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Semi-Local Convergence of a Seventh Order Method with One Parameter for Solving Non-Linear Equations

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Abstract: The semi-local convergence is presented for a one parameter seventh order method to obtain solutions of Banach space valued nonlinear models. Existing works utilized hypotheses up to the eighth derivative to prove the local convergence. But these high order derivatives are not on the method and they may not exist. Hence, the earlier results can only apply to solve equations containing operators that are at least eight times differentiable although this method may converge. That is why, we only apply the first derivative in our convergence result. Therefore, the results on calculable error estimates, convergence radius and uniqueness region for the solution are derived in contrast to the earlier proposals dealing with the less challenging local convergence case. Hence, we enlarge the applicability of these methods. The methodology used does not depend on the method and it is very general. Therefore, it can be used to extend other methods in an analogous way. Finally, some numerical tests are performed at the end of the text, where the convergence conditions are fulfilled.

Keywords: Banach spaces; Fréchet derivative; semi-local convergence; convergence ball

MSC: 65G99; 65H10; 47H99; 49M15

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1. Introduction

Let U and V be Banach spaces, and B be a non-empty, convex and open subset of U. Suppose $F : B \subseteq U \to V$ is derivable in the Fréchet sense. The ultimate aim is to produce a solution u^* for the equation

$$F(u) = 0. \tag{1}$$

A plethora of highly challenging scientific and engineering problems can be modeled as nonlinear equations in the form (1) [1–4]. Overcoming this nonlinearity has long been a significant problem in mathematics. Purely analytical answers to these equations are difficult to provide. Because of this, scientists and researchers often apply the strategy of iterative algorithms to obtain the required solution. Among iterative procedures, the approach given by Newton is widely used to address (1). During the last several years, a lot of new higher order iterative techniques have been developed and are being implemented to deal with nonlinear equations [5–9]. In most of these research works, convergence theorems of iterative schemes have been established using conditions on derivatives of higher order. Additionally, these studies provide no conclusions on the convergence radii, error distances and existence-uniqueness regions for the solution. The study of semi-local analysis of an iterative formula allows to estimate the convergence balls, bounds on error and uniqueness region for a solution. The results of local convergence of efficient iterative procedures have been deduced in [6–18]. In these works, important results containing convergence radii, measurements on error estimates and expanded utility of these iterative approaches have been given. Outcomes of local analysis are valuable because they illustrate the complexity of selecting initial points. Most recently, Liu et al. [7] established a local convergence theorem for a class of sixth and seventh convergence order iterative methods defined on the real line by considering assumptions on the first derivative of F and Lipschitz parameters. These methods are defined for $n = 0, 1, 2, ...,$ by

$$\begin{aligned} s_n &= u_n - F'(u_n)^{-1}F(u_n), \\ z_n &= s_n - \tau\frac{F(u_n) + F(s_n)}{F'(u_n)} - (1-\tau)F'(u_n)^{-1}F(u_n)\frac{F(u_n)}{F(u_n) - F(s_n)}, \\ u_{n+1} &= z_n - \frac{F(z_n)}{[z_n, s_n; F] + [z_n, u_n, u_n; F](z_n - s_n)}, \end{aligned} \tag{2}$$

where u_0 is a starting guess, $\tau \in \mathbb{R}$ and $[., .; F]$ is the first order divided difference, and

$$\begin{aligned} [z_n, s_n; F] = [s_n, z_n; F] &= \frac{F(z_n) - F(s_n)}{z_n - s_n}, \\ [z_n, u_n; F] = [u_n, z_n; F] &= \frac{F(z_n) - F(u_n)}{z_n - u_n}, \\ [z_n, u_n, u_n; F] &= \frac{[z_n, u_n; F] - F'(u_n)}{z_n - u_n}. \end{aligned}$$

The authors also obtained the error bounds and radii of convergence based on their proposed theorem. They concluded that this family has the largest convergence radius for $\tau = 0$ in comparison with the other members (for $\tau = -2$, $\tau = -1.5$, $\tau = 0.5$ and $\tau = 2.5$).

