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Nonlocal Boundary Value Problems for (k,ψ)-Hilfer Fractional Differential Equations and Inclusions

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Abstract: In the present research, single and multi-valued (k,ψ)-Hilfer type fractional boundary value problems of order in $(1,2]$ involving nonlocal integral boundary conditions were studied. In the single-valued case, the Banach and Krasnosel'skiĭ fixed point theorems as well as the Leray–Schauder nonlinear alternative were used to establish the existence and uniqueness results. In the multi-valued case, when the right-hand side of the inclusion has convex values, we established an existence result via the Leray–Schauder nonlinear alternative method for multi-valued maps, while the second existence result, dealing with the non-convex valued right-hand side of the inclusion, was obtained by applying Covitz-Nadler fixed point theorem for multi-valued contractions. The obtained theoretical results are well illustrated by the numerical examples provided.

Keywords: (k,ψ)-Hilfer fractional derivative and integral operators; boundary value problems; existence of solutions; fixed point theorems

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1. Introduction

Fractional calculus, dealing with integral and differential operators of non-integer order, has found interesting applications in many engineering and scientific disciplines such as physics, chemistry, mathematical biology, mechanics, and so forth, see the monographs [1–9]. Usually, fractional derivatives are defined in terms of fractional integral operators with different forms of the kernel function. Examples include Riemann–Liouville, Caputo, Hadamard, Katugampola and Hilfer fractional derivatives. Certain forms of fractional operators contain a number of different fractional operators. For example, the generalized fractional derivative of Katugampola [10,11] includes both Riemann–Liouville and Hadamard fractional derivatives. The Hilfer fractional derivative operator [12] contains Riemann–Liouville as well as Caputo fractional derivative operators. Another fractional derivative operator unifying Caputo, Caputo–Hadamard and Caputo–Erdélyi–Kober fractional derivative operators is the ψ-fractional derivative operator [13]. The (k,ψ)-Hilfer fractional derivative operator introduced in [14] generalizes many of the well-known fractional derivative operators, see [15].

Initial and boundary value problems involving the (k,ψ)-Hilfer fractional derivative operator recently received considerable attention. In [14], an existence and uniqueness result for a (k,ψ)-Hilfer type fractional initial value problem was derived. The authors discussed the existence of solutions for (k,ψ)-Hilfer fractional differential equations and inclusions supplemented with nonlocal boundary conditions in [15].

Motivated by the work presented in [14,15], in the present paper, we studied the existence of solutions for a (k,ψ)-Hilfer type fractional differential equation of order in $(1,2]$, equipped with nonlocal (k,ψ)-Riemann–Liouville fractional integral boundary conditions. In precise terms, we investigated the following (k,ψ)-Hilfer type fractional boundary value problem:

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}z(t) = \mathfrak{f}(t,z(t)), \quad t \in (a,b], \\ z(a) = 0, \quad z(b) = \lambda z(\xi) + \mu\, {}^{k}\mathfrak{J}^{v,\psi}z(\sigma), \end{cases} \tag{1}$$

where ${}^{k,H}D^{\alpha,\beta;\psi}$ denotes the (k,ψ)-Hilfer type fractional derivative operator of order α, $1 < \alpha < 2$, $0 \le \beta \le 1$, $k > 0$, $\mathfrak{f} \in C([a,b] \times \mathbb{R}, \mathbb{R})$, ${}^{k}\mathfrak{J}^{v,\psi}$ is the (k,ψ)-Riemann–Liouville fractional integral of order $v > 0$, $\lambda, \mu \in \mathbb{R}$, and $a < \xi,\ \sigma < b$.

The corresponding multi-valued analogue of (k,ψ)-Hilfer type boundary value problem (1) provided by

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}z(t) \in \mathfrak{F}(t,z(t)), \quad t \in (a,b], \\ z(a) = 0, \quad z(b) = \lambda z(\xi) + \mu\, {}^{k}\mathfrak{J}^{v,\psi}z(\sigma), \end{cases} \tag{2}$$

was also studied. In (2), $\mathfrak{F} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ represents a multi-valued map ($\mathcal{P}(\mathbb{R})$ denotes a family of all nonempty subsets of $\mathbb{R}$), while the other quantities are the same as described in the problem (1).

We applied Krasnosel'skiĭ's fixed point theorem and the Leray–Schauder nonlinear alternative to prove the existence results for the problem (1), while the uniqueness of solutions for the problem (1) was established via Banach's fixed point theorem. The existence results for the multivalued problem (2) for convex-valued and non-convex-valued cases were, respectively, obtained by means of the Leray–Schauder nonlinear alternative for multi-valued maps and the Covitz–Nadler fixed point theorem for multi-valued contractions. Concerning the advantages of the methods employed in the present study over other existing methods, we mention that the tools of the fixed point theory provide a suitable platform to establish the existence theory for boundary value problems once the problem at hand is converted into a fixed point problem.

Here, we recall that Hilfer fractional differential equations find useful applications in real world problems such as filtration processes [16,17], advection–diffusion phenomena [18], glass forming materials [19], etc. On the other hand, the nonlocal integral boundary conditions have potential applications in physical problems such as diffusion processes [20], blood flow problems [21], bacteria self-organization models [22], etc. We anticipated that the modeling of physical phenomena based on the Hilfer fractional derivative would be improved by using the (k,ψ)-Hilfer fractional derivative. Further, the (k,ψ)-Hilfer type boundary value problems considered in this paper correspond to a variety of fractional boundary value problems for different choices of ψ, for details, see [15]. In fact, the results obtained for the problems (1) and (2) are not only new in the given configuration but also correspond to several special cases for an appropriate choice of the values of ψ and the parameters involved in the given problems. Hence, the work established in this paper enriches the existing literature on the class of (k,ψ)-Hilfer boundary value problems.

The remainder of our paper is arranged as follows. In Section 2, we recall some fundamental concepts related to the study of the proposed problems. Section 3 contains an auxiliary result that plays a key role in converting the given problems into equivalent fixed point problems. Section 4 is devoted to the derivation of the main results for the single-valued problem (1), while the existence results for the multi-valued problem (2) are established in Section 5. Illustrative numerical examples demonstrating the applicability of the obtained theoretical results are presented in Section 6. The paper concludes with some interesting observations.

2. Preliminaries

Let us begin this section by introducing some preliminary concepts of fractional calculus.

