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Extended Convergence for Two Sixth Order Methods under the Same Weak Conditions

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Abstract: High-convergence order iterative methods play a major role in scientific, computational and engineering mathematics, as they produce sequences that converge and thereby provide solutions to nonlinear equations. The convergence order is calculated using Taylor Series extensions, which require the existence and computation of high-order derivatives that do not occur in the methodology. These results cannot, therefore, ensure that the method converges in cases where there are no such high-order derivatives. However, the method could converge. In this paper, we are developing a process in which both the local and semi-local convergence analyses of two related methods of the sixth order are obtained exclusively from information provided by the operators in the method. Numeric applications supplement the theory.

Keywords: non-linear equations; Fréchet derivative; convergence; Banach space

MSC: 37N30; 47J25; 49M15; 65H10; 65J15

1. Introduction

The problem most common in applied and computational mathematics, and in the fields of science and engineering generally, is that of finding a solution to a nonlinear equation.

$$F(x) = 0 \tag{1}$$

where $F : \Omega \subseteq X \to Y$ is derivable as per Fréchet, X and Y are complete normed linear spaces and Ω is a non-null, open and convex set.

Researchers have battled for a long time to overcome this nonlinearity. In most of the cases, a direct solution is very hard to obtain. For this reason, the use of an iterative algorithm to arrive at a conclusion has been widely used by researchers and scientists. Newton's method is a well-known iterative method for handling non-linear equations. Many new iterative strategies of higher order for the handling of non-linear equalities have been detected and are being applied in the last few years [1–11]. Theorems of convergence in the majority of these papers, however, are deduced by the application of high-order derivatives. In addition, the results are not discussed in terms of error bounds, convergence radii, or in the region where the solution is unique.

Examining local (LCA) and semi-local analyses (SLA) of an iterative algorithm makes it possible to estimate convergence domains, error estimates, and the unique region of a solution. The local and semi-local convergence results of efficient iterative methods were derived and stated in [9–13]. Important results were presented in these works, which include convergence radii, error estimation measurement, and extended benefits of this iteration approach. The results of this kind of analysis are valuable because they illustrate the complexities of starting point selection. Additionally, the applicability of our analysis

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can be extended to engineering problems such as the shrinking projection methods used for solving variational inclusion problems as in [14–16].

In this article, convergence theorems are developed for two competing methods having sixth order convergence found in [17] and are as stated below:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - 2(F'(x_n) + F'(y_n))^{-1}F(x_n) \\ x_{n+1} &= z_n - (\frac{7}{2}I - 4F'(x_n)^{-1}F'(y_n) + \frac{3}{2}(F'(x_n)^{-1}F'(y_n)^2)F'(x_n)^{-1}F(z_n) \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ A_n &= F'(x_n) + 2F'(\frac{x_n + y_n}{2}) + F'(y_n) \\ z_n &= y_n - 4A_n^{-1}F(y_n) \\ B_n &= F'(x_n) + 2F'(\frac{x_n + z_n}{2}) + F'(z_n) \\ x_{n+1} &= z_n - 4B_n^{-1}F(z_n). \end{aligned} \tag{3}$$

The local convergence of methods (2) and (3) are given in [17]. The order was established assuming that the seventh derivative (at least) of the operator F exists. As a result, these schemes' applicability is limited. In order to observe it, we define F on $\Omega = [-0.5, 1.5]$ by

$$F(t) = \begin{cases} t^3 \ln(t) + 3t^5 - 3t^4, & \text{if} \quad t \neq 0 \\ 0, & \text{if} \quad t = 0 \end{cases} \tag{4}$$

The third derivative is given by

$$F'''(t) = 11 - 72t + 180t^2 + 6\ln(t).$$

Hence, due to the unboundedness of F''', the conclusions on convergence of (2) and (3) are not true for this example. Nor does it provide a formula for the approximation of the error, the region of convergence, or the singleness and exact location of its root x^*. This strengthens our idea to develop the Ball-Convergence-Theory and thus compare the convergence range of (2) and (3) using hypotheses based on F' only. This research provides important formulas for the assessment of errors and convergence radii. The study also discusses the precise position and singleness of x^*.

The rest of the contents are: Section 2 deals with the LCA of the methods (2) and (3). Section 3 discusses the SLA of the methods under consideration. Numerical examples are in Section 4. Concluding comments are also included.