In this document, we offer the semi-local convergence result for a one parameter seventh convergence order iterative method discussed by Amiri et al. [10] to address problem (1). This method is presented for $\alpha \neq 0$ as follows:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \frac{1}{\alpha}F'(x_n)^{-1}F(y_n), \\ v_n &= z_n - F'(x_n)^{-1}((2 - \frac{1}{\alpha} - \alpha)F(y_n) + \alpha F(z_n)), \\ x_{n+1} &= v_n - H(m_n)F'(z_n)^{-1}F(v_n), \end{aligned} \tag{3}$$

where x_0 is a starting estimation, $Q_n = (2 - \frac{1}{\alpha} - \alpha)F(y_n) + \alpha F(z_n)$, $m_n = I - \frac{1}{\alpha}F'(x_n)^{-1}$ $[y_n, z_n; F]$, $H(t) = (1 - \frac{1}{\alpha})I + \alpha(t - (1 - \frac{1}{\alpha})I) + \frac{1}{2}(-\alpha + 6\alpha^2)(t - (1 - \frac{1}{\alpha})I)^2$, $\alpha > 0$ and $[., .; F] : B \times B \to L(U, V)$ is a divided difference of order one. It is shown to be of order seven utilizing eighth order Fréchet derivative of F [5,8–12,18]. The usage of these solvers is restricted due to such hypotheses on derivatives of higher order.

Let us choose the following motivational problem, where $U = V = \mathbb{R}$ and the function T is defined on $B = [-\frac{1}{2}, \frac{3}{2}]$ by

$$T(u) = \begin{cases} u^3 \ln(u^2) + u^5 - u^4, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0 \end{cases}. \tag{4}$$

The definition of T gives that $T'''(u) = 6 \ln u + 60u^2 - 24u + 22$. Hence, we arrive at the conclusion that the convergence theorem for the method (3) suggested by [10] does not apply for this example although the method may converge. This is because the third derivative T''' is unbounded on B. Besides, no results on the convergence domain, bounds on error and uniqueness results were established in the existing article by [10]. However, we propose the semi-local convergence theorem for this method (3) in the more general Banach space case by considering a set of assumptions only on F'. In particular, ω-continuity of the first Fréchet derivative is employed to enhance the utility of these methods.

It is worth noticing that our approach does not depend on method (3). Therefore, due to its generality it can be used on other methods using inverses of linear operators [5–7,9–18].

Other computational pit-falls of high convergence order methods have been reported by Sen et al. [4]. In particular, they illustrated that the computational complexity of high-order methods can be higher than the second-order Newton method. The family of methods (3) can be applied to solve: oxygen diffusion problems in cylindrically shaped sections of tissue [1,3] and melting problems [2].

The outline of this document can be described as: A majorizing sequence is developed to prove the semi-local convergence of method (3) in Section 2. The Semi-local convergence of the considered method (3) is established in Section 3. In Section 4, Numerical experiments are described. Conclusions are also presented in Section 5.

2. Convergence for the Majorizing Sequence

A scalar sequence is developed that is shown to be majorizing for the method (3) in the next section. Set $M_0 = [0, \infty)$.
Suppose:

(i) There exists a function ω_0 with $Dom(\omega_0) = M_0$ and $Range(\omega_0) = \mathbb{R}$ which is continuous and non-decreasing such that the equation

$$\omega_0(u) - 1 = 0$$

has a smallest solution $\rho_0 \in M_0 - \{0\}$. Set $M = [0, \rho_0)$.

(ii) There exist functions ω, ψ with $Dom(\omega) = M, Dom(\psi) = M \times M \times M$ whose range is $\mathbb{R}$ which are continuous and non-decreasing.

It is convenient to define parameters l, s and p by

$$l = 1 - \frac{1}{\alpha}, \quad s = \frac{\alpha(6\alpha - 1)}{2} \quad \text{and} \quad p = \frac{6\alpha - 1}{2\alpha}$$

The notation $S(x, \xi), S[x, \xi]$ stands for the open and closed ball in U with center x and of radius $\xi > 0$.

