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Integral Results Related to Similarly Separable Vectors in Separable Hilbert Spaces

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Abstract: In this work, we use similarly separable vectors in separable Hilbert spaces to provide generalized integral results related to majorization, Niezgoda, and Ćebysév type inequalities. Next, we furnish some refinements of these inequalities. Theorems obtained in this work extend and improve several known results in the literature. An important aspect of our work is that these inequalities are directly related to Arithmetic, Geometric, Harmonic, and Power means. These means have played an important role in many branches of arts and sciences since the last 2600 years.

Keywords: Jensen's inequality; separable Hilbert spaces; similarly separable vectors; majorization; Čebyšev inequality; Niezgoda's inequality

MSC: 26A51; 39B62; 26D15; 26D20; 26D99

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1. Introduction

The core of mathematics is to generalize concepts and results. Therefore, in the proposed research our aim is to generalize some classical and celebrated inequalities including Jensen's inequality, Chebysev's inequality, Andersson's inequality, Slater's inequality etc. For this purpose, we will use the notion of similarly separable vectors in separable Hilbert spaces. This notion of similarly separable vectors (sequences) was introduced by Marek Niezgoda in [1]. This concept is a natural generalization of monotone sequences and synchronous sequences. It plays a central role in proving a class of linear inequalities, such as Chebyshev's inequality and Andersson's inequality.

We begin by recalling the basic integral inequalities for convex functions. Throughout this article, I and $[\beta_0, \beta_1]$ are intervals in $\mathbb{R}$.

We recall the integral version of Jensen's inequality for convex functions [2], p. 58. It relates the value of the integral of a convex function to that of a convex function of the integral.

Proposition 1. *Let $f : [\beta_0, \beta_1] \to \mathbb{R}$ be a continuous function. If $\varphi : [\beta_0, \beta_1] \to \mathbb{R}$ is a nondecreasing, bounded function and $\varphi(\beta_0) \neq \varphi(\beta_1)$; then, the inequality*

$$\phi\left(\frac{\int_{[\beta_0,\beta_1]} f(t)d\varphi(t)}{\int_{[\beta_0,\beta_1]} d\varphi(t)}\right) \leq \frac{\int_{[\beta_0,\beta_1]} \phi(f(t))d\varphi(t)}{\int_{[\beta_0,\beta_1]} d\varphi(t)} \tag{1}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$.

Steffensen presented a generalized form of Jensen's Integral inequality, which we refer to as Jensen–Steffensen's integral inequality [2], p. 59. This may be stated as:

Proposition 2. *Assume φ is continuous or has bounded variation and satisfies $\varphi(\beta_0) \le \varphi(x) \le \varphi(\beta_1)$ $\forall x \in [\beta_0, \beta_1]$, $\varphi(\beta_0) < \varphi(\beta_1)$ and f is continuous and monotonic. Then inequality (1) holds.*

For other variants and related generalized results of the topic, we refer the reader to [1,3–8].

Separable Hilbert Spaces

In this article, we take U as an open subset in a separable Hilbert space H, with a suitable inner product denoted by $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$. It is a known fact that every separable Hilbert space has a countable orthonormal basis [9]. Separable Hilbert spaces possess many interesting properties ([9,10]).

Here, we recall some definitions from [7]: let $\Xi = \{e_i : i \in \bar{\mathbb{N}}\}$ be an ordered basis of H and $\Theta = \{d_i : i \in \bar{\mathbb{N}}\}$ the dual basis of H. For $i, j \in \bar{\mathbb{N}}$, we have $\langle e_i, d_j \rangle = \delta_{ij}$ (Kronecker delta), where $\bar{\mathbb{N}} \subseteq \mathbb{N}$ (the dimensions of H can be finite or infinite). We define Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

In this article, we will be using definitions of Ξ-positive, μ, v-separable, and v-separable vectors as stated below [7]. Additionally, throughout this chapter we assume J_1 and J_2 are index sets with $J_1 \bigcup J_2 = J$.

Definition 1. *A vector $v \in H$ is $\Xi-$positive if $\langle e_i, v \rangle > 0\ \forall\ i$ where $i \in \bar{\mathbb{N}}$.*

We denote $J = \bar{\mathbb{N}}$. And $J_1 \cup J_2 = J$ where J_1 and J_2 be two sets of indices.

Definition 2. *Given $\mu \in \mathbb{R}$ and $v \in H$, a vector $z \in H$ is $\mu, v-$separable w.r.t. a basis Ξ on J_1 and J_2, if $\langle e_i, z - \mu v \rangle \ge 0$ for $i \in J_1$ and $\langle e_j, z - \mu v \rangle \le 0$ for $j \in J_2$.*

Definition 3. *A vector $z \in H$ is v-separable w.r.t. Ξ on J_1 and J_2, if z is $\mu, v-$separable on J_1 and J_2 for some $\mu \in \mathbb{R}$.*

Definition 4. *A map $\psi : I \to \mathbb{R}$ preserves $v-$separability on J_1 and J_2 w.r.t. Ξ, if $\psi(z)$ is $v-$separable on J_1 and J_2 w.r.t. Ξ given that $z \in H$ is $v-$separable on J_1 and J_2 w.r.t. Ξ.*

Definition 5. *Let $f, g, v, y \in V$ and $\lambda, \mu \in \mathbb{R}$. The vectors f, v are said to be similarly separable w.r.t. $(\lambda, g, \Xi; \mu, y, \Theta)$ if:*

(i) *f is $\lambda, g-$separable w.r.t. Ξ on J_1 and J_2,*
(ii) *v is $\mu, y-$separable w.r.t. Θ on J_1 and J_2.*

This article consists of primarily three sections. In first section, we recall the basic definitions and previously proven inequalities. It also provides some basic notions related to similarly separable vectors. Section 2 presents some important results, which include the integral version of Niezgoda's inequality for similarly separable vectors in Hilbert spaces. Section 3 follows by providing a refinement of our main result, which we proved in Section 2. Section 4 includes some applications, where we define and compare different means by making use of our refined inequality.