2. LCA

Set $M = [0, +\infty)$. Certain functions defined on the interval M play a role in the LCA of these methods. Assume:

(i) $\exists$ function $\omega_0 : M \to \mathbb{R}$, which is non-decreasing and continuous such that the function

$$\omega_0(t) - 1$$

admits a smallest positive root ρ_0. Set $M_0 = [0, \rho_0)$.

(ii) $\exists$ a function $\omega : M_0 \to \mathbb{R}$, which is non-decreasing and continuous such that the function

$$g_1(t) - 1$$

admits a smallest positive root $r_1 \in M_0$, where $g_1 : M_0 \to \mathbb{R}$ is

$$g_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)}.$$

(iii) The function $p(t) - 1$ has a smallest positive root $\rho_p \in M_0$, where the function $p : M_0 \to \mathbb{R}$ is given as

$$p(t) = \frac{1}{2}(\omega_0(t) + \omega_0(g_1(t)t)).$$

Set $M_1 = [0, \rho)$, where $\rho = min\{\rho_0, \rho_p\}$.

(iv) The functions $g_2(t) - 1, g_3(t) - 1$ have smallest positive roots $r_2, r_3 \in M_1$, where $g_2 : M_1 \to \mathbb{R}, g_3 : M_1 \to \mathbb{R}$ are given by

$$g_2(t) = g_1(t) + \frac{\bar{\omega}(t)(1 + \int_0^1 \omega_0(\theta t)d\theta)}{2(1-\omega_0(t))(1-p(t))},$$

$$g_3(t) = \left[1 + \frac{1}{2}\left(3\left(\frac{\bar{\omega}(t)}{1-\omega_0(t)}\right)^2 + 2\left(\frac{\bar{\omega}(t)}{1-\omega_0(t)}\right) + 2\right)\frac{(1 + \int_0^1 \omega_0(\theta g_2(t)t)d\theta}{1-\omega_0(t)}\right] g_2(t),$$

$$\bar{\omega}(t) = \begin{cases} \omega_0(t) + \omega_0(g_1(t)t) \\ \omega(t(1+g_1(t))) \end{cases}.$$

Note that in practice, we choose the smallest of the two functions in the formula for the function $\bar{\omega}$.

Define the parameter r as

$$r = min\{r_m\}, \quad m = 1,2,3. \tag{5}$$

The parameter r is shown to be a radius of convergence (RC) for the method (2) (see Theorem 1).

Let $M_2 = [0, r)$. Then, for each $t \in M_2$, the following items hold:

$$0 \le \omega_0(t) < 1 \tag{6}$$

$$0 \le p(t) < 1 \tag{7}$$

$$\text{and} \quad 0 \le g_m(t) < 1. \tag{8}$$

The notation $U(x^*, \alpha)$ stands for the open ball with center x^* and of radius $\alpha > 0$, whereas $U[x^*, \alpha]$ stands for the closure of the ball $U(x^*, \alpha)$.

The scalar functions ω_0 and ω relate x^* to operators appearing on the method (2) or the method (3) are as follows.

Suppose:

(H_1) $\exists$ a solution $x^* \in \Omega$ of the equation $F(x) = 0$ such that $F'(x^*)^{-1} \in \mathscr{L}(Y, X)$.

(H_2) $\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \le \omega_0(\|x - x^*\|)$ for each $x \in \Omega$.
Set $\Omega_0 = U(x^*, \rho_0) \cap \Omega$.

(H_3) $\|F'(x^*)^{-1}(F'(u_2) - F'(u_1))\| \le \omega(\|u_2 - u_1\|)$ for each $u_1, u_2 \in \Omega_0$.

(H_4) $U[x^*, d] \subset \Omega$, where d is specified later.

The conditions (H_1)–(H_2) are utilized first to prove the convergence of the method (2). Let $l_n = \|x_n - x^*\|$.