Let $\Delta \geq 0$ be a given parameter. Moreover, define the sequence $\{a_n\}$ for all $n = 0, 1, 2, ...$ by
$a_0 = 0, b_0 = \Delta$,

$$\begin{aligned}
c_n &= b_n + \frac{\int_0^1 \omega((1-\theta)b_n)d\theta(b_n - a_n)}{|\alpha|(1 - \omega_0(a_n))}, \\
q_n &= |2 - \frac{1}{\alpha}| \int_0^1 \omega((1-\theta)(b_n - a_n))d\theta(b_n - a_n) \\
&\qquad + |\alpha|\left(1 + \int_0^1 \omega_0(b_n + \theta(c_n - b_n))d\theta\right)(c_n - b_n), \\
d_n &= c_n + \frac{q_n}{1 - \omega_0(a_n)}, \\
\alpha_n &= \left(1 + \int_0^1 \omega_0(b_n + \theta(d_n - b_n))d\theta\right)(d_n - b_n) + \int_0^1 \omega((1-\theta)(b_n - a_n))d\theta(b_n - a_n), \\
\gamma_n &= \frac{\psi(a_n, b_n, c_n)}{1 - \omega_0(a_n)}, \\
h_n &= |l| + \gamma_n + |p|\gamma_n^2, \\
a_{n+1} &= d_n + \frac{\alpha_n h_n}{1 - \omega_0(c_n)}, \\
\lambda_{n+1} &= \int_0^1 \omega((1-\theta)(a_{n+1} - a_n))d\theta(a_{n+1} - a_n) + (1 + \omega_0(a_n))(a_{n+1} - b_n) \\
\text{and} \quad b_{n+1} &= a_{n+1} + \frac{\lambda_{n+1}}{1 - \omega_0(a_{n+1})}.
\end{aligned} \tag{5}$$

A general convergence result follows for the sequence $\{a_n\}$.

Lemma 1. *Suppose there exists $\rho > \rho_0$ such that for all $n = 0, 1, 2, \ldots$*

$$\omega_0(a_n) < 1 \tag{6}$$

and

$$a_n < \rho. \tag{7}$$

Then, the sequence $\{a_n\}$ is convergent to some $a^* \in [\eta, \rho]$ such that $a_n \leq b_n \leq c_n \leq d_n \leq a_{n+1} \leq \rho$.

Proof. It follows by the definition of $\{a_n\}$ and the conditions (6) and (7) that this sequence is bounded from above by ρ and non-decreasing. Hence, it is convergent to its unique least upper bound a^*. □

Remark 1. *If the function ω_0 is strictly increasing, then set*

$$\rho = \omega_0^{-1}(1). \tag{8}$$

In this case the condition (6) becomes

$$a_n < \rho. \tag{9}$$

3. Convergence for the Method (3)

The semi-local convergence is based on the majorizing sequence $\{a_n\}$ and the following conditions.

Suppose:

(A_1) There exists a point $x_0 \in B$ and a parameter $\Delta \geq 0$ such that $F'(x_0)^{-1} \in L(V, U)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq \Delta$.

(A_2) $\|F'(x_0)^{-1}(F'(u) - F'(x_0))\| \leq \omega_0(\|u - x_0\|)$ for all $u \in B$. Set $B_0 = S(x_0, \rho_0) \cap B$.

(A_3) $\|F'(x_0)^{-1}(F'(u_2) - F'(u_1))\| \leq \omega(\|u_2 - u_1\|)$ and $\|F'(x_0)^{-1}([u_1, u_2; F] - F'(u_3))\| \leq \psi(\|u_1 - x_0\|, \|u_2 - x_0\|, \|u_3 - x_0\|)$ for all $u_1, u_2, u_3 \in B_0$.

(A_4) The conditions (6) and (7) hold.

(A_5) $S[x_0, a^*] \subset B$.

Next, the main semi-local convergence result is presented under the conditions (A_1)–(A_5) and using $\{a_n\}$ as the majorizing sequence for method (3).

Theorem 1. *Suppose that the conditions (A_1)–(A_5) hold. Then, the sequence $\{x_n\}$ produced by the method (3) and starting at $x_0 \in B$ is convergent to a solution u^* of the equation $F(u) = 0$ such that*

$$\|x_n - u^*\| \leq a^* - a_n, \tag{10}$$

where the sequence $\{a_n\}$ is given by the formula (5).