Definition 1 ([23]). *Let $k, \alpha \in \mathbb{R}^+$. The k-Riemann–Liouville fractional derivative of order α for the function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ is obtained by*

$$^{k}\mathfrak{I}^{\alpha}_{a+}\mathfrak{h}(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-u)^{\frac{\alpha}{k}-1}\mathfrak{h}(u)du,$$

where Γ_k is the k-Gamma function defined by [24]

$$\Gamma_k(z) = \int_0^\infty s^{z-1}e^{-\frac{s^k}{k}}ds,\ z \in \mathbb{C}, \Re(z) > 0,$$

which satisfies the following properties:

$$\lim_{k\to 1}\Gamma_k(\theta) = \Gamma(\theta),\ \Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma\left(\frac{\theta}{k}\right) \text{ and } \Gamma_k(\theta+k) = \theta\Gamma_k(\theta).$$

Definition 2 ([25]). *The k-Riemann-Liouville fractional derivative of order α for the function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ with $k, \alpha \in \mathbb{R}^+$ is defined by*

$$^{k,RL}D^{\alpha}_{a+}\mathfrak{h}(t) = \left(k\frac{d}{dt}\right)^n {}^{k}\mathfrak{I}^{nk-\alpha}_{a+}\mathfrak{h}(t),\ \ n = \left\lceil \frac{\alpha}{k} \right\rceil,$$

where $\left\lceil \frac{\alpha}{k} \right\rceil$ represents the ceiling function of $\frac{\alpha}{k}$.

Definition 3. *Let $\psi : [a,b] \to \mathbb{R}$ be an increasing function with $\psi'(\theta) \neq 0$ for all $\theta \in [a,b]$. Then, the ψ-Riemann–Liouville and (k,ψ)-Riemann–Liouville fractional integrals of order α for the function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$, respectively defined in [2,26], are given by*

$$\mathfrak{I}^{\alpha;\psi}\mathfrak{h}(t) = \frac{1}{\Gamma_k(\alpha)} \int_a^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}\mathfrak{h}(u)du,$$

$$^{k}\mathfrak{I}^{\alpha;\psi}_{a+}\mathfrak{h}(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \psi'(u)(\psi(t)-\psi(u))^{\frac{\alpha}{k}-1}\mathfrak{h}(u)du.$$

Definition 4 ([27]). *The ψ-Hilfer fractional derivative for the function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $\alpha \in (n-1,n], n \in \mathbb{N}$ and type $\beta \in [0,1]$ is defined by*

$$^{H}D^{\alpha,\beta;\psi}\mathfrak{h}(t) = \mathfrak{I}^{\beta(n-\alpha);\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathfrak{I}^{(1-\beta)(n-\alpha);\psi}_{a+}\mathfrak{h}(t),$$

where $\psi : [a,b] \to \mathbb{R}$ is an increasing function such that $\psi \in C^n([a,b],\mathbb{R})$ and $\psi'(\theta) \neq 0,\ \theta \in [a,b]$. On the other hand, the (k,ψ)-Hilfer fractional derivative of order α and type β for the function $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ defined in [14] is

$$^{k,H}D^{\alpha,\beta;\psi}\mathfrak{h}(t) = {}^{k}\mathfrak{I}^{\beta(nk-\alpha);\psi}_{a+}\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^n {}^{k}\mathfrak{I}^{(1-\beta)(nk-\alpha);\psi}_{a+}\mathfrak{h}(t),\ \ n = \left\lceil \frac{\alpha}{k} \right\rceil.$$

Remark 1. *For $\beta \in [0,1]$ and $n-1 < \frac{\alpha}{k} \leq n$, we have $n-1 < \frac{\theta_k}{k} \leq n$, $\theta_k = \alpha + \beta(nk-\alpha)$. Further, one can notice that*

$$\begin{aligned} ^{k,H}D^{\alpha,\beta;\psi}\mathfrak{h}(t) &= {}^{k}\mathfrak{I}^{\theta_k-\alpha;\psi}_{a+}\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^n {}^{k}\mathfrak{I}^{nk-\theta_k;\psi}_{a+}\mathfrak{h}(t) \\ &= {}^{k}\mathfrak{I}^{\theta_k-\alpha;\psi}_{a+}\left({}^{k,RL}D^{\theta_k;\psi}\mathfrak{h}\right)(t). \end{aligned}$$

Now, we recall some useful results involving the (k,ψ)-Hilfer fractional derivative and integral operators.

Lemma 1 ([14]). *Let $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ and ${}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h} \in C^n([a,b],\mathbb{R})$ with $\mu, k \in \mathbb{R}^+ = (0,\infty)$ and $n = \left\lceil \frac{\mu}{k} \right\rceil$. Then,*

$$ {}^k\mathfrak{J}^{\mu;\psi}\left({}^{k,RL}D^{\mu;\psi}\mathfrak{h}(t)\right) = \mathfrak{h}(t) - \sum_{j=1}^{n} \frac{(\psi(t)-\psi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^{n-j} {}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h}(t)\right]_{z=a}. $$

Lemma 2 ([14]). *Let $\theta_k = \alpha + \beta(k-\alpha)$ with $\alpha, k \in \mathbb{R}^+ = (0,\infty)$, $\alpha < k$, and $\beta \in [0,1]$. Then,*

$$ {}^k\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}D^{\theta_k;\psi}\mathfrak{h}\right)(t) = {}^k\mathfrak{J}^{\alpha;\psi}\left({}^{k,H}D^{\alpha,\beta;\psi}\mathfrak{h}\right)(t), \;\; \mathfrak{h} \in C^n([a,b],\mathbb{R}). $$

3. An Auxiliary Result

In this section, we provide an auxiliary result, which helps us in transforming the nonlinear (k,ψ)-Hilfer type fractional boundary value problem (1) into a fixed point problem. The following lemma concerns a linear variant of the (k,ψ)-Hilfer type fractional boundary value problem (1).

Lemma 3. *For $g \in C(a,b) \cap L^1(a,b)$ and $\Lambda \neq 0$, the function $z \in C([a,b],\mathbb{R})$ is a solution of the following problem*

$$ \begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}z(t) = g(t), \; k>0, \; 1<\alpha\le 2, \; \beta \in [0,1], \;\; t \in (a,b], \\ z(a) = 0, \;\;\; z(b) = \lambda z(\xi) + \mu\, {}^k\mathfrak{J}^{v,\psi}z(\sigma), \end{cases} \tag{3} $$

if and only if

$$ z(t) = {}^k\mathfrak{J}^{\alpha;\psi}g(t) + \frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\left[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi}g(\xi) + \mu\, {}^k\mathfrak{J}^{v+\alpha,\psi}g(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi}g(b)\right], \tag{4} $$

where $\theta_k = \alpha + \beta(2k-\alpha)$ and

$$ \Lambda := \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} - \lambda\frac{(\psi(\xi)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} - \mu\frac{(\psi(\sigma)-\psi(a))^{\frac{\theta_k+v}{k}-1}}{\Gamma_k(\theta_k+v)}. \tag{5} $$