2. Generalization of Niezgoda's Inequality

In this section, we generalize Niezgoda's Inequality using Similarly Separable Vectors in Separable Hilbert Spaces. For that purpose, we recall Theorem 3.5 of [1]:

Proposition 3. *Take Ξ as a basis of H with inner product defined as $\langle .,. \rangle : H \times H \to \mathbb{R}$, let Θ is the dual basis of Ξ. Let f, g, v, and y be vectors in H. Denote $\lambda = \langle f, v\rangle / \langle y, v\rangle$, where $\langle y, v\rangle \neq 0$. Under these conditions, the following are equivalent:*

(i) *The vector g is $\lambda, y-$separable w.r.t. Θ on $\jmath_1$ and $\jmath_2$ if $\langle y, v\rangle > 0$ (or w.r.t. Θ on $\jmath_2$ and $\jmath_1$ if $\langle y, v\rangle < 0$).*

(ii) *The inequality*

$$\langle f, g\rangle \langle y, v\rangle \geq \langle f, y\rangle \langle g, v\rangle \tag{2}$$

holds $\forall$ vectors f which are $v-$separable w.r.t. Ξ on $\jmath_1$ and $\jmath_2$.

Remark 1. *This result has many important consequences as stated by Niezgoda in [1]. Niezgoda chose $H = \mathbb{R}^k$ where $\bar{\mathbb{N}} = \{1, \ldots, k\}$ for some fixed $k \in \mathbb{N}$ and standard inner product in Proposition 3 and stated all the related results and corollaries for the discrete version in [1]. Here, we are interested in its integral version.*

Consider a measure space $([\beta_0, \beta_1], \Sigma, \eta)$. Let $w : [\beta_0, \beta_1] \to [0, \infty)$ be a measurable function with $w \not\equiv 0$ on a set of nonzero measure. We define the $w-$weighted L_2 space as $L_2([\beta_0, \beta_1], wd\eta)$, where $wd\eta$ means the measure M defined by $M(A) \equiv \int_A w(x)\mathrm{d}\eta(x)$ and $A \in \Sigma$. The inner product for $L_2([\beta_0, \beta_1], wd\eta)$ is

$$\langle f, g\rangle := \langle f, g\rangle_{L_2([\beta_0,\beta_1], w\,\mathrm{d}\eta)} = \int_{\beta_0}^{\beta_1} w(q) f(q) g(q) d\eta(q). \tag{3}$$

Corollary 1. *Let f, g, v, and y be vectors in $L_2([\beta_0, \beta_1], wd\eta)$. Denote $\lambda = \langle f, v\rangle / \langle y, v\rangle$ where $\langle y, v\rangle > 0$. Assume that Ξ is a basis of $L_2([\beta_0, \beta_1], wd\eta)$ and Θ is the dual basis of Ξ.*

If

(i) *f is $v-$separable w.r.t. Ξ on $\jmath_1$ and $\jmath_2$ and*

(ii) *g is λ, y-separable w.r.t. Θ on $\jmath_1$ and $\jmath_2$,*

then

$$\int_{\beta_0}^{\beta_1} w(q)f(q)y(q)d\eta(q) \int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \leq \int_{\beta_0}^{\beta_1} w(q)f(q)g(q)d\eta(q) \int_{\beta_0}^{\beta_1} w(q)v(q)y(q)d\eta(q). \tag{4}$$

holds.

Remark 2. *Take $y \equiv 1$ and $v \equiv 1$ in (4); then, using Lebesgue measure we obtain the well-known Čebyšev inequality [2], p. 197:*

Corollary 2. *Let $f, g : [\beta_0, \beta_1] \to \mathbb{R}$ s.t f and g are monotonic in the same direction. Let $w : [\beta_0, \beta_1] \to [0, \infty)$ be an integrable function. Then*

$$\int_{\beta_0}^{\beta_1} w(q)f(q)dt \int_{\beta_0}^{\beta_1} w(q)g(q)dt \leq \int_{\beta_0}^{\beta_1} w(q)dt \int_{\beta_0}^{\beta_1} w(q)f(q)g(q)dt \tag{5}$$

if the integrals exist.

The reverse inequality (5) holds if g and f are monotonic in opposite directions.

Equality in (5) holds in either cases iff either one of g or f is constant a.e.

Remark 3. *The inequality (5) still hold under different assumptions. For detailed discussion on inequality (5), we refer [2], pp. 198–199.*

Now, we recall a few important results from [11] as under:

Proposition 4. *A linear functional $\digamma$ in a normed linear space with domain $D(\digamma)$ is continuous if and only if $\digamma$ is bounded.*

We now state the "Riesz Representation Theorem" [11].

Proposition 5. *For each linear functional $\mathcal{F}$ that is bounded on a Hilbert space H, there is an inner product representation written as:*

$$\mathcal{F}(t) = \langle t, v \rangle, \tag{6}$$

where v depends upon $\mathcal{F}$ and has a unique value. The norm of v is:

$$\|v\| = \|\mathcal{F}\| = \sup_{0 \neq t \in D(\mathcal{F})} \frac{|\mathcal{F}(t)|}{\|t\|}.$$

If U is an open convex subset in V where V is a normed linear space, then a convex function ϕ on U generates a supporting hyperplane at every point $t_0 \in U$ [12], p. 128. This implies the presence of a linear functional $\mathcal{F}$ that is continuous on V and is characterized as

$$\phi(t) \geq \phi(t_0) + \mathcal{F}(t - t_0) \; \forall \; t \in U. \tag{7}$$

The functionals $\mathcal{F}$ are known as the support of ϕ at t_0, and the subdifferential of ϕ at the point t_0 is established through the set $\partial\phi(t_0)$ of all functionals $\mathcal{F}$.