Proof. Induction is employed to show

$$\|y_k - x_k\| \leq b_k - a_k, \tag{11}$$

$$\|z_k - y_k\| \leq c_k - b_k \tag{12}$$

$$\|v_k - z_k\| \leq d_k - c_k \tag{13}$$

and

$$\|x_{k+1} - v_k\| \leq a_{k+1} - d_k. \tag{14}$$

The assertion (11) holds true for $k = 0$, since by the condition (A_1)

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \Delta = b_0 - a_0 = b_0 < a^*.$$

Hence, the iterates y_0, z_0, v_0, x_1 are well defined and $y_0 \in S(x_0, a^*)$. Let $u \in S(x_0, a^*)$. It then follows from the conditions (A_1), (A_2) and (6) that

$$\|F'(x_0)^{-1}(F'(u) - F'(x_0))\| \leq \omega_0(\|u - x_0\|) \leq \omega_0(a^*) < 1.$$

Thus, the Banach perturbation lemma [15] asserts that $F'(u)^{-1} \in L(V, U)$ and

$$\|F'(u)^{-1}F'(x_0)\| \leq \frac{1}{1 - \omega_0(\|u - x_0\|)}. \tag{15}$$

Suppose that the assertions hold for all integer values smaller than k.

The first substep of method (3) gives the identity

$$\begin{aligned} F(y_k) &= F(y_k) - F(x_k) - F'(x_k)(y_k - x_k) \\ &= \int_0^1 (F'(x_k + \theta(y_k - x_k))d\theta - F'(x_k))(y_k - x_k). \end{aligned}$$

Then, it follows by the condition (A_3), (15) (for $u = x_k$), (5) and the second sub-step of the method (3),

$$\begin{aligned} \|z_k - y_k\| &= \frac{1}{|\alpha|}\|F'(x_k)^{-1}F(y_k)\| \\ &\leq \frac{1}{|\alpha|}\|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_k)\| \\ &\leq \frac{\int_0^1 \omega((1-\theta)\|y_k - x_k\|)d\theta\|y_k - x_k\|}{|\alpha|(1 - \omega_0(\|x_k - x_0\|))} \\ &\leq \frac{\int_0^1 \omega((1-\theta)(b_k - a_k))d\theta(b_k - a_k)}{|\alpha|(1 - \omega_0(a_k))} \\ &= c_k - b_k \\ \text{and} \quad \|z_k - x_0\| &\leq \|z_k - y_k\| + \|y_k - x_0\| \\ &\leq c_k - b_k + b_k \\ &= c_k < a^*. \end{aligned}$$

Therefore, the assertions (12) holds and the iterate $z_k \in S(x_0, a^*)$.

Similarly, the identity

$$\begin{aligned} F(z_k) - F(y_k) &= \int_0^1 F'(y_k + \theta(z_k - y_k))d\theta(z_k - y_k) \\ &= \int_0^1 (F'(y_k + \theta(z_k - y_k))d\theta - F'(x_0))(z_k - y_k) + F'(x_0)(z_k - y_k) \end{aligned}$$

gives

$$\begin{aligned} \|F'(x_0)^{-1}Q_k\| &= \|(2 - \frac{1}{\alpha})F'(x_0)^{-1}(F(y_k) + \alpha(F(z_k) - F(y_k)))\| \\ &\leq |2 - \frac{1}{\alpha}|\int_0^1 \omega((1-\theta)\|y_k - x_k\|)d\theta\|y_k - x_k\| \\ &\quad + |\alpha|(1 + \int_0^1 \omega_0(\|y_k - x_0\| + \theta\|z_k - y_k\|)d\theta\|z_k - y_k\| \\ &= \bar{q}_k \leq q_k. \end{aligned} \tag{16}$$

Thus, it follows

$$\begin{aligned}\|v_k - z_k\| &\le \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}Q_k\| \\ &\le \frac{\bar{q}_k}{1-\omega_0(a_k)} \\ &\le \frac{q_k}{1-\omega_0(a_k)} = d_k - c_k\end{aligned}$$

and

$$\begin{aligned}\|v_k - x_0\| &\le \|v_k - z_k\| + \|z_k - x_0\| \\ &\le d_k - c_k + c_k \\ &= d_k < a^*.\end{aligned}$$

Hence, the iterate $v_k \in S(x_0, a^*)$ and the assertion (13) holds.

