0^{(0)} - x^* - B_0F'(x_0)^{-1}F(x_0) \\ &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + ((1-a)I - B_0)F'(x_0)^{-1}F(x_0).\end{aligned} \tag{10}$$

Then, by (3), (A_2), (A_3), and (10), we obtain

$$\begin{aligned}\|y_0^{(1)} - x^*\| &\le \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| + \|(1-a)I - B_0\|\|F'(x_0)^{-1}F(x_0)\| \\ &\le \left(\frac{\int_0^1 \phi((1-\theta)\|x_0 - x^*\|)d\theta + p(\|x_0 - x^*\|)\int_0^1 \phi_1(\theta\|x_0 - x^*\|)d\theta}{1 - \phi_0(\|x_0 - x^*\|)}\right)\|x_0 - x^*\| \\ &\le \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\le \|x_0 - x^*\| < r,\end{aligned} \tag{11}$$

which proves $y_0^{(1)} \in U(x^*, r)$, where we have used the approximation

$$\begin{aligned}\|(1-a)I - B_0\| &= \left\| \frac{9}{12}F'(x_0)^{-1}(F'(y_0) - F'(x_0)) - \frac{3a-2}{3}I \right\| \\ &\le \frac{3(\phi_0(\|x_0 - x^*\|) + \phi_0(\|y_0 - x^*\|))}{4(1 - \phi_0(\|x_0 - x^*\|))} + \frac{|3a-2|}{3} \\ &\le p(\|x_0 - x^*\|).\end{aligned}$$

Similarly, by the third sub-step of method (2),

$$\begin{aligned}
\|y_0^{(2)} - x^*\| &\le \|y_0^{(1)} - x^*\| + \|g(x_0, y_0^{(0)})\| \|F'(x_0)^{-1}F(y_0^{(1)})\| \\
&\le \left(1 + \frac{s(\|x_0 - x^*\|)\int_0^1 \phi_1(\theta\|y_0^{(1)} - x^*\|)d\theta}{1 - \phi_0(\|x_0 - x^*\|)}\right)\|y_0^{(1)} - x^*\| \\
&\le q(\|x_0 - x^*\|)\psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\le \psi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\le \|x_0 - x^*\| < r,
\end{aligned} \tag{12}$$

where we have used the approximation

$$\begin{aligned}
\|g(x_0, y_0^{(0)})\| &= \left\|\frac{1}{2}(5I - 3F'(x_0)^{-1}F'(y_0^{(0)}))\right\| \\
&= \left\|I + \frac{3}{2}F'(x_0)^{-1}(F'(x_0) - F'(y_0^{(0)}))\right\| \\
&\le 1 + \frac{3}{2}\frac{\phi_0(\|x_0 - x^*\|) + \phi_0(\|y_0^{(0)} - x^*\|)}{1 - \phi_0(\|x_0 - x^*\|)} \\
&\le s(\|x_0 - x^*\|).
\end{aligned}$$

Equation (12) proves that $y_0^{(2)} \in U(x^*,\ r)$. Using similar approximations as in (12), for each $j = 3, 4, \ldots$, we have

$$\begin{aligned}
\|y_0^{(j)} - x^*\| &\le q(\|x_0 - x^*\|)\psi_{j-1}(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\le \psi_j(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\le \|x_0 - x^*\|.
\end{aligned} \tag{13}$$

Therefore, at the $(p+1)$-th step of method (2),

$$\begin{aligned}
\|x_1 - x^*\| &= \|y_0^{(p)} - x^*\| \\
&\le \psi_p(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\le \|x_0 - x^*\|.
\end{aligned}$$

which shows that $x_1 \in U(x^*,\ r)$. Now, simply replace $y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(p)}$ by $y_k^{(0)}, y_k^{(1)}, \ldots, y_k^{(p)}$ in the preceding estimations to obtain

$$\begin{aligned}
\|x_{k+1} - x^*\| &\le \|y_k^{(p)} - x^*\| \\
&\le \psi_p(\|x_k - x^*\|)\|x_k - x^*\| \\
&\le \|x_k - x^*\|.
\end{aligned}$$

Hence, $x_{k+1} \in U(x^*,\ r)$ for each $k = 1, 2, \ldots$, and moreover $\lim_{k\to\infty} x_k = x^*$. □

Proposition 1. *Assume that*

(i) The point $x^* \in U[x^*, r]$ is the simple solution of (1), and satisfies the conditions (A_1) and (A_2).
(ii) There exists $b \ge r$, such that

$$\int_0^1 \phi_0((1-\theta)b)d\theta < 1. \tag{14}$$

Set $D_1 = D \cap U[x^*, b]$. Then, x^* is the only solution of Equation (1) in the domain D_1.