Now, we consider Hilbert spaces: if V is a Hilbert space, then the continuous linear functional $\mathcal{F}$ as defined in (7) would be bounded by Proposition 4 and hence we fulfill all the requirements of Proposition 5. Bringing in use the Riesz representation theorem, we have a unique representation of all such functionals $\mathcal{F}$ as $\mathcal{F}(t) = \langle t, v \rangle$ for $t \in V$ such that $\|\mathcal{F}\| = \|v\|$.

In this case inequality, (7) becomes

$$\phi(t) \geq \phi(t_0) + \langle t - t_0, v \rangle \quad \text{for all} \quad t \in U. \tag{8}$$

The set of all such vectors v (termed subgradients) constitute the subdifferential $\partial\phi(t_0)$. When V is in $\mathbb{R}^k$, the inequality (8) becomes

$$\phi(\mathbf{z}) \geq \phi(\mathbf{z_0}) + \langle \mathbf{z} - \mathbf{z_0}, \Phi(\mathbf{z_0}) \rangle \quad \text{for all} \quad \mathbf{z} \in U, \tag{9}$$

where $\Phi(\mathbf{z}) = (\Phi(z_1), \ldots, \Phi(z_k))$ for $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{R}^k$ and the set of all functions Φ (usually called subgradients) constitute of the subdifferential $\partial\phi(\mathbf{z_0})$ (see, e.g., [12,13]).

We now present our first result:

Theorem 1. *Consider an open subset U of H. Let $\psi : U \to \mathbb{R}$ be a convex function defined on U. Let $\partial\psi : U \to \mathbb{R}$ be the subdifferential of ψ and let $\Psi \in \partial\psi$. Assume that Ξ is a basis of H with inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ and Θ is the dual basis of Ξ. Denote $\lambda = \langle g - f, v \rangle / \langle y, v \rangle$ where f, g, v, and y are vectors in H with $\langle y, v \rangle \neq 0$. If*

(i) f is $v-$separable w.r.t. Ξ on J_1 and J_2,
(ii) $g - f$ is $\lambda, y-$separable w.r.t. Θ on J_1 and J_2 and
(iii) Φ preserves $v-$separability w.r.t. Ξ on J_1 and J_2.

(a) If $\langle g - f, v \rangle = 0$, then

$$\langle \psi(g) - \psi(f), 1 \rangle \geq 0 \tag{10}$$

holds.

(b) If $\langle g - f, v \rangle \geq 0$ and $\langle \Phi(f), y \rangle \geq 0$ then inequality (10) holds.

Proof.

(a) Using the definition of subdifferential, we have:

$$\psi(g) - \psi(f) \geq \langle g - f, \Phi(f) \rangle, \tag{11}$$

Consider conditions (i) and (iii), we note that the vector $\Phi(f)$ is $v-$separable w.r.t. Ξ on J_1 and J_2. Using Proposition 3, we get

$$\langle g-f,\Phi(f)\rangle \geq \frac{1}{\langle y,v\rangle}\langle g-f,v\rangle\langle\Phi(f),y\rangle \tag{12}$$

since $\langle y,v\rangle > 0$. So, if $\langle g-f,v\rangle = 0$, then (10) follows from (11) and (12).

(b) Clearly, (10) holds whenever $\langle g-f,v\rangle \geq 0$ and $\langle\Phi(f),y\rangle \geq 0$ by using inequalities (11) and (12).

□

Remark 4. *Theorem 2.2 of [8] becomes a special case of our result by choosing $H = \mathbb{R}^k$ with weighted inner product on $\mathbb{R}^k$ for positive real weights $\mathbf{p} = (p_1,\ldots,p_k)$ and $\mathbf{x} = (x_1,\ldots,x_k), \mathbf{y} = (y_1,\ldots,y_k) \in \mathbb{R}^k$ defined as:*

$$\langle\mathbf{x},\mathbf{y}\rangle = \sum_{i=1}^{k} p_i x_i y_i, \tag{13}$$

Additionally, we can easily obtain its corollaries and examples. Here, we are interested in one of its consequences in integral version.

Remark 5. *In Theorem 1, by choosing $H = L_2([\beta_0,\beta_1], wd\eta)$ with inner product as defined in (3), we get the following integral majorization inequality:*

Corollary 3. *Consider an open interval I of $\mathbb{R}$ and let $\psi : I \to \mathbb{R}$ be a convex function, and $\partial\psi : I \to \mathbb{R}$ be the subdifferential of ψ and $\Psi \in \partial\psi$.*

Let $([\beta_0,\beta_1],\Sigma,\eta)$ be a measure space with positive finite measure η, and $f, g : [\beta_0,\beta_1] \to I$ be two functions s.t. $g, f \in L_2([\beta_0,\beta_1], wd\eta)$, where w be a non-negative measurable function on $[\beta_0,\beta_1]$ with $w \not\equiv 0$ on a set of nonzero measure.

Assume that Ξ is an ordered basis in $L_2([\beta_0,\beta_1], wd\eta)$ and Θ is the dual basis of Ξ. Let v and y be vectors in $L_2([\beta_0,\beta_1], wd\eta)$and the inner product is given by (3). Denote $\lambda = \langle g-f,v\rangle/\langle y,v\rangle$ with $\langle y,v\rangle > 0$. If

(i) f is $v-$separable w.r.t. Ξ on J_1 and J_2,
(ii) $g-f$ is $\lambda, y-$separable w.r.t. Θ on J_1 and J_2 and
(iii) Φ preserves $v-$separability w.r.t. Ξ on J_1 and J_2.