Proof. Assume that z is a solution of the problem (3). Operating on both sides of equation in (3), the fractional integral ${}^k\mathfrak{J}^{\alpha;\psi}$ and using Lemmas 1 and 2, we obtain

$$ \begin{aligned} {}^k\mathfrak{J}^{\alpha;\psi}\left({}^{k,H}D^{\alpha,\beta;\psi}z\right)(t) &= {}^k\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}D^{\theta_k;\psi}z\right)(t) \\ &= z(t) - \frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)}\left[\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right){}^k\mathfrak{J}^{2k-\theta_k;\psi}z(t)\right]_{w=a} \\ &\quad - \frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k-k)}\left[{}^k\mathfrak{J}^{2k-\theta_k;\psi}z(t)\right]_{w=a}. \end{aligned} $$

Consequently

$$ z(t) = {}^k\mathfrak{J}^{\alpha;\psi}g(t) + c_0\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} + c_1\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k-k)}, \tag{6} $$

where

$$ c_0 = \left[\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right){}^k\mathfrak{J}^{2k-\theta_k;\psi}z(t)\right]_{w=a}, \;\; c_1 = \left[{}^k\mathfrak{J}^{2k-\theta_k;\psi}z(t)\right]_{w=a}. $$

By the condition $z(a) = 0$, we obtain $c_1 = 0$, since $\frac{\theta_k}{k} - 2 < 0$ as shown by Remark 1. Now, using the nonlocal condition: $z(b) = \lambda z(\xi) + \mu \, {}^k\mathfrak{I}^{v,\psi} z(\sigma)$ and the following result from [14]:

$$ {}^k\mathfrak{I}^{v,\psi}(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1} = \frac{\Gamma_k(\theta_k)}{\Gamma_k(\theta_k + v)}(\psi(t) - \psi(a))^{\frac{\theta_k+v}{k}-1}, \tag{7} $$

we find that

$$ c_0 = \frac{1}{\Lambda}\Big[\lambda \, {}^k\mathfrak{I}^{\alpha;\psi} g(\xi) + \mu \, {}^k\mathfrak{I}^{v+\alpha,\psi} g(\sigma) - {}^k\mathfrak{I}^{\alpha;\psi} g(b)\Big]. $$

Substituting the above value of c_0 and $c_1 = 0$ in (6), we obtain the solution (4). By carrying out direct computation we can easily establish the converse of this lemma. The proof is completed. □

4. The Single-Valued Problem

Let $C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the norm $\|z\| = \sup_{t\in[a,b]} |z(t)|$. By using Lemma 3, an operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ can be defined as

$$ \begin{aligned} (\mathcal{A}z)(t) &= \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda \, {}^k\mathfrak{I}^{\alpha;\psi}\mathfrak{f}(\xi_i, z(\xi_i)) + \mu \, {}^k\mathfrak{I}^{v+\alpha,\psi}\mathfrak{f}(\sigma, z(\sigma)) - {}^k\mathfrak{I}^{\alpha;\psi}\mathfrak{f}(b, z(b))\Big] \\ &\quad + {}^k\mathfrak{I}^{\alpha;\psi}\mathfrak{f}(t, z(t)), \quad t \in [a,b]. \end{aligned} \tag{8} $$

Notice that the fixed points of the operator $\mathcal{A}$ will be solutions of the nonlocal (k,ψ)-Hilfer type fractional boundary value problem (1).

For computational convenience, we set the following notation:

$$ \begin{aligned} \mathbb{G} &= \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \\ &\quad + |\mu|\frac{(\psi(b) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)}\Bigg]. \end{aligned} \tag{9} $$

4.1. Existence and Uniqueness Result

In this subsection, we make use of Banach's fixed point theorem [28] to prove a uniqueness result for the (k,ψ)-Hilfer type fractional boundary value problem (1).

Theorem 1. *Suppose that*

(H_1) $|\mathfrak{f}(t,z) - \mathfrak{f}(t,y)| \le \mathfrak{L}|z - y|$, $\mathfrak{L} > 0$ *for each* $t \in [a,b]$ *and* $z, y \in \mathbb{R}$.

Then, there exists a unique solution for the problem (1) on $[a,b]$ *provided that*

$$ \mathfrak{L}\mathbb{G} < 1, \tag{10} $$

where $\mathbb{G}$ *is defined by (9).*

Proof. Transform the problem (1) into a fixed point problem $\mathcal{A}z = z$, where the operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is defined in (8). We verify the hypothesis of Banach's fixed point theorem to deduce that the operator $\mathcal{A}$ has a unique fixed point.

Let us first show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{z \in C([a,b],\mathbb{R}) : \|z\| \le r\}$ with

$$ r \ge \frac{Q\mathbb{G}}{1 - \mathfrak{L}\mathbb{G}}, \quad \sup_{t\in[a,b]} |\mathfrak{f}(t,0)| = Q < \infty. \tag{11} $$

By (H_1), we have

$$
\begin{aligned}
|\mathfrak{f}(t,z(t))| &\le |\mathfrak{f}(t,z(t))-\mathfrak{f}(t,0)|+|\mathfrak{f}(t,0)| \\
&\le \mathfrak{L}|z(t)|+Q\le \mathfrak{L}\|z\|+Q\le \mathfrak{L}r+Q.
\end{aligned}
$$

For any $z\in B_r$, we have

$$
\begin{aligned}
|(\mathcal{A}z)(t)| &\le \sup_{t\in[a,b]}\Bigg\{\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\ {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(\xi,z(\xi))|+|\mu|\ {}^k\mathfrak{J}^{v+\alpha,\psi}|\mathfrak{f}(\sigma,z(\sigma))| \\
&\quad +{}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(b,z(b))|\Big]+{}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(t,z(t))|\Bigg\} \\
&\le {}^k\mathfrak{J}^{\alpha;\psi}(|\mathfrak{f}(t,z(t))-\mathfrak{f}(t,0)|+|\mathfrak{f}(t,0)|) \\
&\quad +\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big(|\lambda|\ {}^k\mathfrak{J}^{\alpha;\psi}(|\mathfrak{f}(\xi,z(\xi))-\mathfrak{f}(\xi,0)|+|\mathfrak{f}(\xi,0)|) \\
&\quad +|\mu|\ {}^k\mathfrak{J}^{v+\alpha,\psi}(|\mathfrak{f}(\sigma,z(\sigma))-\mathfrak{f}(\sigma,0)|+|\mathfrak{f}(\sigma,0)|) \\
&\quad +{}^k\mathfrak{J}^{\alpha;\psi}(|\mathfrak{f}(b,z(b))-\mathfrak{f}(b,0)|+|\mathfrak{f}(b,0)|)\Big) \\
&\le \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}+\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&\quad +|\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)}+\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\Bigg\}(\mathfrak{L}\|z\|+Q) \\
&\le (\mathfrak{L}r+Q)\mathbb{G}\le r,
\end{aligned}
$$

where (11) has been applied. Consequently, $\|\mathcal{A}z\|\le r$, which means that $\mathcal{A}B_r\subset B_r$.