Theorem 1. *Assume the conditions (H_1)–(H_4) hold and the initial guess $x_0 \in U(x^*, d)$ for $d = r$. Then, the following assertion holds:*

$$\lim_{n\to\infty} x_n = x^*$$

Proof. The iterates $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ shall be shown to exist in the ball $U(x^*, r)$ by mathematical induction. Let $u \in U(x^*, r)$, but arbitrary. By utilizing item (6) and the hypotheses (H_1), (H_2),

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq \omega_0(\|u - x^*\|) \leq \omega_0(r) < 1.$$

Then, it follows by the standard Lemma due to Banach [12,18] involving linear operators that their inverses $F'(u)^{-1} \in \mathscr{L}(Y, X)$ with

$$\|F'(u)^{-1}F'(x^*)\| \leq \frac{1}{1 - \omega_0(\|u - x^*\|)}. \tag{9}$$

If we choose $u = x_0$, then the iterate y_0 exists by the first sub-step of the method (2) if $k = 0$, since by hypothesis $x_0 \in U(x^*, r)$. Moreover, we have

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1} - x^* = F'(x_0)^{-1}[\int_0^1 F'(x^* + \vartheta(x_0 - x^*))d\vartheta - F'(x_0)](x_0 - x^*),$$

which gives by (H_3), (8) (for $m = 1$), (9) (for $u = x_0$) and (5) that

$$\|y_0 - x^*\| \leq \frac{\int_0^1 \omega((1-\vartheta)l_0)d\vartheta}{1 - \omega_0(l_0)} \leq g_1(l_0)l_0 \leq l_0 < r. \tag{10}$$

Thus, the iterate $y_0 \in U(x^*, r)$. Then, by (5), (7), (H_2) and (10), we obtain

$$\begin{aligned}\|(2F'(x^*))^{-1}(F'(x_0) + F'(y_0) - 2F'(x^*))\| &\leq \tfrac{1}{2}(\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|) \\ &\leq \frac{1}{2}(\omega_0(l_0) + \omega_0(\|y_0 - x^*\|)) \\ &\leq p(l_0) \leq p(r) < 1,\end{aligned}$$

so

$$\|(F'(x_0) + F'(y_0))^{-1}F'(x^*)\| \leq \frac{1}{2(1 - p(l_0))}. \tag{11}$$

Hence, the iterate z_0 exists by the second sub-step of the method (2) and

$$\begin{aligned}z_0 - x^* &= y_0 - x^* + (F'(x_0)^{-1} - 2(F'(x_0) + F'(y_0))^{-1})F(x_0) \\ &= y_0 - x^* + F'(x_0)^{-1}(F'(x_0) + F'(y_0) - 2F'(x_0))(F'(x_0) + F'(y_0))^{-1}F(x_0),\end{aligned}$$

thus

$$\begin{aligned}\|z_0 - x^*\| &\leq \|y_0 - x^*\| + \frac{\bar{\omega}_n(1 + \int_0^1 \omega_0(\vartheta l_0))d\vartheta\|x_n - x^*\|}{2(1 - \omega_0(l_0))(1 - p(l_0))} \\ &\leq g_2(l_0)l_0 \leq l_0,\end{aligned} \tag{12}$$

since

$$\bar{\omega}_n = \begin{cases} \omega_0(l_0) + \omega_0(\|y_0 - x^*\|) \\ \omega(l_0 + \|y_0 - x^*\|) \end{cases},$$

$$F(x_0) = F(x_n) - F(x^*) = \int_0^1 [F'(x^* + \vartheta(x_0 - x^*))d\vartheta - F'(x^*) + F'(x^*)](x_n - x^*),$$

hence, $\|F'(x^*)^{-1}F(x_0)\| \leq (1 + \int_0^1 \omega_0(\vartheta\|x_0 - x^*\|)d\vartheta)l_n.$

It also follows by (12) that the iterate $z_0 \in U(x^*, r)$. Furthermore, the iterate x_1 exists by the third sub-step of the method (2) for $k = 0$. By the third sub-step, it follows in turn

$$x_1 - x^* = z_0 - x^* - \frac{1}{2}[3(I - F'(x_n)^{-1}F'(y_n))^2 + (I - F'(x_n)^{-1}F'(y_n)) + 2I]F'(x_n)^{-1}F(z_n)$$

leading to

$$l_1 \leq [1 + \frac{1}{2}(3(\frac{\bar{\omega}_n}{1-\omega_0(l_n)})^2 + 2(\frac{\bar{\omega}_n}{1-\omega_0(l_n)}) + 2)\frac{(1+\int_0^1 \omega_0(\vartheta\|z_n - x^*\|)d\vartheta)}{1-\omega_0(l_n)}]\|z_n - x^*\| \\ \leq g_3(l_n)l_n \leq l_n. \tag{13}$$