Then:

(a) If $\langle g-f,v\rangle = 0$, then

$$\int_{\beta_0}^{\beta_1} w(q)\psi(f(q))d\eta(q) \leq \int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) \tag{14}$$

holds.

(b) If $\langle g-f,v\rangle \geq 0$ and $\langle\psi(f),w\rangle \geq 0$ then inequality (14) holds.

Let us introduce some notations here that will be used in our next result. We denote this set of assumptions by **S**.

S: $I = \cup_{I_k}(a_i,b_i)$, $I^c = [\beta_0,\beta_1]\backslash I = \cup_{i=1}^{k+1}[b_{i-1},a_i]$ and $|I^c| = \sum_{i=1}^{k+1}(a_i - b_{i-1})$ where $\beta_0 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq a_{k+1} = \beta_1$ is a partition of the interval $[\beta_0,\beta_1]$.

We now present our main result:

Theorem 2. *Consider an open interval $U \subset H$ and let $\psi : U \to \mathbb{R}$ be a convex function. Let $\partial\psi : U \to \mathbb{R}$ be the subdifferential of ψ and let $\Psi \in \partial\psi$. Let $([\beta_0,\beta_1],\Sigma,\eta)$ and (X,Ω,μ) be* *two measure space with positive finite measures η and μ, respectively. Let $g : [\beta_0, \beta_1] \to U$ and $f : X \times [\beta_0, \beta_1] \to U$ be two functions s.t. $g, f \in L_2([\beta_0, \beta_1], wd\eta)$, where w is a non-negative measurable function on $[\beta_0, \beta_1]$ with $w \not\equiv 0$ on set of measure nonzero. Moreover, suppose the conditions in* **S** *hold true. Further, we assume that Ξ, Θ, y, and v are as in Theorem 1 and the inner product is given by (3). Denote $\lambda = \langle g - f(p,\cdot), v\rangle / \langle y, v\rangle$ for $s \in X$ with $\langle y, v\rangle > 0$. If*

(i) $f(p,\cdot)$ *is $v-$separable w.r.t. Ξ on $\jmath_1$ and $\jmath_2$,*
(ii) $g - f(p,\cdot)$ *is $0, u-$separable w.r.t. Θ on $\jmath_1$ and $\jmath_2$,*
(iii) $\langle g - f(p,\cdot), v\rangle = 0$,
(iv) Φ *preserves $v-$separability w.r.t. Ξ on $\jmath_1$ and $\jmath_2$,*
(v) $v(q) = \gamma \ \forall\, t \in I^c$ *where γ is a non-zero constant,*

then

$$\psi\left(\frac{1}{\gamma\int_{I^c} w(q)d\eta(q)}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right]\right)$$
$$\le \frac{1}{\int_{I^c} w(q)d\eta(q)}\left(\int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q)\right) \quad (15)$$

holds, where $\mu(X)$, $\int_{I^c} w(q)d\eta(q) > 0$.

Proof. For $s \in X$, by (iii) we have $\langle g - f(p,\cdot), v\rangle / \langle y, v\rangle = 0$. Using the aforementioned conditions, it follows from Corollary 3 that the following inequality holds for each $s \in X$

$$\int_{\beta_0}^{\beta_1} w(q)\psi(f(p,q))d\eta(q) \le \int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q). \quad (16)$$

Additionally, we consider the fact that, since $\langle g - f(p,\cdot), v\rangle = 0$ for each $s \in X$, we have

$$\begin{aligned}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \int_I w(q)v(q)f(p,q)d\eta(q)\right] &= \int_{I^c} w(q)v(q)f(p,q)d\eta(q) \\ &= \gamma\int_{I^c} w(q)f(p,q)d\eta(q).\end{aligned}$$

Now, we consider the L.H.S. of inequality (15). Applying Integral Jensen's inequality twice and using the aforementioned fact with inequality (16), we get

$$\begin{aligned}&\int_{I^c} w(q)d\eta(q)\psi\left(\frac{1}{\gamma\int_{I^c} w(q)d\eta(q)}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q)\right.\right.\\ &\left.\left.-\frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right]d\mu(p)\right)\\ &= \int_{I^c} w(q)d\eta(q)\psi\left(\frac{1}{\mu(X)}\int_X \frac{\gamma}{\gamma\int_{I^c} w(q)d\eta(q)}\int_{I^c} w(q)f(p,q)d\eta(q)d\mu(p)\right)\\ &\le \int_{I^c} w(q)d\eta(q)\frac{1}{\mu(X)}\int_X \psi\left(\frac{1}{\int_{I^c} w(q)d\eta(q)}\int_{I^c} w(q)f(p,q)d\eta(q)\right)d\mu(p)\\ &\le \frac{\int_{I^c} w(q)d\eta(q)}{\int_{I^c} w(q)d\eta(q)}\frac{1}{\mu(X)}\int_X\int_{I^c} w(q)\psi(f(p,q))d\eta(q)d\mu(p)\\ &\le \frac{1}{\mu(X)}\int_X\left(\int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \int_I w(q)\psi(f(p,q))d\eta(q)\right)d\mu(p)\\ &= \int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q).\end{aligned}$$

□

The discrete version similar to the above inequality (15) was discussed in [14], which is stated below.