In order to show that $\mathcal{A}$ is a contraction, let $z,y\in C([a,b],\mathbb{R})$. Then, for $t\in[a,b]$, we obtain

$$
\begin{aligned}
|(\mathcal{A}z)(t)-(\mathcal{A}y)(t)| &\le {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(t,z(t))-\mathfrak{f}(t,y(t))| \\
&\quad +\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big(|\lambda|{}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(\xi,z(\xi))-\mathfrak{f}(\xi,y(\xi)| \\
&\quad +|\mu|\ {}^k\mathfrak{J}^{\alpha+v;\psi}|\mathfrak{f}(\sigma,z(\sigma))-\mathfrak{f}(\sigma,y(\sigma)| \\
&\quad +{}^k\mathfrak{J}^{\alpha;\psi}(|\mathfrak{f}(b,z(b))-\mathfrak{f}(b,y(b))|\Big) \\
&\le \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}+\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&\quad +|\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)}+\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\Bigg\}\mathfrak{L}\|x-y\| \\
&= \mathfrak{L}\mathbb{G}\|x-y\|.
\end{aligned}
$$

Thus, $\|\mathcal{A}x-\mathcal{A}y\|\le\mathfrak{L}\mathbb{G}\|x-y\|$, which shows that the operator $\mathcal{A}$ is a contraction in view of the condition (10). Hence, by Banach's fixed point theorem, the operator $\mathcal{A}$ has a unique fixed point, which is indeed a unique solution of the problem (1) on $[a,b]$. This finishes the proof. □

4.2. Existence Results

In this subsection, we present two existence results for problem (1), which are proved with the aid of Krasnosel'skiĭ's fixed point theorem [29] and nonlinear alternative of the Leray–Schauder type [30].

Theorem 2. *Let the continuous function* $\mathfrak{f} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *satisfy assumption* (H_1). *In addition, we suppose that*

(H_2) $|\mathfrak{f}(t,z)| \le \varphi(t), \quad \forall (t,z) \in [a,b] \times \mathbb{R}$, *and* $\varphi \in C([a,b], \mathbb{R}^+)$.

Then, there exists at least one solution for the problem (1) on $[a,b]$, *if* $\mathbb{G}_1\mathfrak{L} < 1$, *where*

$$\mathbb{G}_1 := \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\left[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]. \tag{12}$$

Proof. Set $\sup_{t\in[a,b]} \varphi(t) = \|\varphi\|$ and $B_\rho = \{z \in C([a,b],\mathbb{R}) : \|z\| \le \rho\}$, with $\rho \ge \|\varphi\|\mathbb{G}$. We define on B_ρ two operators $\mathcal{A}_1$, $\mathcal{A}_2$ by

$$\mathcal{A}_1 z(t) = {}^k\mathfrak{J}^{\alpha;\psi}\mathfrak{f}(t,z(t)), \quad t \in [a,b],$$

$$\begin{aligned}(\mathcal{A}_2 z)(t) &= \frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi}\mathfrak{f}(\xi,z(\xi)) + \mu\, {}^k\mathfrak{J}^{v+\alpha,\psi}\mathfrak{f}(\sigma,z(\sigma)) \\ &\quad - {}^k\mathfrak{J}^{\alpha;\psi}\mathfrak{f}(b,z(b))\Big], \quad t \in [a,b].\end{aligned}$$

For any $z, y \in B_\rho$, we have

$$\begin{aligned} & |(\mathcal{A}_1 z)(t) + (\mathcal{A}_2 y)(t)| \\ \le\ & \sup_{t\in[a,b]}\left\{\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\, {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(\xi,y(\xi))| + |\mu|\, {}^k\mathfrak{J}^{v+\alpha,\psi}|\mathfrak{f}(\sigma,y(\sigma))|\right. \\ & \left. + {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(b,y(b))|\Big] + {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(t,z(t))|\right\} \\ \le\ & \left\{\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\left[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right.\right. \\ & \left.\left. + |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]\right\}\|\varphi\| \\ =\ & \mathbb{G}\|\varphi\| \le \rho. \end{aligned}$$

Therefore, $\|(\mathcal{A}_1 z) + (\mathcal{A}_2 y)\| \le \rho$, which shows that $\mathcal{A}_1 z + \mathcal{A}_2 y \in B_\rho$. In the next step, by using (12), we can easily show that the operator $\mathcal{A}_2$ is a contraction mapping.

Observe that the continuity of $\mathfrak{f}$ implies that of the operator $\mathcal{A}_1$. Additionally, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1 z\| \le \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\varphi\|.$$

Next, we establish equicontinuity of the operator $\mathcal{A}_1$. For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, it is easy show that

$$\begin{aligned} & |(\mathcal{A}_1 z)(t_2) - (\mathcal{A}_1 z)(t_1)| \\ \le\ & \frac{\|\varphi\|}{\Gamma_k(\alpha+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2)-\psi(a))^{\frac{\alpha}{k}} - (\psi(t_1)-\psi(a))^{\frac{\alpha}{k}}|], \end{aligned}$$

which, in the limit $t_2 - t_1 \to 0$, tends to zero independently of $z \in B_\rho$. Thus, $\mathcal{A}_1$ is equicontinuous. Consequently, using the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is completely continuous. Thus, the hypothesis of Krasnosel'skiĭ's fixed point theorem is verified, which guarantees that the problem (1) has at least one solution on $[a,b]$. The proof is complete. □

Theorem 3. *Assume that*

(H_3) $\exists$ a continuous nondecreasing function $\chi : [0,\infty) \to (0,\infty)$ and a function $\sigma \in C([a,b],\mathbb{R}^+)$ satisfying $|\mathfrak{f}(t,z)| \leq \sigma(t)\chi(|z|), \forall (t,z) \in [a,b]\times\mathbb{R}$;

(H_4) $\exists$ a constant $\mathfrak{K} > 0$ such that $\dfrac{\mathfrak{K}}{\chi(\mathfrak{K})\|\sigma\|\mathbb{G}} > 1$.

Then, there exists at least one solution on $[a,b]$ for the problem (1).