Thus, the iterate $x_1 \in U(x^*, r)$. Exchange x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding calculations to see that the following estimates hold:

$$\|y_k - x^*\| \leq g_1(l_k)l_k \leq l_k < r,$$
$$\|z_k - x^*\| \leq g_2(l_k)l_k \leq l_k$$

and

$$l_{k+1} \leq g_3(l_k)l_k \leq l_k.$$

Therefore, the iterates $\{x_k\}, \{y_k\}, \{z_k\} \in U(x^*, r)$. Finally, from

$$l_{k+1} \leq \xi l_k < r, \quad \xi = g_3(l_0) \in [0, 1)$$

it follows $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. □

The following proposition is to determine the uniqueness of this solution x^*.

Proposition 1. *Assume:*

(i) *$\exists$ a solution $\bar{x} \in U(x^*, \rho_2)$ for some $\rho_2 > 0$.*

(ii) *The hypothesis (H_2) holds on $U(x^*, \rho_2)$.*

(iii) *There exists $\rho_3 > \rho_2$ such that*

$$\int_0^1 \omega_0(\vartheta\rho_3)d\vartheta < 1.$$

Set $\Omega_2 = U(x^*, \rho_3) \cap \Omega$. Then, the only solution of (1) in the region Ω_2 is x^*.

Proof. Assume $\exists\ \bar{x} \in \Omega_2$ with $F(\bar{x}) = 0$. It follows that for

$$Q = \int_0^1 F'(x^* + \vartheta(\bar{x} - x^*))d\vartheta,$$

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 \omega_0(\vartheta\|\bar{x} - x^*\|)d\vartheta$$
$$\leq \int_0^1 \omega_0(\vartheta\rho_3)d\vartheta < 1,$$

thus, $\bar{x} = x^*$ by the identity $\bar{x} - x^* = Q^{-1}(F(\bar{x} - F(x^*)) = Q^{-1}(0) = 0$ and the invertibility of the operator Q. □

The LCA of the method (3) is obtained analogously, but the functions g_2 and g_3 are given instead by

$$
\begin{aligned}
g_2(t) &= \left[\frac{\int_0^1 \omega((1-\vartheta)g_1(t)t)d\vartheta}{1-\omega_0(g_1(t)t)} + \frac{(1+\int_0^1 \omega_0(\vartheta g_1(t)t)d\vartheta)(\bar{\omega}(t)+2(\frac{(1+g_1(t))t}{2}))}{(1-\omega_0(g_1(t)t))(1-q(t))} \right] g_1(t),\\
q(t) &= \frac{1}{4}(\omega_0(t) + 2\omega_0\left(\frac{(1+g_1(t))t}{2}\right) + \omega_0(g_1(t)t)),\\
q^1(t) &= \frac{1}{4}[\omega_0(t) + 2\omega_0\left(\frac{(1+g_2(t))t}{2}\right) + \omega_0(g_2(t)t)]\\
\text{and} \quad g_3(t) &= \left[\frac{\int_0^1 \omega((1-\vartheta)g_2(t)t)d\vartheta}{1-\omega_0(g_2(t)t)} + \frac{(1+\int_0^1 \omega_0(\vartheta g_2(t)t)d\vartheta)(\omega^1(t)+2\omega(\frac{(1+g_2(t))t}{2}))}{(1-\omega_0(g_2(t)t))(1-q^1(t))} \right] g_2(t),\\
\text{where} \quad \omega^1(t) &= \begin{cases} \omega_0(t) + \omega_0(g_2(t)t) \\ \omega((1+g_2(t))t) \end{cases}.
\end{aligned}
$$

