foundations

MDPI

Article

Simpson's Type Inequalities for s-Convex Functions via a Generalized Proportional Fractional Integral

Henok Desalegn [1], Jebessa B. Mijena [2], Eze R. Nwaeze [3,*] and Tadesse Abdi [1]

[1] Department of Mathematics, Addis Ababa University, Addis Ababa P. O. Box 1176, Ethiopia; henokdesalegn@aau.edu.et (H.D.); tadesse.abdi@aau.edu.et (T.A.)
[2] Department of Mathematics, Georgia College and State University, Milledgeville, GA 31061, USA; jebessa.mijena@gcsu.edu
[3] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA
* Correspondence: enwaeze@alasu.edu

Abstract: In this paper, we give new Simpson's type integral inequalities for the class of functions whose derivatives of absolute values are s-convex via generalized proportional fractional integrals. Some results in the literature are particular cases of our results.

Keywords: Simpson's inequality; s-convex function; generalized proportional fractional integrals; Hölder's inequality

MSC: 26D15; 25D10; 26A51; 26A33

check for updates

Citation: Desalegn, H.; Mijena, J.B.; Nwaeze, E.R.; Abdi, T. Simpson's Type Inequalities for s-Convex Functions via Generalized Proportional Fractional Integral. *Foundations* **2022**, *2*, 607–616. https://doi.org/10.3390/foundations2030041

Academic Editor: Sotiris K. Ntouyas

Received: 7 June 2022
Accepted: 20 July 2022
Published: 25 July 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction and Preliminaries

Simpson's inequality is given by

$$\left|\frac{1}{3}\left[\frac{g(a)+g(b)}{2}+2g\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{(b-a)^4}{2880}\|g^4\|_\infty,$$

where $g : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\|g^{(4)}\|_\infty = \sup_{x\in(a,b)}\left|g^{(4)}\right| < \infty$. This inequality has been studied and generalized by many scholars see for instance [1–7] and references cited therein.

Definition 1. *The function $g : [0,\infty) \to \mathbb{R}$ is a convex function if*

$$g(\lambda x+(1-\lambda)y) \leq \lambda g(x)+(1-\lambda)g(y)$$

holds for ever $x,y \in [0,\infty]$ and $\lambda \in [0,1]$. If the inequality in Definition 1 is reversed, then g is a concave function.

Definition 2. *The function $g : [0,\infty) \to \mathbb{R}$ is s-convex function (in the second sense) if*

$$g(\lambda x+(1-\lambda)y) \leq \lambda^s g(x)+(1-\lambda)^s g(y)$$

for ever $x,y \in [0,\infty)$ and $\lambda \in [0,1]$.

Remark 1. *Definition 2 reduces to Definition 1 when $s = 1$. In the current paper I^0 denote the interior of an interval I and $L_1([a,b])$ represent all integrable functions.*

Theorem 1 ([8]). *Suppose that $g : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$ and let $a, b \in [0,\infty),\ a < b$. If $g \in L_1([a,b])$, then we have:*

$$2^{s-1} g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{g(a)+g(b)}{s+1}. \tag{1}$$

Definition 3 ([9]). *For an integrable function g on $[a,b]$, $a \geq 0$ and $\beta > 0$, the right- and left-sided Riemann–Liouville fractional integral of order β are respectively given by*

$$\mathcal{J}^{\beta}_{a^+} g(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} g(t)dt, \quad x > a, \tag{2}$$

and

$$\mathcal{J}^{\beta}_{b^-} g(x) = \frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1} g(t)dt, \quad t < b, \tag{3}$$

where $\Gamma(\beta) = \displaystyle\int_0^\infty e^{-x} x^{\beta-1} dx$ is the gamma function.