Proof. Let us first show that the operator $\mathcal{A}$ defined by (8) maps bounded sets into $C([a,b],\mathbb{R})$. For $r > 0$, let $B_r = \{z \in C([a,b],\mathbb{R}) : \|z\| \leq r\}$. Then, for $t \in [a,b]$, we obtain

$$\begin{aligned}
|(\mathcal{A}z)(t)| &\leq \sup_{t\in[a,b]}\Bigg\{\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\ {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(\xi,z(\xi))| + |\mu|\ {}^k\mathfrak{J}^{v+\alpha,\psi}|\mathfrak{f}(\sigma,z(\sigma))| \\
&\quad + {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(b,z(b))|\Big] + {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(t,z(t))|\Bigg\} \\
&\leq \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&\quad + |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\Bigg\}\|\sigma\|\chi(\|z\|),
\end{aligned}$$

which implies that

$$\|\mathcal{A}z\| \leq \chi(r)\|\sigma\|\mathbb{G}.$$

Next, it is shown that the operator $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C([a,b],\mathbb{R})$. For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $z \in B_r$, we obtain

$$\begin{aligned}
&|(\mathcal{A}z)(t_2) - (\mathcal{A}z)(t_1)| \\
\leq\ & \frac{1}{\Gamma_k(\alpha)}\Bigg|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1} - (\psi(t_1)-\psi(s))^{\frac{\alpha}{k}-1}]\mathfrak{f}(s,z(s))ds \\
&+ \int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1}\mathfrak{f}(s,z(s))ds\Bigg| \\
&+ \frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\ {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(\xi,z(\xi))| \\
&+ |\mu|\ {}^k\mathfrak{J}^{v+\alpha,\psi}|\mathfrak{f}(\sigma,z(\sigma))| + {}^k\mathfrak{J}^{\alpha;\psi}|\mathfrak{f}(b,z(b))|\Big] \\
\leq\ & \frac{\|\sigma\|\chi(r)}{\Gamma_k(\alpha+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2)-\psi(a))^{\frac{\alpha}{k}} - (\psi(t_1)-\psi(a))^{\frac{\alpha}{k}}|], \\
&+ \frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&+ |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\|\sigma\|\chi(r),
\end{aligned}$$

which, as $t_2 - t_1 \to 0$, tends to zero independently of $z \in B_r$. As a consequence, we deduce by the Arzelá-Ascoli theorem that the operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is completely continuous.

Finally, we prove that the set of all solutions to the equation $z = \lambda \mathcal{A} z$ is bounded for $\lambda \in (0,1)$.

As in the first step, one can obtain

$$\frac{\|z\|}{\chi(\|z\|)\|\sigma\|\mathbb{G}} \leq 1.$$

By (H_4), we can find $\mathfrak{K} > 0$ such that $\|z\| \neq \mathfrak{K}$. Consider the set

$$M = \{z \in C([a,b],\mathbb{R}) : \|z\| < \mathfrak{K}\}.$$

Observe that the operator $\mathcal{A} : \overline{M} \to C([a,b],\mathbb{R})$ is continuous and completely continuous. Thus, there does not exist any $z \in \partial M$ satisfying $z = \lambda \mathcal{A} z$ for some $\lambda \in (0,1)$ by the given choice of M. So, $\mathcal{A}$ has a fixed point $z \in \overline{M}$ by the application of the nonlinear alternative of the Leray–Schauder type, which means that there exists at least one solution for the problem (1) on $[0,1]$. This finishes the proof. □

5. The Multi-Valued Problem

For each $z \in C([a,b],\mathbb{R})$, we define the set of selections of $\mathfrak{F}$ as

$$S_{\mathfrak{F},z} := \{f \in L^1([a,b],\mathbb{R}) : f(t) \in \mathfrak{F}(t, z(t)) \text{ on } [a,b]\}.$$

Definition 5. *A continuous function z is said to be a solution of the (k,ψ)-Hilfer type nonlocal integral fractional boundary value problem (2), if it satisfies the boundary conditions $z(a) = 0$, $z(b) = \lambda z(\xi) + \mu\, {}^{k}\mathfrak{J}^{v,\psi} z(\sigma)$, and there exists an integrable function f with $f(t) \in \mathfrak{F}(t,z)$ for a.e. $t \in [a,b]$ such that z satisfies the differential equation ${}^{k,H}D^{\alpha,\beta;\psi} z(t) = f(t)$ on $[a,b]$.*

Our first result for the multi-valued problem (2) is concerned with the case when the multi-valued map $\mathfrak{F}$ has convex values, and relies on the nonlinear alternative of the Leray–Schauder type for multi-valued maps [30].

Theorem 4. *Suppose that:*

(G_1) *$\mathfrak{F} : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory, where $\mathcal{P}_{cp,c}(\mathbb{R}) = \{\mathfrak{R} \in \mathcal{P}(\mathbb{R}) : \mathfrak{R}$ is compact and convex$\}$;*

(G_2) *$\exists$ a continuous nondecreasing function $\mathcal{H} : [0,+\infty) \to (0,+\infty)$ and a positive continuous real valued function q such that, $\forall\, (t,z) \in [a,b] \times \mathbb{R}$,*

$$\|\mathfrak{F}(t,z)\|_{\mathcal{P}} := \sup\{|f| : f \in \mathfrak{F}(t,z)\} \leq q(t)\mathcal{H}(\|z\|);$$

(G_3) *$\exists$ a constant $\mathfrak{K} > 0$ such that*

$$\frac{\mathfrak{K}}{\|q\|\mathcal{H}(\mathfrak{K})\mathbb{G}} > 1,$$

where $\mathbb{G}$ is defined by (9).

Then, the multi-valued problem (2) has at least one solution on $[a,b]$.

Proof. We define an operator $\mathcal{F} : C([a,b],\mathbb{R}) \longrightarrow \mathcal{P}(C([a,b],\mathbb{R}))$ by

$$\mathcal{F}(z) = \left\{ \begin{array}{l} h \in C([a,b],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda \Gamma_k(\theta_k)} \Big[\lambda\, {}^{k}\mathfrak{J}^{\alpha;\psi} f(\xi) + \mu\, {}^{k}\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^{k}\mathfrak{J}^{\alpha;\psi} f(b)\Big] \\ + {}^{k}\mathfrak{J}^{\alpha;\psi} f(t),\ t \in [a,b],\ f \in S_{\mathfrak{F},z}. \end{array} \right. \end{array} \right\}$$

Notice that the fixed points of $\mathcal{F}$ are solutions to the problem (2).

We split the proof into several steps.

Step 1. $\mathcal{F}(z)$ *is convex, for each* $z \in C([a,b],\mathbb{R}$,

Since $S_{F,z}$ is convex, this step is obvious, and so the proof is omitted.

Step 2. Bounded sets are mapped by $\mathcal{F}$ *into bounded sets in* $C([a,b],\mathbb{R})$.

Let $B_r = \{z \in C([a,b],\mathbb{R}) : \|z\| \leq r\}, r > 0$. Then, for each $h \in \mathcal{F}(z)$, $z \in B_r$, there exists $f \in S_{\mathfrak{F},z}$ such that

$$h(t) = \frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f(b)\Big] + {}^k\mathfrak{J}^{\alpha;\psi} f(t).$$

Further, for $t \in [a,b]$, we have

$$\begin{aligned}
|h(t)| &\leq \sup_{t\in[a,b]}\Bigg\{\frac{(\psi(t)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\, {}^k\mathfrak{J}^{\alpha;\psi}|f(\xi)| + |\mu|\, {}^k\mathfrak{J}^{\alpha+v;\psi}|f(\sigma)| + {}^k\mathfrak{J}^{\alpha;\psi}|f(b)|\Big] \\
&\quad + {}^k\mathfrak{J}^{\alpha;\psi}|f(t)|\Bigg\} \\
&\leq \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&\quad + |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\Bigg\}\|q\|\mathcal{H}(\|z\|),
\end{aligned}$$

which implies that

$$\|h\| \leq \mathcal{H}(r)\|q\|\mathbb{G}.$$

Step 3. $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([a,b],\mathbb{R})$.

Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $z \in B_r$. Then, for each $h \in \mathcal{F}(z)$, we find that

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\Bigg|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1} - (\psi(t_1)-\psi(s))^{\frac{\alpha}{k}-1}]f(s)ds \\
&\quad + \int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1}f(s)ds\Bigg| \\
&\quad + \frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Big[|\lambda|\, {}^k\mathfrak{J}^{\alpha;\psi}|f(\xi)| \\
&\quad + |\mu|\, {}^k\mathfrak{J}^{\alpha+v;\psi}|f(\sigma)| + {}^k\mathfrak{J}^{\alpha;\psi}|f(b)|\Big] \\
&\leq \frac{\|q\|\mathcal{H}(r)}{\Gamma_k(\alpha+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2)-\psi(a))^{\frac{\alpha}{k}} - (\psi(t_1)-\psi(a))^{\frac{\alpha}{k}}|], \\
&\quad + \frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)}\Bigg[|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&\quad + |\mu|\frac{(\psi(b)-\psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha+v+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\Bigg]\|q\|z(r),
\end{aligned}$$

which shows that $|h(t_2) - h(t_1)| \to 0$ independently of $z \in B_r$ as $t_2 - t_1 \to 0$. Therefore, $\mathcal{F} : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ is completely continuous by the application of the Arzelá–Ascoli theorem.

In the next step, we show that $\mathcal{F}$ has a closed graph, which is equivalent to the fact that $\mathcal{F}$ is a upper semi-continuous multivalued map by Proposition 1.2 in [31].

Step 4. $\mathcal{F}$ has a closed graph.

Let $z_n \to z_*, h_n \in \mathcal{F}(z_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathcal{F}(z_*)$. Since $h_n \in \mathcal{F}(z_n)$, there exists $v_n \in S_{\mathfrak{F},z_n}$ such that, for each $t \in [a,b]$,

$$h_n(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f_n(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f_n(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_n(b)\Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_n(t).$$

Thus, we must show that there exists $v_* \in S_{\mathfrak{F},z_*}$ such that, for each $t \in [a,b]$,

$$h_*(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f_*(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f_*(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_*(b)\Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_*(t).$$

Let us consider the linear operator $\Omega : L^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ given by

$$f \mapsto \Omega(f)(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f(b)\Big] + {}^k\mathfrak{J}^{\alpha;\psi} f(t).$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$, as $n \to \infty$. Consequently, $\Omega \circ S_{\mathfrak{F}}$ is a closed graph operator, by a Lazota–Opial result [32]. Further, we obtain $h_n(t) \in \Omega(S_{\mathfrak{F},z_n})$. Since $z_n \to z_*$, we obtain

$$h_*(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f_*(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f_*(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_*(b)\Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_*(t),$$

for some $v_* \in S_{\mathfrak{F},z_*}$.

Step 5. We prove that there exists an open set $\mathcal{U} \subseteq C([a,b],\mathbb{R})$ such that, for any $\nu \in (0,1)$ and all $z \in \partial\mathcal{U}$, we have $z \notin \nu\mathcal{F}(z)$.

Assume that $z \in \nu\mathcal{F}(z)$ for $\nu \in (0,1)$. Then, there exists $f \in L^1([a,b],\mathbb{R})$ with $f \in S_{\mathfrak{F},z}$ such that, for $t \in [a,b]$, we have

$$z(t) = \nu\frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Lambda\Gamma_k(\theta_k)}\Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f(b)\Big] + \nu\, {}^k\mathfrak{J}^{\alpha;\psi} f(t).$$

Then, as in the second step, one can obtain

$$\frac{\|z\|}{\|q\|\mathcal{H}(\|z\|)\mathbb{G}} \leq 1.$$

By (H_3), there exists $\mathfrak{K}$ such that $\|z\| \neq \mathfrak{K}$. Let us set

$$\mathcal{U} = \{z \in C([a,b],\mathbb{R}) : \|z\| < \mathfrak{K}\}.$$

From the preceding arguments, $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{P}(C([a,b],\mathbb{R}))$ is a compact and upper semi-continuous multivalued map with convex closed values. By definition of $\mathcal{U}$, there does not exist any $z \in \partial\mathcal{U}$ such that $z \in \nu\mathcal{F}(z)$ for some $\nu \in (0,1)$. Hence, it follows by the nonlinear alternative of the Leray–Schauder type for multi-valued maps [30] that $\mathcal{F}$ has a fixed point $z \in \overline{\mathcal{U}}$, which is indeed a solution to the multi-valued problem (2). The proof is complete. $\square$

Now, we apply the fixed point theorem for multivalued contractive maps suggested by Covitz and Nadler [33] to show that there exists a solution to the problem (2) when $\mathfrak{F}$ is not necessarily a convex valued map.

Theorem 5. *Assume that*

(A_1) $\mathfrak{F} : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathfrak{f}(\cdot, z) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $z \in \mathbb{R}$, *where* $\mathcal{P}_{cp}(\mathbb{R}) = \{\mathfrak{V} \in \mathcal{P}(\mathbb{R}) : \mathfrak{V} \text{ is compact}\}$;

(A_2) $\exists$ *a function* $m \in C([a,b], \mathbb{R}^+)$ *such that*

$$H_d(\mathfrak{F}(t,z), \mathfrak{F}(t,\bar{z})) \leq m(t)|z - \bar{z}|,$$

with $d(0, \mathfrak{F}(t,0)) \leq m(t)$ *for almost all* $t \in [a,b]$ *and* $z, \bar{z} \in \mathbb{R}$.

Then, the problem (2) has at least one solution on $[a,b]$, *provided that*

$$\delta := \mathbb{G}\|m\| < 1, \tag{13}$$

where $\mathbb{G}$ *is given by* (9).