Corollary 4. *Consider an open interval $I \subseteq \mathbb{R}$ and define $\psi : I \to \mathbb{R}$ to be a convex function.. Let $\partial\psi : I \to \mathbb{R}$ be the subdifferential of ψ and $\Psi \in \partial\psi$. Suppose $\mathbf{b} = (b_1, \ldots, b_m) \in I^m$ and $X = (\mathbf{x}_\gamma) = (x_{\iota\gamma})$ is an $n \times m$ matrix s.t. $x_{\iota\gamma} \in I$ and $(\mathbf{x}_\gamma)$ is a monotonic m-tuple $\forall \iota \in \{1, \ldots, n\}$, $\gamma \in \{1, \ldots, m\}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ s.t. $\langle \mathbf{u}, \mathbf{v} \rangle > 0$. For each $\iota \in J_n$, if*

(i) $\mathbf{x}_{\cdot}^{\cdot}$ *is* $\mathbf{v}$*-separable w.r.t.* Ξ *on* J_1 *and* J_2*,*
(ii) $\mathbf{b} - \mathbf{x}_{\cdot}^{\cdot}$ *is* 0*,* $\mathbf{u}$*-separable w.r.t.* Θ *on* J_1 *and* J_2*,*
(iii) $\langle \mathbf{b} - \mathbf{x}_{\cdot}^{\cdot}, \mathbf{v} \rangle = 0$*,*
(iv) Φ *preserves* $\mathbf{v}$*-separability w.r.t.* Ξ *on* J_1 *and* J_2*.*

Then

$$\psi\left(\sum_{\gamma=1}^{m} \epsilon p_\gamma v_\gamma b_\gamma - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1} \epsilon p_\gamma v_\gamma \sum_{\iota=1}^{n} w_\iota x_{\iota\gamma} - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m} \epsilon p_\gamma v_\gamma \sum_{\iota=1}^{n} w_\iota x_{\iota\gamma}\right)$$
$$\leq \frac{1}{p_\kappa}\sum_{\gamma=1}^{m} p_\gamma \psi(b_\gamma) - \frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1} p_\gamma \sum_{\iota=1}^{n} w_\iota \psi(x_{\iota\gamma}) - \frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m} p_\gamma \sum_{\iota=1}^{n} w_\iota \psi(x_{\iota\gamma}), \quad (17)$$

holds, where $\epsilon = \frac{1}{p_\kappa v_\kappa}$ with $v_\kappa \neq 0$ for $\kappa \in \{1, \ldots, m\}$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are a real $n-$tuple s.t. w_ι represents the weights and satisfies the condition

$$0 \leq W_\iota \leq W_n \quad \text{for} \quad \iota \in \{1, \ldots, n\}, \quad (18)$$

where $W_\iota = \sum_{\iota=1}^{n} w_\iota$ and $W_n > 0$.

Remark 6. *In Corollary 4, if we simply put $\kappa = m$ and further consider the case of positive real weights w_i, then we will get Niezgoda's result as stated in Theorem 3.1 of [7].*

3. Refinements

Let (X, Ω, μ) be a measure space where μ is positive finite measure.
Additionally, $\xi \subset X$ with $\mu(\xi), \mu(\xi^c) > 0$. We take

$$W_\xi = \frac{\mu(\xi)}{\mu(X)}, \quad W_{\xi^c} = \frac{\mu(\xi^c)}{\mu(X)} = 1 - W_\xi.$$

We denote $\mathcal{A} = \int_{I^c} w(q) d\eta(q)$.

Theorem 3. *The following refinement of inequality (15) is valid under the conditions of Theorem 2*

$$\mathcal{A}\,\psi\left(\frac{1}{\gamma\,\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right]\right)$$
$$\leq F(f,g,\psi;\xi) \leq \int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q), \quad (19)$$

where

$$F(f,g,\psi;\xi) =$$
$$W_\xi \mathcal{A}\psi\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_\xi f(p,q)d\mu(p)d\eta(q)\right)\right]$$
$$+ \quad W_{\xi^c}\mathcal{A}\psi\left[\frac{1}{\gamma\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} f(p,q)d\mu(p)d\eta(q)\right)\right].$$

Proof. Using proving techniques of [15], we first apply Jensen's inequality for convex functions to obtain

$$\begin{aligned}
& \mathcal{A}\,\psi.\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right)\right] \\
= \quad & \mathcal{A}\,\psi\left[W_\xi\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right)\right] \\
+ \quad & \mathcal{A}\,\psi\left[W_{\xi^c}\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right)\right] \\
\leq \quad & W_\xi\,\mathcal{A}\,\psi.\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_\xi f(p,q)d\mu(p)d\eta(q)\right)\right] \\
+ \quad & W_{\xi^c}\,\mathcal{A}\,\psi\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} f(p,q)d\mu(p)d\eta(q)\right)\right] \\
= \quad & F(f,g,\psi;\xi)
\end{aligned}$$

for any ξ, which proves the first inequality in (19).

By inequality (15), we also have

$$\begin{aligned}
& F(f,g,\psi;\xi) \\
= \quad & W_\xi\,\mathcal{A}\,\psi\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_\xi f(p,q)d\mu(p)d\eta(q)\right)\right] \\
+ \quad & W_{\xi^c}\,\mathcal{A}\,\psi\left[\frac{1}{\gamma\,\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} f(p,q)d\mu(p)d\eta(q)\right)\right] \\
\leq \quad & W_\xi\int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q) \\
+ \quad & W_{\xi^c}\int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q) \\
= \quad & \int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q),
\end{aligned}$$

for any ξ; thus, the second inequality in (19) holds. □

Remark 7. *Theorem 3 gives us the following inequalities*

$$\begin{aligned}
\mathcal{A}\,\psi\left(\frac{1}{\gamma\,\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) \;-\; \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right]\right) \\
\leq \inf_{\{\xi:0<\mu(\xi)<\mu(X)\}} F(f,g,\psi;\xi),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\{\xi:0<\mu(\xi)<\mu(X)\}} F(f,g,\psi;\xi) \leq \\
\int_{\beta_0}^{\beta_1} w(q)\psi(g(q))d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)\int_X \psi(f(p,q))d\mu(p)d\eta(q).
\end{aligned}$$

4. Applications to Integral Means

Using the integral form of Jensen's Inequality, Haluska and Hutník introduced a class of generalized weighted quasi-arithmetic means in the integral form $M_{[\beta_0,\beta_1],g}(w,f)$ [16]. They used the definition suggested by F. Qi of quasi-arithmetic non-symmetrical weighted mean [17] stated below.