Proof. Consider that $\overline{x} \in D_1$ with $F(\overline{x}) = 0$. Define $M = \int_0^1 F(\overline{x} + \theta(x^* - \overline{x}))d\theta$. Now, using the conditions (A_1), (A_2), and Equation (14), we have

$$\begin{aligned} \|F'(x^*)^{-1}(M - F'(x^*))\| &\leq \int_0^1 \phi_0((1-\theta)\|\overline{x} - x^*\|)d\theta \\ &\leq \int_0^1 \phi_0((1-\theta)b)d\theta < 1. \end{aligned}$$

So, $\overline{x} = x^*$ by the invertibility of M, since $M(\overline{x} - x^*) = F(\overline{x}) - F(x^*) = 0 - 0 = 0$. □

Remark 1. *The convergence order of method* (2) *was proved in* [4] *using the Taylor's series expansions. Instead of using these stronger conditions, the term computational order of convergence (COC)* [11] *is defined as*

$$COC = \ln\left\|\frac{x_{r+2} - x^*}{x_{r+1} - x^*}\right\| \Big/ \ln\left\|\frac{x_{r+1} - x^*}{x_r - x^*}\right\|, \quad \textit{for each } r = 0, 1, 2, \ldots.$$

Note that, to compute COC, the knowledge of the exact solution (x^*) *is required, but that may not be known explicitly. In that case, the order of convergence can be determined using the approximated computational order of convergence (ACOC)* [11], *which is expressed below*

$$ACOC = \ln\left\|\frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r}\right\| \Big/ \ln\left\|\frac{x_{r+1} - x_r}{x_r - x_{r-1}}\right\|, \quad \textit{for each } r = 1, 2, \ldots.$$

Apparently, no computation of derivative(s) is involved to determine the order of convergence of an iterative technique, either by using COC or ACOC.

Remark 2. *In view of the condition* (A_1), *and the following estimate,*

$$\begin{aligned} ||F'(x^*)^{-1}F'(x)|| =& ||F'(x^*)^{-1}(F'(x) - F'(x^*)) + I|| \\ \leq& 1 + ||F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ \leq& 1 + \phi_0(\|x - x^*\|), \end{aligned}$$

the condition $\|F'(x^*)^{-1}F'(x)\| \leq \phi_1(\|x - x^*\|)$ *can be dropped and replaced by*

$$\|F'(x^*)^{-1}F'(x)\| \leq 1 + \phi_0(\|x - x^*\|).$$

3. Numerical Results

To verify the theoretical deductions, we provide here the real parameters or functions, as well as the estimated radius of convergence, for each of the following numerical examples, in particular by taking $a = \frac{2}{3}$ and $p = 3$ in method (2).

Example 1. *Consider the following Hammerstein Equation* [8]*:*

$$x(s) = \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt, \tag{15}$$

where

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t, \end{cases}$$

is termed as the Green's function, and defined as the kernel of Equation (15), *in the domain* $[0,1] \times [0,1]$. *In particular, we observe that*

$$\left\|\int_0^1 G(s,\ t)dt\right\| \leq \frac{1}{8}.$$

By defining a mapping $F : D \subseteq C[0,1] \to C[0,1]$ *as*

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt,$$

we simply have

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\left(\frac{3}{2}x(t)^{1/2} + x(t)\right)dt.$$

In fact, $x^*(s) = 0$ *is the solution of Equation* (15)*, and moreover using* $F'(x^*(s)) = 1$*, we in turn find that*

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\right),$$

and consequently we can choose

$$\phi_0(t) = \phi(t) = \frac{1}{8}\left(\frac{3}{2}t^{1/2} + t\right), \text{ and } \phi_1(t) = 1 + \phi_0(t).$$

Finally, we obtain

$$r_0 = 1.2781,\ r_1 = 0.7917,\ r_2 = 0.3063,\ r_3 = 0.1249, \text{ and } r = 0.1249.$$