Definition 4 ([10]). *Suppose that the function g is integrable on $[a,b]$ and $\delta \in (0,1]$. Then for all $\beta \in \mathbb{C},\ Re(\beta) \geq 0$*

$$\left(\mathcal{J}^{\beta,\delta}_{a^+} g\right)(y) = \frac{1}{\delta^{\beta}\Gamma(\beta)}\int_a^y (y-u)^{\beta-1} g(u)du, \quad y > a, \tag{4}$$

and

$$\left(\mathcal{J}^{\beta,\delta}_{b^-} g\right)(y) = \frac{1}{\delta^{\beta}\Gamma(\beta)}\int_y^b (u-y)^{\beta-1} g(u)du, \quad y < b. \tag{5}$$

The notations $\left(\mathcal{J}^{\beta,\delta}_{a^+} g\right)(y)$ and $\left(\mathcal{J}^{\beta,\delta}_{b^-} g\right)(y)$ are called respectively left- and right-sided generalized proportional fractional integral operators of order β.

Remark 2. *Definition 4 becomes the Riemann–Liouville fractional integrals given in Definition 3 for $\delta = 1$.*

For Riemann–Liouville fractional integrals, Chen and Huang in [11] obtained the following Simpson's type inequality for s-convex functions.

Theorem 2. *Let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta}_{b^-} g\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{b-a}{2^{s+1}} M_1(\beta,s)\Big[|g'(a)| + |g'(b)|\Big],\end{aligned} \tag{6}$$

where

$$M_1(\beta,s) = \int_0^1 \left|\frac{t^{\beta}}{2} - \frac{1}{3}\right| \Big[(1+t)^s + (1-t)^s\Big] dt.$$

Theorem 3. *Let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{2(s+1)^{\frac{1}{q}}}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\left|g'(b)\right|^q+\left|g'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|g'(a)\right|^q+\left|g'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned}\tag{7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Let $g: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0 such that $g'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If $|g'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{(2^{s+1}-1)|g'(b)|^q+|g'(a)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|g'(a)|^q+|g'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],\end{aligned}\tag{8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 5. *Let $g: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0 such that $g'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If $|g'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{2}M_2(\beta)\Big[M_3(\beta,s)+M_4(\beta,s)\Big],\end{aligned}\tag{9}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{aligned}M_2(\beta)&=\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|dt\right)^{\frac{1}{p}},\\M_3(\beta,s)&=\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s|g'(b)|^q+\left(\frac{1-t}{2}\right)^s|g'(a)|^q\right]dt,\\M_4(\beta,s)&=\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s|g'(a)|^q+\left(\frac{1-t}{2}\right)^s|g'(b)|^q\right]dt.\end{aligned}$$

In this paper, we introduce new Simpson's inequalities for s-convex function in the second sense via a generalized proportional fractional integral which is the generalization of the result obtained by Chen and Huang [11]. These types of inequalities can be used to estimate the bounds of both regular and fractional integrals. The paper is organize as follows: In Section 2, we state our main results on inequalities of Simpson's type for s-convex functions via generalized proportional fractional integral. Finally, Section 3 is devoted to the conclusion of our work.

2. Main Results

The following Lemma is required to prove our main results.

Lemma 1. *Let $g : I \to \mathbb{R}$ be an absolutely continuous mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. Then we have the following equality:*

$$\begin{aligned} &\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right] \\ &= \frac{b-a}{2}\int_0^1 \left[\left(\frac{t^\beta}{2} - \frac{1}{3}\right) g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t^\beta}{2}\right) g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right] dt. \end{aligned} \tag{10}$$

Proof. Let

$$\begin{aligned} M &= \int_0^1 \left[\left(\frac{t^\beta}{2} - \frac{1}{3}\right) g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t^\beta}{2}\right) g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right] dt \\ &= \int_0^1 \left(\frac{1}{3} - \frac{t^\beta}{2}\right) g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt + \int_0^1 \left(\frac{t^\beta}{2} - \frac{1}{3}\right) g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \\ &= M_5 + M_6. \end{aligned}$$