Let $[\beta_0,\beta_1] \subset \mathbb{R}$ where $\beta_0 < \beta_1$. Denote the vector space of all real Lebesgue measurable functions defined on $[\beta_0,\beta_1]$ by $C_1([\beta_0,\beta_1])$, and the classical Lebesgue measure and $C_1^+([\beta_0,\beta_1])$ denote the positive cone of $C_1([\beta_0,\beta_1])$. Let $\|w\|_{[\beta_0,\beta_1]}$ denote the finite C_1-norm of a function $w \in C_1^+([\beta_0,\beta_1])$.

Definition 6. *Let $(w,f) \in C_1^+([\beta_0,\beta_1]) \times C_1^+([\beta_0,\beta_1])$ and $g : [0,\infty] \to \mathbb{R}$ be a real continuous and strictly monotone function. The generalized weighted quasi-arithmetic mean of a function f with respect to weight function w is a number $M_{[\beta_0,\beta_1],g}(w,f) \in \mathbb{R}$ where*

$$M_{[\beta_0,\beta_1],g}(w,f) = g^{-1}\left(\frac{1}{\|w\|_{[\beta_0,\beta_1]}} \int_{[\beta_0,\beta_1]} w(x)g(f(x))dx\right), \tag{20}$$

where g^{-1} denotes the inverse to the function g.

Means $M_{[\beta_0,\beta_1],g}(w,f)$ include various two variable integral means frequently used as special cases when considering the suitable function w, f and g. For instance:

(a) *Weighted Arithmetic Mean*: For the identity function $g(x) = x = I(x)$, we obtain

$$M_{[\beta_0,\beta_1],g}(w,f) = A_{[\beta_0,\beta_1]}(w,f) = \frac{1}{\|w\|_{[\beta_0,\beta_1]}} \int_{[\beta_0,\beta_1]} w(x)f(x)dx.$$

(b) *Weighted Harmonic Mean*: for $g(x) = x^{-1}$, we have

$$M_{[\beta_0,\beta_1],g}(w,f) = H_{[\beta_0,\beta_1]}(w,f) = \left(\frac{1}{\|w\|_{[\beta_0,\beta_1]}} \int_{[\beta_0,\beta_1]} \frac{w(x)}{f(x)}dx\right)^{-1}.$$

(c) *Weighted Power Mean of order r*: for $g(x) = x^r$, we obtain

$$M_{[\beta_0,\beta_1],g}(w,f) = M^{[r]}(f;w;a,b) = \begin{cases} \left(\frac{1}{\|w\|_{[\beta_0,\beta_1]}} \int_{[\beta_0,\beta_1]} w(x)f(x)^r dx\right)^{1/r}; \\ \exp\left(\frac{1}{\|w\|_{[\beta_0,\beta_1]}} \int_{[\beta_0,\beta_1]} w(x)\ln f(x)dx\right). \end{cases}$$

When $r = 0$, we get the *weighted geometric mean*.

Using the assumptions of Theorem 2 where $S \in \{X, \xi, \xi^c\}$, we define the following notations. Denote $\mathcal{A} = \int_{I^c} w(q)d\eta(q)$.

Arithmetic Mean

$$\begin{aligned} A'_g &= \frac{\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q)\right]}{\gamma\mathcal{A}}, \\ A_S &= \frac{1}{\mu(S)}.\frac{\int_I w(q)v(q)\int_S f(p,q)d\mu(p)d\eta(q)}{\gamma\mathcal{A}}, \\ \tilde{A}_S &= A'g - A_S. \end{aligned}$$

Geometric Mean

$$\begin{aligned} G'_g &= \exp\left(\frac{\int_{\beta_0}^{\beta_1} w(q)\ln(g(q))d\eta(q)}{\mathcal{A}}\right), \\ G_S &= \exp\left(\frac{1}{\mu(S)}.\frac{\int_I w(q)\int_S \ln(f(p,q))d\mu(p)d\eta(q)}{\mathcal{A}}\right), \\ \tilde{G}_S &= \frac{G'_g}{G_S}. \end{aligned}$$

Harmonic Mean

$$
\begin{aligned}
H'_g &= \left[\frac{\int_{\beta_0}^{\beta_1} w(q)v(q)\frac{1}{g(q)}d\eta(q)}{\gamma\mathcal{A}}\right]^{-1}, \\
H_S &= \left[\frac{1}{\mu(S)}.\frac{\int_I w(q)v(q)\int_S \frac{1}{f(p,q)}d\mu(p)d\eta(q)}{\gamma\mathcal{A}}\right]^{-1}, \\
\frac{1}{\tilde{H}_S} &= \frac{1}{H'_g} - \frac{1}{H_S}.
\end{aligned}
$$

Power Mean

$$
\begin{aligned}
M'_g &= \frac{\int_{\beta_0}^{\beta_1} w(q)v(q)g^r(q)d\eta(q)}{\gamma\mathcal{A}}, \\
M_S^{[r]} &= \frac{1}{\mu(S)}.\frac{\int_I w(q)v(q)\int_S f^r(p,q)d\mu(p)d\eta(q)}{\gamma\mathcal{A}}, \\
\tilde{M}_S^{[r]} &= (M'_g - M_S^{[r]})^{1/r}.
\end{aligned}
$$

We assume that ln and exp have the natural domain.