This time the RC $\bar{r}$ is provided again by the formula (5), but with the new functions g_2 and g_3. Then, similarly under the conditions (H_1)–(H_4) with $d = \bar{r}$, it follows

$$
\begin{aligned}
\|(4F'(x^*))^{-1}(A_n - 4F'(x^*))\| &\leq \tfrac{1}{4}\Big[\|F'(x^*)^{-1}(F'(x_n) - F'(x^*))\| + 2\|F'(x^*)^{-1}(F'\left(\tfrac{x_n+y_n}{2}\right) - F'(x^*))\|\\
&\quad + \|F'(x^*)^{-1}(F'(y_n) - F'(x^*))\|\Big]\\
&\leq q(l_n) = q_n < 1,\\
\|A_n^{-1}F'(x^*)\| &\leq \frac{1}{1-q_n},\\
\|F'(x^*)^{-1}(A_n - 4F'(y_n))\| &\leq \|F'(x^*)^{-1}(F'(x_n) - F'(y_n))\| + 2\|F'(x^*)^{-1}(F'(\tfrac{x_n+y_n}{2}) - F'(y_n))\|\\
&\leq \bar{\omega}_n + 2\omega\left(\frac{\|y_n - x_n\|}{2}\right),\\
\|F'(x^*)^{-1}F(y_n)\| &= \| \textstyle\int_0^1 F'(x^*)^{-1}[F'(x^* + \vartheta(y_n - x^*))d\vartheta - F'(x^*) + F'(x^*)](y_n - x^*)\|\\
&\leq (1 + \int_0^1 \omega_0(\vartheta\|y_n - x^*\|)d\vartheta)\|y_n - x^*\|,\\
z_n - x^* &= y_n - x^* - F'(y_n)^{-1}F(y_n) + F'(y_n)^{-1}(A_n - 4F'(y_n))A_n^{-1}F(y_n),\\
\|z_n - x^*\| &\leq \left[\frac{\int_0^1 \omega((1-\vartheta)\|y_n-x^*\|)d\vartheta}{1-\omega_0(\|y_n-x^*\|)} + \frac{(1+\int_0^1 \omega_0(\vartheta\|y_n-x^*\|)d\vartheta)(\bar{\omega}_n+2\omega(\frac{\|y_n-x_n\|}{2}))}{(1-\omega_0(\|y_n-x^*\|))(1-q_n)}\right]\|y_n - x^*\|\\
&\leq g_2(l_n)l_n \leq l_n,\\
x_{n+1} - x^* &= z_n - x^* - F'(z_n)^{-1}F(z_n) + F'(z_n)^{-1}(B_n - 4F'(z_n))F(z_n),\\
l_{n+1} &\leq \left[\frac{\int_0^1 \omega((1-\vartheta)\|z_n-x^*\|)d\vartheta}{1-\omega_0(\|z_n-x^*\|)} + \frac{(1+\int_0^1 \omega_0(\vartheta\|z_n-x^*\|)d\vartheta)(\bar{\omega}_n^1+2\omega(\frac{\|y_n-x_n\|}{2}))}{(1-\omega_0(\|z_n-x^*\|))(1-q_n^1)}\right]\|z_n - x^*\|\\
&\leq g_3(l_n)l_n \leq l_n,\\
\text{where} \quad \bar{\omega}_n^1 &= \begin{cases} \omega_0(l_n) + \omega_0(\|z_n - x^*\|) \\ \omega((l_n + \|z_n - x^*\|) \end{cases}.
\end{aligned}
$$

Therefore, under the above-mentioned changes, the conclusions of the Theorem 1 hold, but for the method (3). The results of the Proposition (1) obviously also apply to the method (3). Therefore, we can provide the corresponding result for the method (3).

Theorem 2. *Assume the conditions (H_1)–(H_4) hold for $d = \bar{r}$ and the initial guess $x_0 \in U(x^*, \bar{r})$. Then, the following assertion holds:*

$$\lim_{n\to+\infty} x_n = x^*.$$

Proof. It follows from Theorem 1 under the preceding changes. □

Remark 1. *Under the conditions* (H_1)*–*(H_4)*, we can set* $\rho_2 = r$ *or* $\rho_2 = \bar{r}$ *in Proposition 1 depending on which method is used.*