We compute M_5 and M_6 by using integration by parts and by change of variable. For this, we get

$$\begin{aligned} M_5 &= \int_0^1 \left(\frac{1}{3} - \frac{t^\beta}{2}\right) g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\ &= \frac{2}{a-b}\int_0^1 \left(\frac{1}{3} - \frac{t^\beta}{2}\right) d\left(g\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right) \\ &= \frac{2}{b-a}\left[\frac{1}{6}g(a) + \frac{1}{3}g\left(\frac{a+b}{2}\right) - \frac{\beta}{2}\left(\frac{2}{b-a}\right)^\beta \int_a^{\frac{a+b}{2}} \left(\left(\frac{a+b}{2}\right) - x\right)^{\beta-1} g(x)dx\right] \\ &= \frac{2}{b-a}\left[\frac{1}{6}g(a) + \frac{1}{3}g\left(\frac{a+b}{2}\right) - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\frac{1}{\Gamma(\beta)\delta^\beta} \int_a^{\frac{a+b}{2}} \left(\left(\frac{a+b}{2}\right) - x\right)^{\beta-1} g(x)dx\right] \\ &= \frac{2}{b-a}\left[\frac{1}{6}g(a) + \frac{1}{3}g\left(\frac{a+b}{2}\right) - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

Using similar argument as outlined above, we obtain:

$$\begin{aligned} M_6 &= \int_0^1 \left(\frac{t^\beta}{2} - \frac{1}{3}\right) g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \\ &= \frac{2}{b-a}\left[\frac{1}{6}g(b) + \frac{1}{3}g\left(\frac{a+b}{2}\right) - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

By adding M_5 and M_6, we get the desired identity. □

Theorem 6. *Let $\delta, \beta > 0$ and let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|$ is s-convex on $[a,b]$, for $s \in (0,1]$, then the following inequalities hold:*

$$\begin{aligned} &\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{b-a}{2^{s+1}} M_7(\beta,s)\left[|g'(a)| + |g'(b)|\right] \\ &\leq \frac{b-a}{3(s+1)}\left[|g'(a)| + |g'(b)|\right], \end{aligned} \tag{11}$$

where
$$M_7(\beta,s) = \int_0^1 \left|\frac{t^\beta}{2} - \frac{1}{3}\right| \Big[(1+t)^s + (1-t)^s\Big] dt.$$

Proof. Using Lemma 1 and s-convexity of $|g'|$, we have:

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1 \left[\left|\left(\frac{t^\beta}{2} - \frac{1}{3}\right)\right|\left|g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| + \left|\left(\frac{1}{3} - \frac{t^\beta}{2}\right)\right|\left|g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\right] dt \\
&= \frac{b-a}{2}\int_0^1 \left|\left(\frac{t^\beta}{2} - \frac{1}{3}\right)\right|\left[\left|g'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| + \left|g'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\right] dt \\
&\leq \frac{b-a}{2}\int_0^1 \left[\left|\left(\frac{t^\beta}{2} - \frac{1}{3}\right)\right|\left[\left(\frac{1+t}{2}\right)^s |g'(b)| + \left(1 - \frac{1+t}{2}\right)^s |g'(a)| + \left(\frac{1+t}{2}\right)^s |g'(a)| + \left(1 - \frac{1+t}{2}\right)^s |g'(b)|\right]\right] dt \\
&= \frac{b-a}{2^{s+1}}\int_0^1 \left|\frac{t^\beta}{2} - \frac{1}{3}\right| \Big((1+t)^s + (1-t)^s\Big)\Big[|g'(a)| + |g'(b)|\Big].
\end{aligned}$$

This complete the proof of the first inequality of (11). The second inequality of (11) follows since $\left|\frac{t^\beta}{2} - \frac{1}{3}\right| \leq \frac{1}{3}$ for ever $t \in [0,1]$ and

$$M_7(\beta,s) \leq \frac{1}{3}\int_0^1 \Big[(1+t)^s + (1-t)^s\Big] dt = \frac{2^{s+1}}{3(s+1)}, \tag{12}$$

which proves the second inequality. □

Corollary 1. *Let $\delta, \beta > 0$ and let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|$ is convex on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{4} M_7(\beta,1)\Big[|g'(a)| + |g'(b)|\Big] \\
&\leq \frac{b-a}{6}\Big[|g'(a)| + |g'(b)|\Big].
\end{aligned} \tag{13}$$

Proof. Taking $s = 1$ in Theorem 6 we have the result. □

Remark 3. *In Theorem 6 if we put*

a. $\delta = 1$, *then the first inequality* (11) *coincide with Theorem* 2 *and the second inequality coincide with Corollary 8 in* [12].

b. $\delta = \beta = 1$, *then the first inequality* (11) *coincide with Theorem 7 in* [13].