Using assumptions and refinement from Theorem 3, we obtain relationships between the following means:

1. Arithmetic and Geometric mean:

Theorem 4. *Under the assumptions of Theorem 3 we have*

$$\tilde{G}_X \leq \tilde{A}_\xi^{W_\xi}.\tilde{A}_{\xi^c}^{W_{\xi^c}} \leq \tilde{A}_X.$$

Proof. In (19), let $\Psi(x) = -\ln(x)$ to obtain

$$
\begin{aligned}
&-\ln\left(\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X f(p,q)d\mu(p)d\eta(q)\right]\right) \\
&\leq W_\xi\left(-\ln\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_\xi f(p,q)d\mu(p)d\eta(q)\right]\right) \\
&+W_{\xi^c}\left(-\ln\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)d\eta(q) - \frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} f(p,q)d\mu(p)d\eta(q)\right]\right) \\
&\leq -\left(\frac{\int_{\beta_0}^{\beta_1} w(q)\ln(g(q))d\eta(q)}{\mathcal{A}} - \frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X \ln(f(p,q))d\mu(p)d\eta(q)}{\mathcal{A}}\right).
\end{aligned}
$$

Using our defined notations, we have

$$
\begin{aligned}
- \quad & \ln(A'_g - A_X) \leq -W_\xi\ln(A'_g - A_\xi) - W_{\xi^c}\ln(A'_g - A_{\xi^c}) \\
\leq - \quad & \left[\ln\left(\exp\frac{\int_{\beta_0}^{\beta_1} w(q)\ln(g(q))d\eta(q)}{\mathcal{A}}\right) - \ln\left(\exp\frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X \ln(f(p,q))d\mu(p)d\eta(q)}{\mathcal{A}}\right)\right]
\end{aligned}
$$

$$\Rightarrow -\ln\tilde{A}_X \leq -(W_\xi\ln\tilde{A}_\xi + W_{\xi^c}\ln\tilde{A}_{\xi^c}) \leq -[\ln G'_g - \ln G_X]$$

Using the porperty of $-\ln$ gives us,

$$-\ln\tilde{A}_X \leq -(\ln\tilde{A}_{\xi}^{W_{\xi}} + \ln\tilde{A}_{\xi^c}^{W_{\xi}^c}) \leq -\ln\tilde{G}_X$$

$$\Rightarrow \ln\tilde{A}_X \geq \ln(\tilde{A}_{\xi}^{W_{\xi}}.\tilde{A}_{\xi^c}^{W_{\xi}^c}) \geq \ln\tilde{G}_X$$

$$\Rightarrow \tilde{G}_X \leq \tilde{A}_{\xi}^{W_{\xi}}.\tilde{A}_{\xi^c}^{W_{\xi^c}} \leq \tilde{A}_X.$$

□

2. Geometric and Harmonic mean:

Theorem 5. *Under the assumptions of Theorem 3 we have*

$$\frac{1}{\tilde{G}_X} \leq \frac{1}{\tilde{H}_{\xi}^{W_{\xi}}\tilde{H}_{\xi^c}^{W_{\xi^c}}} \leq \frac{1}{\tilde{H}_X}.$$

Proof. In (19) replace $g(q) \leftrightarrow \frac{1}{g(q)}$ and $f(p,q) \leftrightarrow \frac{1}{f(p,q)}$ and take $\Psi(x) = -\ln(x)$ to get

$$\begin{aligned}
&-\ln\left(\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)\frac{1}{g(q)}d\eta(q) \;-\; \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X \frac{1}{f(p,q)}d\mu(p)d\eta(q)\right]\right) \\
&\leq W_{\xi}\left(-\ln\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)\frac{1}{g(q)}d\eta(q) \;-\; \frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_{\xi} \frac{1}{f(p,q)}d\mu(p)d\eta(q)\right]\right) \\
&+W_{\xi^c}\left(-\ln\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)\frac{1}{g(q)}d\eta(q) \;-\; \frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} \frac{1}{f(p,q)}d\mu(p)d\eta(q)\right]\right) \\
&\leq -\frac{\int_{\beta_0}^{\beta_1} w(q)\ln\left(\frac{1}{g(q)}\right)d\eta(q)}{\mathcal{A}} + \frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X \ln\left(\frac{1}{f(p,q)}\right)d\mu(p)d\eta(q)}{\mathcal{A}}.
\end{aligned}$$

Using our defined notations, we have

$$\begin{aligned}
&-\ln[(H_g')^{-1} - (H_X)^{-1}] \\
&\leq -W_{\xi}\ln[(H_g')^{-1} - (H_{\xi})^{-1}] - W_{\xi^c}\ln[(H_g')^{-1} - (H_{\xi^c})^{-1}] \\
&\leq -\left(\frac{\int_{\beta_0}^{\beta_1} w(q)(\ln 1 - \ln g(q))d\eta(q)}{\mathcal{A}}\right) \\
&-\left(\frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X(\ln 1 - \ln f(p,q))d\mu(p)d\eta(q)}{\mathcal{A}}\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\ln(\tilde{H}_X)^{-1} \leq -W_{\xi}\ln(\tilde{H}_{\xi})^{-1} - W_{\xi^c}\ln(\tilde{H}_{\xi^c})^{-1} \\
&\leq -\left(-\frac{\int_{\beta_0}^{\beta_1} w(q)\ln g(q)d\eta(q)}{\mathcal{A}} + \frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X \ln f(p,q)d\mu(p)d\eta(q)}{\mathcal{A}}\right).
\end{aligned}$$

Multiplying the last inequality by (ln) (exp), we obtain

$$-\ln\frac{1}{(\tilde{H}_X)} \leq -W_\xi \ln\frac{1}{(\tilde{H}_\xi)} - W_{\xi^c}\ln\frac{1}{(\tilde{H}_{\xi^c})}$$
$$\leq -\left(-\ln\exp\frac{\int_{\beta_0}^{\beta_1} w(q)\ln g(q)d\eta(q)}{\mathcal{A}} + \ln\exp\frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X \ln f(p,q)d\mu(p)d\eta(q)}{\mathcal{A}}\right)$$

$$\Rightarrow -\ln\frac{1}{(\tilde{H}_X)} \leq -\ln(\frac{1}{\tilde{H}_\xi})^{W_\xi} - \ln(\frac{1}{\tilde{H}_{\xi^c}})^{W_{\xi^c}} \leq -[-\ln G'_g + \ln(G_X)].$$