Then, the identity $F(v_k) = F(v_k) - F(y_k) + F(y_k)$ gives similarly

$$\begin{aligned}\|F'(x_0)^{-1}F(v_k)\| \le (1 + \int_0^1 \omega_0(\|y_k - x_0\| + \theta\|v_k - y_k\|)d\theta)\|v_k - y_k\| \\ + \int_0^1 \omega((1-\theta)\|y_k - x_k\|)d\theta\|y_k - x_k\| \\ \le \bar{\alpha}_k \le \alpha_k.\end{aligned} \tag{17}$$

Moreover, it can be written

$$\begin{aligned}m_k &= (I - \frac{1}{\alpha})I + \frac{1}{\alpha}(I - F'(x_k)^{-1}[y_k, z_k; F]) \\ &= lI + \frac{1}{\alpha}F'(x_k)^{-1}(F'(x_k) - [y_k, z_k; F]), \\ m_k - l &= \frac{1}{\alpha}\bar{\gamma}_k,\end{aligned}$$

so

$$\begin{aligned}H(m_k) &= lI + \alpha(\frac{1}{\alpha}\bar{\gamma}_k) + s(\frac{1}{\alpha}\bar{\gamma}_k)^2 \\ &= lI + \bar{\gamma}_k + p\bar{\gamma}_k{}^2\end{aligned} \tag{18}$$

by the definition of the parameters l and p. Consequently, by the estimate (18), the second condition in (A_3), (5), it is obtained

$$\begin{aligned}\|H(m_k)\| &\le |l| + \|\bar{\gamma}_k\| + |p|\|\bar{\gamma}_k\|^2 \\ &\le |l| + \gamma_k + |p|\gamma_k{}^2 = h_k,\end{aligned}$$

where we also used

$$\begin{aligned}\bar{\gamma}_k &= \|F'(x_k)^{-1}(F'(x_k) - [y_k, z_k; F])\| \\ &\le \frac{\psi(\|x_k - x_0\|, \|y_k - x_0\|, \|z_k - x_0\|)}{1 - \omega_0(\|x_k - x_0\|} \\ &\le \frac{\psi(a_k, b_k, c_k)}{1-\omega_0(a_k)} = \gamma_k.\end{aligned}$$

Then, by the fourth sub-step of the method (3), (17), (15) (for $u = z_k$) and (5),

$$\begin{aligned}\|x_{k+1} - v_k\| &\le \|H(m_k)\|\|F'(z_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(v_k)\| \\ &\le \frac{h_k\alpha_k}{1-\omega_0(c_k)} = a_{k+1} - d_k\end{aligned}$$

and

$$\begin{aligned}\|x_k - x_0\| &\leq \|x_{k+1} - v_k\| + \|v_k - x_0\| \\ &\leq a_{k+1} - d_k + d_k = a_{k+1} < a^*.\end{aligned}$$

Therefore, the iterate $x_{k+1} \in S(x_0, a^*)$ and the estimate (14) holds. The first sub-step of method (3) gives

$$\begin{aligned}F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(y_k - x_k) \\ &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) + F'(x_k)(x_{k+1} - y_k)\end{aligned}$$

leading to

$$\begin{aligned}\|F'(x_0)^{-1}F(x_{k+1})\| \leq & \int_0^1 \|F'(x_0)^{-1}(F'(x_k + \theta(x_{k+1} - x_k))d\theta - F'(x_k))(x_{k+1} - x_k) \\ & + F'(x_0)^{-1}(F'(x_k) - F(x_0))(x_{k+1} - y_k) + F'(x_0)^{-1}F'(x_0)(x_{k+1} - y_k)\| \\ \leq & \int_0^1 \omega((1-\theta)\|x_{k+1} - x_k\|)d\theta\|x_{k+1} - x_k\| \\ & + (1 + \omega_0(\|x_k - x_0\|))\|x_{k+1} - y_k\| \\ \leq & \int_0^1 \omega((1-\theta)(a_{k+1} - a_k))d\theta(a_{k+1} - a_k) + (1 + \omega_0(a_k))(a_{k+1} - b_k) \\ = & \bar{\lambda}_{k+1},\end{aligned} \tag{19}$$

so

$$\begin{aligned}\|y_{k+1} - x_{k+1}\| &\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \\ &\leq \frac{\bar{\lambda}_{k+1}}{1 - \omega_0(\|x_{k+1} - x_0\|)} \\ &\leq \frac{\lambda_{k+1}}{1 - \omega_0(a_{k+1})} = b_{k+1} - a_{k+1} \\ \text{and} \quad \|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq b_{k+1} - a_{k+1} + a_{k+1} = b_{k+1} < a^*.\end{aligned}$$