Example 2. *Next, consider an equation due to Kepler* [17]*:*

$$F(x) = x - \beta\sin(x) - K = 0,$$

where $0 \le \beta < 1, 0 \le K \le \pi$*. Different choices of values of* β *and* K *are given in* [17]*. In particular, we set* $K = 0.1$ *and* $\beta = 0.27$*. Then, we have the solution* $x^* \approx 0.13682853547099\ldots$*. Notice that*

$$F'(x) = 1 - \beta\cos(x).$$

So,

$$\begin{aligned} |F'(x^*)^{-1}(F'(x) - F'(y))| &= \frac{|\beta(\cos(x) - \cos(y))|}{|1 - \beta\cos(x^*)|} \\ &= \frac{2\beta|\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})|}{|1-\beta\cos(x^*)|} \\ &\le \frac{\beta}{|1-\beta\cos(x^*)|}|x-y|, \end{aligned}$$

and

$$|F'(x^*)^{-1}F'(x)| = \frac{|1-\beta\cos(x)|}{|1-\beta\cos(x^*)|} \le \frac{1+\beta}{|1-\beta\cos(x^*)|}.$$

Then, we can choose

$$\phi_0(t) = 0.36859\,t,\ \phi(t) = 0.36859\,t, \text{ and } \phi_1(t) = 1.73373.$$

The computed values of parameters are given by

$$r_0 = 0.7634,\ r_1 = 0.5879,\ r_2 = 0.2489,\ r_3 = 0.1081, \text{ and } r = 0.1081.$$

Example 3. *The Van der Waals Equation* [3,9] *of state for vapor is expressed as:*

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT, \tag{16}$$

where all constants, appearing in the above equation, have a physical meaning whose values can be found in [3]. *Then, we must solve the equation*

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0,$$

in V. *In particular, for* $P = 10,000$ *units and* $T = 800$ *units, the solution of the resulting equation is* $V = 36.9167\ldots$. *So, we have*

$$\phi_0(t) = 0.386121\, t,\ \phi(t) = 0.386121\, t,\ \text{and}\ \phi_1(t) = 1 + \phi_0(t).$$

and consequently, we obtain the estimates

$$r_0 = 1.0359,\ r_1 = 0.7288,\ r_2 = 0.4069,\ r_3 = 0.2419,\ \text{and}\ r = 0.2419.$$

Example 4. *Now, consider the system* [11], *which governs the motion of an object in three dimensions, and which is expressed by the following set of ordinary differential equations:*

$$\begin{aligned} &f_1'(x) - f_1(x) - 1 = 0,\\ &f_2'(y) - (e-1)y - 1 = 0,\\ &f_3'(z) - 1 = 0, \end{aligned} \tag{17}$$

where $(x, y, z) \in D = \overline{B}(0, 1)$, *and* $f_1(0) = f_2(0) = f_3(0) = 0$. *For any* $t = (x, y, z) \in D$, *the solution of the given system* (17) *is defined by the function* $F : D \to \mathbb{R}^3$, *where* $F := (f_1, f_2, f_3)$, *and*

$$F(t) = (e^x - 1,\ \frac{e-1}{2}y^2 + y,\ z)^T,$$

and therefore

$$F'(t) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, we choose

$$\phi_0(t) = (e-1)t,\ \phi(t) = e^{\frac{1}{e-1}} t,\ \text{and}\ \phi_1(t) = e^{\frac{1}{e-1}}.$$

and consequently, we obtain the estimates as

$$r_0 = 0.1544,\ r_1 = 0.1227,\ r_2 = 0.05096,\ r_3 = 0.02167,\ \text{and}\ r = 0.02167.$$

Example 5. *Lastly, we look at the example given in the introduction section. Observe that* $x^* = 1/\pi$ *is the zero of this function. In this particular problem, we can choose*

$$\phi_0(t) = 146.66290\, t,\ \phi(t) = 146.66290\, t,\ \text{and}\ \phi_1(t) = 1 + \phi_0(t).$$

Then, we obtain

$$r_0 = 2.727 \times 10^{-3},\ r_1 = 1.919 \times 10^{-3},\ r_2 = 1.071 \times 10^{-3},$$

$$r_3 = 6.369 \times 10^{-4},\ \text{and}\ r = 6.369 \times 10^{-4}.$$

4. Conclusions

A generalized $(p + 1)$-step iterative technique with a convergence order of $2p + 1$ is comprehensively analyzed for its local convergence in the Banach spaces. Assuming the conditions of the first-order derivatives only, contrary to the usual approach of using Taylor's series expansions, we establish the generalized results in order to determine the convergence region of the given technique. Consequently, the applicability of the technique is extended to a wider section of problems. Moreover, the numerical estimation of the upper bounds of the convergence radius satisfactorily favors our analysis.

Author Contributions: The contributions of all authors are similar. All authors worked together to develop the present manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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