Theorem 7. *Let $\delta, \beta > 0$ and let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|^q$ is s-convex on $[a,b]$ for $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{2^{\beta-1}\delta^\beta\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathcal{J}^{\beta,\delta}_{a^+} g\left(\frac{a+b}{2}\right) + \mathcal{J}^{\beta,\delta}_{b^-} g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1 \left|\frac{t^\beta}{2} - \frac{1}{3}\right|^p dt\right)^{\frac{1}{p}} \left[\left(\frac{|g'(b)|^q + |g'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{|g'(a)|^q + |g'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{14}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. Using Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}_{a^+}^{\beta,\delta}g\left(\frac{a+b}{2}\right)+\mathcal{J}_{b^-}^{\beta,\delta}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\left(\frac{t^\beta}{2}-\frac{1}{3}\right)\right|\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\left(\frac{1}{3}-\frac{t^\beta}{2}\right)\right|\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&=\frac{b-a}{2}\int_0^1\left|\left(\frac{t^\beta}{2}-\frac{1}{3}\right)\right|\left[\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|g'|^q$ is s-convex, by using change of variable and Equation (1), we have

$$\int_0^1\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt=\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}|g'(x)|^q dx\leq\left[\frac{|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{s+1}\right] \tag{15}$$

and

$$\int_0^1\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}|g'(x)|^q dx\leq\left[\frac{|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{s+1}\right]. \tag{16}$$

Hence by using inequalities (15) and (16) we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}_{a^+}^{\beta,\delta}g\left(\frac{a+b}{2}\right)+\mathcal{J}_{b^-}^{\beta,\delta}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof is complete. □

Corollary 2. *Let $\delta,\beta>0$ and let $g: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0 such that $g'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If $|g'|^q$ is convex on $[a,b]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}_{a^+}^{\beta,\delta}g\left(\frac{a+b}{2}\right)+\mathcal{J}_{b^-}^{\beta,\delta}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{2}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. We have the result by taking $s=1$ in Theorem 7. □

Remark 4. *In Theorem 7 if we put*

a. $\delta=1$, *then the inequality* (14) *coincide with Theorem* 3.
b. $\delta=\beta=1$, *then the inequality* (14) *coincide with Theorem 8 in* [13].

Theorem 8. *Let $\delta, \beta > 0$ and let $g : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|^q$ is $s-$convex on $[a,b]$, for $s \in (0,1]$ and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta,\delta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta,\delta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{(2^{s+1}-1)|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right] \qquad (17) \\
&\leq \frac{b-a}{6}\left[\left(\frac{(2^{s+1}-1)|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Hölder's inequality, s-convexity of $|g'|^q$ and Lemma 1, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta,\delta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta,\delta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\left(\frac{t^{\beta}}{2}-\frac{1}{3}\right)\right|\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\left(\frac{1}{3}-\frac{t^{\beta}}{2}\right)\right|\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&= \frac{b-a}{2}\int_0^1\left|\left(\frac{t^{\beta}}{2}-\frac{1}{3}\right)\right|\left[\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(\frac{1+t}{2}\right)^s|g'(b)|^q+\left(1-\frac{1+t}{2}\right)^s|g'(a)|^q dt\right)^{\frac{1}{q}}+\right. \\
&\qquad \left.\int_0^1\left(\left(\frac{1+t}{2}\right)^s|g'(a)|^q+\left(1-\frac{1+t}{2}\right)^s|g'(b)|^q dt\right)^{\frac{1}{q}}\right] \\
&= \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{(2^{s+1}-1)|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is the desired first inequality. We get the second inequality from the first inequality due to the fact that $\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right| \leq \frac{1}{3}$ for all $t \in [0,1]$ and $\int_0^1 (1+t)^s dt = \dfrac{2^{s+1}-1}{(s+1)}$ and $\int_0^1 (1-t)^s dt = \dfrac{1}{(s+1)}$.
That concludes the proof. □