Proof. By the assumption (A_1), the set $S_{\mathfrak{F},z}$ is nonempty for each $z \in C([a,b], \mathbb{R})$. Hence, by implementing Theorem III.6 [34], $\mathfrak{F}$ has a measurable selection. We now prove that $\mathcal{F}(z) \in \mathcal{P}_{cl}(C([a,b], \mathbb{R}))$ for each $z \in C([a,b], \mathbb{R})$. Consider $\{z_n\}_{n \geq 0} \in \mathcal{F}(z)$ such that $z_n \to z$ $(n \to \infty)$ in $C([a,b], \mathbb{R})$. Then, we have $z \in C([a,b], \mathbb{R})$ and there exists $v_n \in S_{\mathfrak{F},z_n}$ such that, for each $t \in [a,b]$,

$$z_n(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\Lambda \Gamma_k(\theta_k)} \Big[\lambda \, {}^k\mathfrak{J}^{\alpha;\psi} f_n(\xi) + \mu \, {}^k\mathfrak{J}^{\alpha+v;\psi} f_n(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_n(b) \Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_n(t).$$

Since $\mathfrak{F}$ has compact values, there exists v_n which converges to v in $L^1([a,b], \mathbb{R})$. Hence, $v \in S_{\mathfrak{F},z}$ and for each $t \in [a,b]$, we have

$$z_n(t) \to z(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\Lambda \Gamma_k(\theta_k)} \Big[\lambda \, {}^k\mathfrak{J}^{\alpha;\psi} f(\xi) + \mu \, {}^k\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f(b) \Big] + {}^k\mathfrak{J}^{\alpha;\psi} f(t).$$

Thus, $z \in \mathcal{F}(z)$.

Next, we show that

$$H_d(\mathcal{F}(z), \mathcal{F}(\bar{z})) \leq \delta \|z - \bar{z}\|, \quad \delta < 1, \quad \text{for each } z, \bar{z} \in C^2([a,b], \mathbb{R}).$$

Let $z, \bar{z} \in C^2([a,b], \mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then $\exists$ $v_1(t) \in \mathfrak{F}(t, z(t))$ such that, for each $t \in [a,b]$,

$$h_1(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\Lambda \Gamma_k(\theta_k)} \Big[\lambda \, {}^k\mathfrak{J}^{\alpha;\psi} f_1(\xi) + \mu \, {}^k\mathfrak{J}^{\alpha+v;\psi} f_1(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_1(b) \Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_1(t).$$

Using (A_2), we obtain

$$H_d(\mathfrak{F}(t,z), \mathfrak{F}(t,\bar{z})) \leq m(t)|z(t) - \bar{z}(t)|.$$

So, $\exists$ $w \in \mathfrak{F}(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|z(t) - \bar{z}(t)|, \quad t \in [a,b].$$

Define $\mathcal{V} : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|z(t) - \bar{z}(t)|\}.$$

According to Proposition III.4 [34], the multivalued operator $\mathcal{V}(t) \cap \mathfrak{F}(t, \bar{z}(t))$ is measurable, and thus $\exists$ a function $v_2(t)$ which is a measurable selection for $\mathcal{V}$. So $v_2(t) \in \mathfrak{F}(t, \bar{z}(t))$ and for each $t \in [a, b]$, we have $|v_1(t) - v_2(t)| \leq m(t)|z(t) - \bar{z}(t)|$.

For each $t \in [a, b]$, let us define

$$h_2(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\Lambda \Gamma_k(\theta_k)} \Big[\lambda\, {}^k\mathfrak{J}^{\alpha;\psi} f_2(\xi) + \mu\, {}^k\mathfrak{J}^{\alpha+v;\psi} f_2(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_2(b) \Big] + {}^k\mathfrak{J}^{\alpha;\psi} f_2(t).$$

Then, we have

$$\begin{aligned} |h_1(t) - h_2(t)| \ \leq\ & \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Lambda| \Gamma_k(\theta_k)} \Big[|\lambda|\, {}^k\mathfrak{J}^{\alpha;\psi}(|f_1(s) - f_2(s)|)(\xi) \\ & + |\mu|\, {}^k\mathfrak{J}^{\alpha+v;\psi}(|f_1(s) - f_2(s)|)(\sigma) + {}^k\mathfrak{J}^{\alpha;\psi}(|f_1(s) - f_2(s)|)(b) \Big] \\ & + {}^k\mathfrak{J}^{\alpha;\psi}(|f_1(s) - f_2(s)|)(t) \\ \leq\ & \Bigg\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Lambda| \Gamma_k(\theta_k)} \Bigg[|\lambda| \frac{(\psi(\xi) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \\ & + |\mu| \frac{(\psi(b) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \Bigg] \Bigg\} \|m\| \|z - \bar{z}\| \\ =\ & \mathbb{G} \|m\| \|z - \bar{z}\|. \end{aligned}$$

Hence

$$\|h_1 - h_2\| \leq \mathbb{G} \|m\| \|z - \bar{z}\|.$$

Analogously, by interchanging the roles of z and $\bar{z}$, we obtain

$$H_d(\mathcal{F}(z), \mathcal{F}(\bar{z})) \leq \mathbb{G} \|m\| \|z - \bar{z}\|.$$

So, $\mathcal{F}$ is a contraction, and thus $\mathcal{F}$ has a fixed point z by application of the Covitz and Nadler theorem [33]. Consequently, there exists at least one solution on $[a, b]$ to the problem (2). The proof is finished. $\square$

6. Examples

In this section, some examples, illustrating the obtained theoretical results in the previous section, are presented.

Consider the following (k, ψ)-Hilfer type nonlocal integral fractional boundary value problem

$$\begin{cases} {}^{\frac{13}{10}, H}D^{\frac{8}{5}, \frac{3}{4}; \sqrt{t}+1} z(t) = \mathfrak{f}(t, z(t)), & \dfrac{1}{3} < t < \dfrac{7}{3}, \\ z\left(\dfrac{1}{3}\right) = 0, \qquad z\left(\dfrac{7}{3}\right) = \dfrac{1}{\pi} z\left(\dfrac{2}{3}\right) + \dfrac{6}{17}\, {}^{\frac{13}{10}}\mathfrak{J}^{\frac{4}{3}, \sqrt{w}+1} z\left(\dfrac{5}{3}\right). \end{cases} \tag{14}$$

Here, $k = 13/10$, $\alpha = 8/5$, $\beta = 3/4$, $\psi(t) = \sqrt{t} + 1$, $a = 1/3$, $b = 7/3$, $\lambda = 1/\pi$, $\xi = 2/3$, $\mu = 6/17$, $v = 4/3$, $\sigma = 5/3$, and we can compute that $\theta_{\frac{13}{10}} = 47/20$, $\Lambda \approx 0.9298349818$, $\mathbb{G} \approx 1.182496327$, $\mathbb{G}_1 \approx 0.5758310296$.