That is the induction for the assertions (11)–(14) is terminated. The condition (A_4) implies that the sequence $\{a_k\}$ is complete. Consequently, by (11)–(14) the sequence $\{x_k\}$ is also complete and convergent to some $u^* \in S[x_0, a^*]$ (since the ball $S[x_0, a^*]$ is a closed set). Furthermore, by the continuity of the operator F and if $k \to \infty$ in (19), we deduce that $F(u^*) = 0$. Finally, for $i \geq 0$ the estimate

$$\|x_{k+i} - x_k\| \leq a_{k+i} - a_k$$

shows the assertion (10) by letting $i \to \infty$. □

Next, the uniqueness of the convergence region is determined.

Proposition 1. *Suppose:*
There exists a solution $\mu \in S(x_0, \rho_1)$ for some $\rho_1 > 0$; condition (A_2) holds on the ball $S(x_0, \rho_1)$ and there exists $\rho_2 \geq \rho_1$ such that

$$\int_0^1 \omega_0(\theta\rho_2)d\theta < 1. \tag{20}$$

Set $B_1 = S[x_0, \rho_1] \cap B$. Then, the Equation (1) is uniquely solvable by μ in the region B_1.

Proof. Suppose that there exists $\mu_0 \in B_1$ with $F(\mu_0) = 0$. Define the linear operator $G = \int_0^1 F'(\mu + \theta(\mu_0 - \mu))(\mu_0 - \mu)d\theta$.
Then, by the application of the conditions (A_2) and (20) it follows that

$$\|F'(x_0)^{-1}(G - F'(x_0))\| \leq \int_0^1 \omega_0(\theta\|\mu_0 - \mu\|)d\theta$$
$$\int_0^1 \omega_0(\theta\rho_2)d\theta < 1.$$

Therefore, the operator is invertible and $\mu_0 - \mu = G^{-1}(F(\mu_0) - F(\mu)) = G^{-1}(0 - 0) = 0$. Therefore, we conclude that $\mu = \mu_0$. □

Remark 2. *(a) The parameter ρ_0 given by the condition (i) of the Lemma 1 in closed form can replace a^* in the condition (A_4).*

(b) Proposition 1 is not using all the conditions of the Theorem 1. But if this is the case, then set $\rho_1 = a^*$ or $\rho_1 = \rho_0$.

(c) The second condition in (A_3) involving the function ψ can be dropped as follows: Suppose that there exists a function $\psi_0 : M \times M \to \mathbb{R}$ continuous and non-decreasing such that for all $y, z \in B_0$

$$\|F'(x_0)^{-1}([y, z; F] - F'(x_0))\| \leq \psi_0(\|y - x_0\|, \|z - x_0\|).$$

Then, in view of the estimate

$$\|F'(x_0)^{-1}(F'(x) - [y, z; F]\| \leq \|F'(x_0)^{-1}(F'(x) - F'(x_0))\| + \|F'(x_0)^{-1}([y, z; F] - F'(x_0))\|,$$

the function ψ can be defined by

$$\psi(t_1, t_2, t_3) = \omega_0(t_1) + \psi_0(t_2, t_3) \quad \textit{for all} \quad t_1, t_2, t_3 \in [0, \rho_0).$$

Moreover, if the divided difference is defined by

$$[y, z; F] = \int_0^1 F'(z + \theta(y - z))d\theta,$$

then, the function ψ can be defined by

$$\psi(t_1, t_2, t_3) = \omega_0(t_1) + \frac{1}{2}(\psi_0(t_2) + \psi_0(t_3)), \tag{21}$$

since $\psi_0(t_2, t_3) = \frac{1}{2}(w_0(t_2) + \omega_0(t_3))$. In fact, (21) is the choice for the function ψ used in the examples.