Corollary 3. *Let $\delta, \beta > 0$ and let $g : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $g' \in L_1([a,b])$, where $a, b \in I^0$ with $a < b$. If $|g'|^q$ is convex on $[a,b]$ and $q > 1$, then the following inequalities hold:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta,\delta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta,\delta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{3|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{6}\left[\left(\frac{(3)|g'(b)|^q+|g'(\frac{a+b}{2})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(a)|^q+|g'(\frac{a+b}{2})|^q}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{18}
$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. Taking $s=1$ in Theorem 8 we get the result. □

Remark 5. *In Theorem 8 if we put*

a. $\delta=1$, *then the first inequality* (17) *coincide with Theorem* 4.
b. $\delta=\beta=1$, *then the first inequality* (17) *coincide with Theorem 9 in* [13].

Theorem 9. *Let* $\delta,\beta>0$ *and let* $g: I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I^0 *such that* $g'\in L_1([a,b])$, *where* $a,b\in I^0$ *with* $a<b$. *If* $|g'|^q$ *is s-convex on* $[a,b]$ *for* $s\in(0,1]$ *and* $q>1$, *then the following inequalities hold:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}^{\beta,\delta}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{J}^{\beta,\delta}_{b^-}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2^{1+\frac{s}{q}}}M_8(\beta)\left[\left(M_9(\beta,s)|g'(b)|^q+M_{10}(\beta,s)|g'(a)|^q\right)^{\frac{1}{q}}+\left(M_9(\beta,s)|g'(a)|^q+M_{10}(\beta,s)|g'(b)|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{2^{1+\frac{s}{q}}}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{3(s+1)}|g'(b)|^q+\frac{1}{3(s+1)}|g'(a)|^q\right)^{\frac{1}{q}}+\left(\frac{2^{s+1}-1}{3(s+1)}|g'(a)|^q+\frac{1}{3(s+1)}|g'(b)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{19}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$
\begin{aligned}
M_8(\beta)&=\left(\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}, \\
M_9(\beta,s)&=\int_0^1\left|\frac{t^{\beta}}{2}-\frac{1}{3}\right|(1+t)^s dt, \\
M_{10}(\beta,s)&=\int_0^1\left|\frac{1}{3}-\frac{t^{\beta}}{2}\right|(1-t)^s dt.
\end{aligned}
$$

Proof. Using Lemma 1, s-convexity of $|g'|^q$, the power mean inequality and Hölder's inequality, we get

$$
\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}_{a^+}^{\beta,\delta}g\left(\frac{a+b}{2}\right)+\mathcal{J}_{b^-}^{\beta,\delta}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\left(\frac{t^\beta}{2}-\frac{1}{3}\right)\right|\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\left(\frac{1}{3}-\frac{t^\beta}{2}\right)\right|\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\leq \frac{b-a}{2}\left[\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|\left|g'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|\left|g'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{2}\left[\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s|g'(b)|^q+\left(1-\frac{1+t}{2}\right)^s|g'(a)|^q dt\right]\right)^{\frac{1}{q}}+\right. \\
&\qquad\left.\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|\left[\left(\frac{1+t}{2}\right)^s|g'(a)|^q+\left(1-\frac{1+t}{2}\right)^s|g'(b)|^q dt\right]\right)^{\frac{1}{q}}\right] \\
&=\frac{b-a}{2^{1+\frac{s}{q}}}\left[\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|\left[(1+t)^s|g'(b)|^q+(1-t)^s|g'(a)|^q dt\right]\right)^{\frac{1}{q}}+\right. \\
&\qquad\left.\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|\left[(1+t)^s|g'(a)|^q+(1-t)^s|g'(b)|^q dt\right]\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

which completes the first inequality. We get the second inequality from the first inequality due to the facts that:

$$\left|\frac{t^\beta}{2}-\frac{1}{3}\right|<\frac{1}{3}$$

for all $t\in[0,1]$,

$$\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|(1+t)^s dt\leq\frac{2^{s+1}-1}{3(s+1)}$$

and

$$\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|(1-t)^s dt\leq\frac{1}{3(s+1)}.$$