Example 1. *Let $\mathfrak{f}$ be a nonlinear unbounded Lipschitz function defined by*

$$\mathfrak{f}(t, z) = \left(\frac{z^2 + 10|z|}{2(6 + |z|)} \right) \cos^2 \pi w + \frac{1}{3} w^4 + 2w^3 + 1. \tag{15}$$

Clearly, $\mathfrak{f}$ *satisfies the Lipschitz condition with constant* $\mathfrak{L} = 5/6$ *as*

$$|\mathfrak{f}(t,z_1) - \mathfrak{f}(t,z_2)| \leq \frac{5}{6}|z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{R}$ *and* $t \in [1/3, 7/3]$*. Moreover,* $\mathfrak{L}\mathbb{G} \approx 0.9854136058 < 1$*. Hence, by Theorem 1, the problem* (14) *with* $\mathfrak{f}$ *given in* (15)*, has a unique solution on* $[1/3, 7/3]$*.*

Example 2. *Let a nonlinear bounded Lipschitz function* $\mathfrak{f}$ *be given by*

$$\mathfrak{f}(t,z) = \left(\frac{10|z|}{6+|z|}\right)e^{-|3w-1|} + \frac{1}{4}w^3 + \frac{1}{3}w^2 + \frac{4}{5}, \tag{16}$$

Note that

$$|\mathfrak{f}(t,z)| \leq 10e^{-|3w-1|} + \frac{1}{4}w^3 + \frac{1}{3}w^2 + \frac{4}{5} := \varphi(t),$$

for all $t \in [1/3, 7/3]$*. Observe that* $\mathfrak{f}$ *satisfies the Lipschitz condition* $|\mathfrak{f}(t,z_1) - \mathfrak{f}(t,z_2)| \leq (10/6)|z_1 - z_2|$*, with the Lipschitz constant* $\mathfrak{L} = 5/3$*. Using the given data, we obtain* $\mathfrak{L}\mathbb{G}_1 \approx 0.9597183827 < 1$*. As a consequence, the conclusion of Theorem 2 applies and hence, the problem* (14) *with* $\mathfrak{f}$ *given by (16) has at least one solution on* $[1/3, 7/3]$*. It is imperative to notice that the uniqueness of the solution for this problem cannot be guaranteed since* $\mathfrak{L}\mathbb{G} \approx 1.970827212 > 1$*.*

Example 3. *Consider the function*

$$\mathfrak{f}(t,z) = \sigma(t)(Ag_1(z) + B), \tag{17}$$

where $\sigma : [1/3, 7/3] \to \mathbb{R}^+$*,* $g_1 : \mathbb{R} \to \mathbb{R}$ *with* $|g_1(z)| \leq |z|$*,* $0 \leq A < 1/(\|\sigma\|\mathbb{G})$ *and* $B > 0$*. Then, we have*

$$|\mathfrak{f}(t,z)| \leq \|\sigma\|(A|z| + B).$$

Setting $\chi(u) = A|u| + B$*, we can find a constant* $\mathfrak{K}$ *satisfying the condition* (H_4) *of Theorem 3 as*

$$\mathfrak{K} > \frac{B\|\sigma\|\mathbb{G}}{1 - A\|\sigma\|\mathbb{G}}.$$

By applying Theorem 3, we deduce that the problem (14) *with* $\mathfrak{f}$ *given in* (17) *has at least one solution on* $[1/3, 7/3]$*.*

Example 4. *Let* $\mathfrak{f}$ *be defined by*

$$\mathfrak{f}(t,z) = \sigma(t)(Ag_2(z) + B), \tag{18}$$

where $\sigma : [1/3, 7/3] \to \mathbb{R}^+$*,* $g_2 : \mathbb{R} \to \mathbb{R}$ *with* $|g_2(z)| \leq z^2$*, and* A, B *are positive constants with* $AB < 1/(4\|\sigma\|^2\mathbb{G}^2)$*. Then, we obtain*

$$|\mathfrak{f}(t,z)| \leq \|\sigma\|\left(Az^2 + B\right).$$

Choosing a function $\chi(u) = Au^2 + B$*, we find a constant*

$$\mathfrak{K} \in \left(\frac{1 - \sqrt{1 - 4AB\|\sigma\|^2\mathbb{G}^2}}{2A\|\sigma\|\mathbb{G}}, \frac{1 + \sqrt{1 - 4AB\|\sigma\|^2\mathbb{G}^2}}{2A\|\sigma\|\mathbb{G}}\right),$$

satisfying the condition (H_4) *of Theorem 3. Thus, all the assumptions of Theorem 3 are satisfied. Hence, the problem* (14) *with* $\mathfrak{f}$ *given by* (18) *has at least one solution on* $[1/3, 7/3]$*.*

Example 5. *Assume that the first equation of* (14) *is replaced by*

$$ {}^{\frac{13}{10},H}D^{\frac{8}{5},\frac{3}{4};\sqrt{t}+1}z(t) \in \mathfrak{F}(t,z(t)), \quad \frac{1}{3} < t < \frac{7}{3}, \tag{19} $$

where

$$ \mathfrak{F}(t,z) = \left[0, \frac{1}{(3t+2)}\left(\frac{z^2+2|z|}{1+|z|} + \sin^2 t\right)\right]. $$

Observe that $\mathfrak{F}(t,z)$ *is a measurable set. Additionally,*

$$ H_d(\mathfrak{F}(t,z), \mathfrak{F}(t,\bar{z})) \leq \frac{2}{(3t+2)}|z-\bar{z}|. $$

Now, we set $m(t) = 2/(3t+2)$ *such that* $d(0,\mathfrak{F}(t,0)) \leq 1/(3t+2) < 2/(3t+2) = m(t)$ *for almost all* $t \in [1/3, 7/3]$. *From* $\delta = \mathbb{G}\|m\| \approx 0.7883308847 < 1$, *we deduce that the* (k,ψ)*-Hilfer type fractional inclusion* (19) *with nonlocal integral boundary conditions given in* (14), *has at least one solution on* $[1/3, 7/3]$.

7. Conclusions

We presented the existence criteria for solutions to the (k,ψ)-Hilfer type fractional differential equations and inclusions of order in $[1,2]$ complemented with nonlocal integral boundary conditions. We first transformed the nonlinear (k,ψ)-Hilfer type fractional boundary value problem into a fixed point problem. For the single-valued case, we established existence and uniqueness results by applying the Banach contraction mapping principle, Krasnosel'skiĭ fixed point theorem and the Leray–Schauder nonlinear alternative. Our first existence result dealing with the convex-valued multi-valued map involved in the inclusion was established by applying the Leray–Schauder nonlinear alternative for multivalued maps, while the existence result for the non-convex valued multivalued map in the inclusion was obtained by applying the Covitz–Nadler fixed point theorem for contractive multivalued maps. It is worthwhile to mention that the work established for (k,ψ)-Hilfer fractional differential equations supplemented with nonlocal (k,ψ)-Riemann–Liouville fractional integral boundary conditions is more general and significant as the (k,ψ)-Riemann–Liouville and (k,ψ)-Caputo fractional derivatives are special cases of the (k,ψ)-Hilfer fractional derivative. Moreover, the (k,ψ)-Riemann–Liouville fractional integral operator used in the boundary conditions is of a more general nature.

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