4. Numerical Examples

Example 1. *Let $B = S(x_0, 1 - a)$ for some $a \in [0, 1)$. Define the cubic polynomial*

$$F(u) = u^3 - a.$$

If one chooses $x_0 = 1$ and defines $[\delta_1, \delta_2; F] = \frac{F(\delta_2) - F(\delta_1)}{\delta_2 - \delta_1}$ for $\delta_1 \neq \delta_2$, then the conditions (A_1)–(A_2) are verified for $\Delta = \frac{1-a}{3}$, $\omega_0(t) = (3 - a)t$, $\rho = \rho_0 = \frac{1}{3-a}$, $B_0 = S(x_0, \frac{1}{3-a})$, $\omega(t) = 2(1 + \frac{1}{3-a})$ and $\psi(t_1, t_2, t_3) = (3 - a)(t_1 + \frac{1}{2}(t_2 + t_3))$. Choose $a = 0.95$. Tables 1–3 verifies condition (6) and (7) for $\alpha = 0.75, 1, 1.25$ respectively.

Estimates for $\omega_0(a_n)$ and a_n.

Table 1. $\alpha = 0.75$.

n	0	1	2	3	4	5	6	7	8
$\omega_0(a_n)$	0	0.0380825	0.0438734	0.0446644	0.04477	0.0447841	0.044786	0.0447862	0.0447862
a_n	0	0.0185768	0.0214017	0.0217875	0.021839	0.0218459	0.0218468	0.0218469	0.0218469

Here, $a^* = 0.0218469$.

Table 2. $\alpha = 1$.

n	0	1	2	3	4	5	6	7
$\omega_0(a_n)$	0	0.0368743	0.0409929	0.0413345	0.0413615	0.0413636	0.0413638	0.0413638
a_n	0	0.0179875	0.0199966	0.0201632	0.0201764	0.0201774	0.0201775	0.0201775

$a^* = 0.0201775$.

Table 3. $\alpha = 1.25$.

n	0	1	2	3	4	5	6	7
$\omega_0(a_n)$	0	0.0376078	0.042654	0.0431496	0.0431953	0.0431994	0.0431998	0.0431998
a_n	0	0.0183453	0.0208068	0.0210486	0.0210709	0.0210729	0.0210731	0.0210731

$a^* = 0.0210731$.

Hence, we can conclude that (6) and (7) holds, so (A_4) holds. Therefore, all the conditions (A_1)–(A_5) holds and hence by Theorem 1, the sequence $\{x_n\}$ generated by the method (3) converges to a solution u^* of the equation $F(u) = 0$ in $S[x_0, a^*]$.

Example 2. *The utility of our results in the real world can be successfully established by considering the quartic equation for fractional conversion which represents the fraction of the nitrogen-hydrogen feed that gets converted to ammonia. At 500 °C and 250 atm, this equation can be formulated as*

$$F(u) = u^4 - 7.79075u^3 + 14.7445u^2 + 2.511u - 1.674.$$

Let $B = (0.3, 0.4)$ and choose $x_0 = 0.3$. Define $[\delta_1, \delta_2; F] = \frac{F(\delta_2) - F(\delta_1)}{\delta_2 - \delta_1}$, for $\delta_1 \neq \delta_2$. Then, the conditions (A_1)–(A_2) are valid if

$$\|F'(x_0)^{-1}F(x_0)\| = 0.0217956 = \Delta,$$

$\omega_0 = \omega = 1.56036$, $\rho_0 = 0.640877$, $B_0 = S[x_0, \rho] \cap B$ and $\psi(t_1, t_2, t_3) = \omega_0(t_1) + \frac{1}{2}(\omega_0(t_2) + \omega_0(t_3))$. Tables 4–6 verifies the conditions (6) and (7) of Lemma 1.

Estimates for $\omega_0(a_n)$ and a_n.

Table 4. $\alpha = 0.75$.

n	0	1	2	3	4	5	6	7
$\omega_0(a_n)$	0	0.0366798	0.0404137	0.040727	0.0407527	0.0407548	0.0407549	0.0407549
a_n	0	0.0235073	0.0259002	0.026101	0.0261175	0.0261188	0.0261189	0.0261189

$a^* = 0.0261189$.