Using the property of $-\ln$ we have,

$$-\ln\frac{1}{(\tilde{H}_X)} \leq -(\ln\frac{1}{\tilde{H}_\xi^{W_\xi}} + \ln\frac{1}{\tilde{H}_{\xi^c}^{W_{\xi^c}}}) \leq -\ln\frac{(G_X)}{G'_g}.$$

On simplification, we obtain

$$-\ln\frac{1}{(\tilde{H}_X)} \leq -\ln(\frac{1}{\tilde{H}_\xi^{W_\xi}}\frac{1}{\tilde{H}_{\xi^c}^{W_{\xi^c}}}) \leq -\ln\frac{1}{\tilde{G}_X}$$

$$\Rightarrow \frac{1}{\tilde{G}_X} \leq \frac{1}{\tilde{H}_\xi^{W_\xi}\tilde{H}_{\xi^c}^{W_{\xi^c}}} \leq \frac{1}{\tilde{H}_X}.$$

□

3. Power Mean and Arithmetic mean:

Theorem 6. *Let all the assumptions of Theorem 3 be valid.*

(i) *For $r \leq 1$, we have*

$$\tilde{M}_X^{[r]} \leq W_\xi \tilde{M}_\xi^{[r]} + W_{\xi^c}^{[r]}\tilde{M}_{\xi^c} \leq \tilde{A}_X. \tag{21}$$

(ii) *The above inequalities are reversed in case $r \geq 1$.*

Proof.

(i) In (19), replace $g(q) \leftrightarrow (g(q))^r$ and $f(p,q) \leftrightarrow (f(p,q))^r$ and take $\Psi(x) = x^{\frac{1}{r}}$, to obtain

$$\left[\frac{1}{\gamma\mathcal{A}}\left(\int_{\beta_0}^{\beta_1} w(q)v(q)g(q)^r d\eta(q) - \frac{1}{\mu(X)}\int_I w(q)v(q)\int_X (f(p,q))^r d\mu(p)d\eta(q)\right)\right]^{\frac{1}{r}}$$
$$\leq W_\xi\left(\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)(g(q))^r d\eta(q)\frac{1}{\mu(\xi)}\int_I w(q)v(q)\int_\xi (f(p,q))^r d\mu(p)d\eta(q)\right]\right)^{\frac{1}{r}}$$
$$+W_{\xi^c}\left(\frac{1}{\gamma\mathcal{A}}\left[\int_{\beta_0}^{\beta_1} w(q)v(q)(g(q))^r d\eta(q)\frac{1}{\mu(\xi^c)}\int_I w(q)v(q)\int_{\xi^c} (f(p,q))^r d\mu(p)d\eta(q)\right]\right)^{\frac{1}{r}}$$
$$\leq \frac{\int_{\beta_0}^{\beta_1} w(q)((g(q))^r)^{\frac{1}{r}}d\eta(q)}{\mathcal{A}} - \frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X ((f(p,q))^r)^{\frac{1}{r}}d\mu(p)d\eta(q)}{\mathcal{A}}.$$

Using our defined notations, we have

$$(M'_g - M_X^{[r]})^{\frac{1}{r}} \leq W_\xi(M'_g - M_\xi^{[r]})^{\frac{1}{r}} + W_{\xi^c}(M'_g - M_{\xi^c}^{[r]})^{\frac{1}{r}}$$
$$\leq \frac{\int_{\beta_0}^{\beta_1} w(q)(g(q))d\eta(q)}{\mathcal{A}} - \frac{1}{\mu(X)}.\frac{\int_I w(q)\int_X (f(p,q))d\mu(p)d\eta(q)}{\mathcal{A}},$$

$$\Rightarrow (M'_g - M_X^{[r]})^{\frac{1}{r}} \leq W_\xi(M'_g - M_\xi^{[r]})^{\frac{1}{r}} + W_{\xi^c}(M'_g - M_{\xi^c}^{[r]})^{\frac{1}{r}} \leq (A'_g - A_X)$$

$$\Rightarrow \tilde{M}_X^{[r]} \leq W_\xi \tilde{M}_\xi^{[r]} + W_{\xi^c} \tilde{M}_{\xi^c}^{[r]} \leq \tilde{A}_X.$$

(ii) Incase $r \geq 1$, the inequalities in (21) are reversed since $\Psi(x) = x^{\frac{1}{r}}$ is concave.

□

5. Conclusions and Future Ideas

Marek Niezgoda stated all the results in n-dimensional real spaces (finite dimensional Hilbert spaces). We extended the idea by using separable Hilbert spaces, covering both the case of the finite dimensional and infinite dimensional, thus providing generalized integral results related to majorization, Niezgoda, and Ćebysév type inequalities. More concretely, using a concept of similarly separable vectors, Niezgoda stated all the results for the sequences, i.e., he provided discrete inequalities. We stated these results for functions taken from weighted L_2 spaces, i.e., we provided these results for integral inequalities. We also provided some refinements of these inequalities. Our proved inequalities are directly related to the Arithmetic, Geometric, Harmonic, and Power Means.

In the future, we can also provide a generalization of Mercer's inequality [6] using functions with non-decreasing increments. These results will be the generalization of results stated in [18].

Additionally, we can further extend all the stated results by using the Isotonic Linear Functional [2] and hence as an application we may state relations between some generalized means as given in [15].

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