This concludes the proof. □

Corollary 4. *Let $\delta,\beta>0$ and let $g:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0 such that $g'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If $|g'|^q$ is convex on $[a,b]$ and $q>1$, then the following inequalities hold:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{2^{\beta-1}\delta^{\beta}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathcal{J}_{a^+}^{\beta,\delta}g\left(\frac{a+b}{2}\right)+\mathcal{J}_{b^-}^{\beta,\delta}g\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq\frac{b-a}{2^{1+\frac{s}{q}}}M_8(\beta)\left[\left(M_9(\beta,1)|g'(b)|^q+M_{10}(\beta,1)|g'(a)|^q\right)^{\frac{1}{q}}+\left(M_9(\beta,1)|g'(a)|^q+M_{10}(\beta,1)|g'(b)|^q\right)^{\frac{1}{q}}\right] \qquad (20)\\
&\leq\frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}|g'(b)|^q+\frac{1}{6}|g'(a)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{2}|g'(a)|^q+\frac{1}{6}|g'(b)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By taking $s=1$ in Theorem 9 we have the desired result. □

Remark 6. *In Theorem 9 if we put*

a. $\delta = 1$, *then the first inequality* (19) *coincide with Theorem 5.*

b. $\delta = \beta = 1$, *then the first inequality* (19) *coincide with Theorem 10 in* [13].

3. Conclusions

Our results have introduced a new integral inequality of Simpson's type integral inequalities using s-convexity via generalized proportional fractional integrals. The inequalities obtained are generalizations of Simpson's type inequality that are given for the Riemann–Liouville fractional integrals in [13]. Similar inequalities could possibly be established for more generalized fractional integrals such as Riemann–Liouville fractional integrals of a function with respect to another generalized function and to a proportional fractional integral of a function with respect to another function.

Author Contributions: Conceptualization, H.D. and T.A.; writing—original draft preparation, H.D. and T.A.; writing—review and editing, H.D., J.B.M., T.A. and E.R.N.; supervision, E.R.N. and J.B.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Acknowledgments: Many thanks to the anonymous referees whose comments and suggestions improved this final version of our paper. In addition, the first author acknowledges Addis Ababa University, Department of Mathematics and International Science Program(ISP), Uppsala University for their support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of simpson's type for s-convex functions with applications. *Res. Rep. Collect.* **2009**, *12*, 9.
2. Alomari, M.; Hussain, S. Two inequalities of simpson type for quasi-convex functions and applications. *Appl. Math. E-Notes* **2011**, *11*, 110–117.
3. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
4. Kashuri, A.; Meftah, B.; Mohammed, P.O. Some weighted simpson type inequalities for differentiable s-convex functions and their applications: Some weighted simpson type inequalities. *J. Fract. Calc. Nonlinear Syst.* **2020**, *1*, 75–94. [CrossRef]
5. Kermausuor, S. Simpson's type inequalities for strongly (s, m)-convex functions in the second sense and applications. *Open J. Math. Sci.* **2019**, *3*, 74–83. [CrossRef]
6. Kermausuor, S. Simpson's type inequalities via the katugampola fractional integrals for s-convex functions. *Kragujev. J. Math.* **2021**, *45*, 709–720. [CrossRef]
7. Rangel-Oliveros, Y.; Nwaeze, E.R. Simpson's type inequalities for exponentially convex functions with applications. *Open J. Math. Sci.* **2021**, *5*, 84–94. [CrossRef]
8. Dragomir, S.S.; Fitzpatrick, S. The Hadamard inequalities for s-convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696. [CrossRef]
9. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
10. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [CrossRef]
11. Chen, J.; Huang, X. Some new inequalities of simpson's type for s-convex functions via fractional integrals. *Filomat* **2017**, *31*, 4989–4997. [CrossRef]
12. Matłoka, M. Some inequalities of simpson type for h-convex functions via fractional integrals. *Abstr. Appl. Anal.* **2015**, *2015*, 956850. [CrossRef]
13. Sarikaya, M.Z.; Set, E.; Ozdemir, M.E. On new inequalities of simpson's type for s-convex functions. *Comput. Math. Appl.* **2010**, *60*, 2191–2199. [CrossRef]