Table 5. $\alpha = 1$.

n	0	1	2	3	4	5	6	7
$\omega_0(a_n)$	0	0.0358561	0.0385603	0.0387014	0.0387085	0.0387089	0.0387089	0.0387089
a_n	0	0.0229794	0.0247124	0.0248029	0.0248074	0.0248076	0.0248077	0.0248077

$a^* = 0.0248077$.

Table 6. $\alpha = 1.25$.

n	0	1	2	3	4	5	6	7
$\omega_0(a_n)$	0	0.0363564	0.0396474	0.0398455	0.0398568	0.0398574	0.0398575	0.0398575
a_n	0	0.0233	0.0254091	0.0255361	0.0255433	0.0255437	0.0255438	0.0255438

$a^* = 0.0255438.$

Hence, we can conclude that condition (A_4) holds and therefore all the conditions of Theorem 1 are fulfilled. Hence, the sequence $\{x_n\}$ generated by the method (3) converges to a solution u^* of equation the $F(u) = 0$ in $S[x_0, a^*]$.

Example 3. *Next, we reconsider the motivational example from the introduction part of this study. Choose $x_0 = 0.9955$. $[\delta_1, \delta_2; F]$ is a divided difference of order one and defined as $[\delta_1, \delta_2; F] = \frac{F(\delta_2) - F(\delta_1)}{\delta_2 - \delta_1}$, for $\delta_1 \neq \delta_2$. Conditions (A_1) and (A_2) are verified for*

$$\|F'(x_0)^{-1}F(x_0)\| = 0.00456182 = \Delta,$$

$\omega_0 = 12.8089$, $\rho_0 = 0.0780704$, $B_0 = S(x_0, \rho_0) \cap B$, $\omega = 1.12091$ *and* $\psi(t_1, t_2, t_3) = \omega_0(t_1) + \frac{1}{2}(\omega_0(t_2) + \omega_0(t_3))$. *Tables 7–9 verifies the conditions (6) and (7) of Lemma 1 for* $\alpha = 0.75, 1$ *and* 1.25 *respectively.*

Estimates for $\omega_0(a_n)$ and a_n.

Table 7. $\alpha = 0.75$.

n	0	1	2	3	4	5
$\omega_0(a_n)$	0	0.0591554	0.0600971	0.0601084	0.0601086	0.0601086
a_n	0	0.0046183	0.00469182	0.00469271	0.00469272	0.00469272

$a^* = 0.00469272.$

Table 8. $\alpha = 1$.

n	0	1	2	3	4
$\omega_0(a_n)$	0	0.0589339	0.0596338	0.0596393	0.0596394
a_n	0	0.00460101	0.00465565	0.00465609	0.00465609

$a^* = 0.00465609.$

Table 9. $\alpha = 1.25$.

n	0	1	2	3	4	5
$\omega_0(a_n)$	0	0.059072	0.0599212	0.0599287	0.0599287	0.0599287
a_n	0	0.00461179	0.00467809	0.00467867	0.00467868	0.00467868

$a^* = 0.00467868.$

From Tables 7–9, we can easily verify that condition (A_4) holds. Therefore, all the assumptions (A_1)–(A_5) of Theorem 1 are satisfied and hence, we can conclude that the sequence $\{x_n\}$ generated by the method (3) converges to a solution u^* of the equation $F(u) = 0$ in $S[x_0, a^*]$.

5. Conclusions

The semi-local convergence theorem for method (3) is established by applying generalized Lipschitz condition only on the first derivative. As a result, estimates on convergence balls, measurable error distances and the existence-uniqueness regions for the solution are deduced. At the end, the suggested theoretical outcomes are verified on application problems. It is found that method (3) has the smallest convergence radius for $\alpha = 1$. As, the value of α drifts away from one, the convergence radii of the balls increases. This approach is so general and independent of the method that it can be employed on other high convergence order methods including multi-point ones. That will be our future focus. In particular, we will also provide necessary conditions for the semi-local convergence of old and new single and multi-step high convergence order methods.

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Abbreviations

The following abbreviations are used in this manuscript:

ω, ω_0	Lipschitz constants
$L(U,V)$	Set of Linear operators from U to V
$\{a_n\}$	Scalar sequence

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