3. SLA

If the role of x^* is replaced by x_0 in the calculations of the previous section, one can introduce the SLA utilizing majorizing sequences. These sequences are defined for some $\lambda \geq 0$, respectively, by $t_0 = 0$, $s_0 = \lambda$,

$$\begin{aligned}
p_n &= p(t_n) = \frac{1}{2}(\omega_0(t_n) + \omega_0(s_n)), \\
u_n &= s_n + \frac{\omega(s_n - t_n)(s_n - t_n)}{(1 - \omega_0(t_n))(1 - p_n)}, \\
a_n &= (1 + \int_0^1 \omega_0(t_n + \vartheta(u_n - t_n))d\vartheta)(u_n - t_n) + (1 + \omega_0(t_n))(s_n - t_n), \\
t_{n+1} &= u_n + \frac{1}{2}\left[3(\frac{\omega(s_n - t_n)}{1 - \omega_0(t_n)})^2 + 2(\frac{\omega(s_n - t_n)}{1 - \omega_0(t_n)}) + 2\right]\frac{a_n}{1 - \omega_0(t_n)}, \\
\mu_{n+1} &= \int_0^1 \omega((1 - \vartheta)(t_{n+1} - t_n))d\vartheta(t_{n+1} - t_n) + (1 + \omega_0(t_n))(t_{n+1} - s_n), \\
s_{n+1} &= t_{n+1} + \frac{\mu_{n+1}}{1 - \omega_0(t_{n+1})}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\psi(t_n) &= \frac{1}{4}(\omega_0(t_n) + 2\omega_0(\frac{s_n - t_n}{2}) + \omega_0(s_n)), \\
u_n &= s_n + 4\frac{\int_0^1 \omega((1 - \vartheta)(s_n - t_n))d\vartheta(s_n - t_n)}{1 - \psi(t_n)}, \\
b_n &= (1 + \int_0^1 \omega_0(s_n + \vartheta(u_n - s_n))d\vartheta)(u_n - s_n) + \int_0^1 \omega((1 - \vartheta)(s_n - t_n))d\vartheta(s_n - t_n), \\
\psi_1(t_n) &= \frac{1}{4}(\omega_0(t_n) + 2\omega_0(\frac{t_n + u_n}{2}) + \omega_0(u_n)), \\
t_{n+1} &= u_n + 4\frac{b_n}{1 - \psi_1(t_n)}, \\
s_{n+1} &= t_{n+1} + \frac{\mu_{n+1}}{1 - \omega_0(t_{n+1})}.
\end{aligned} \tag{15}$$

These sequences majorize $\{x_n\}$ (see Theorem 3). However, first, we develop some convergence conditions for them.

Lemma 1. *Assume for each* $n = 0, 1, 2, \ldots$

$$\omega_0(t_n) < 1, \quad p(t_n) < 1 \quad \text{and} \quad t_n \leq \xi \quad \text{for some} \quad \xi \geq 0. \tag{16}$$

Then, the sequence $\{t_n\}$ *given by the method (2) is bounded from above by* ξ*, non-decreasing and is convergent to some* $\xi^* \in [0, \xi]$.

Proof. The result is implied immediately from the formula (14) and the condition (16). □

Lemma 2. *Suppose that for each* $n = 0, 1, 2, \ldots$

$$\omega_0(t_n) < 1, \quad \psi(t_n) < 1, \quad \psi_1(t_n) < 1 \quad \text{and} \quad t_n \leq \xi_1 \quad \text{for some} \quad \xi_1 \geq 0. \tag{17}$$

Then, the sequence $\{t_n\}$ given by the formula (15) is bounded from above by ξ_1 and is convergent to some $\xi_1^* \in [0, \xi_1]$.

Proof. The result is implied immediately by the formula (15) and the condition (17). □

Remark 2. *A possible choice for the upper bounds ξ or ξ_1 is ρ_0 given in (i) of Section 2.*

The following conditions are used for both methods. Suppose:

(C_1) There exists an element $x_0 \in \Omega$ and a parameter $\lambda \geq 0$ with $F'(x_0)^{-1} \in \mathscr{L}(Y, X)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq \lambda$.

(C_2) $\|F'(x_0)^{-1}(F'(u) - F'(x_0))\| \leq \omega_0(\|x - x_0\|)$ for each $u \in \Omega$.
Set $\Omega_1 = U(x_0, \rho_0) \cap \Omega$.

(C_3) $\|F'(x_0)^{-1}(F'(u_2) - F'(u_1))\| \leq \omega(\|u_2 - u_1\|)$ for each $u_1, u_2 \in \Omega_1$.

(C_4) Conditions (16) and (17) hold for the methods (2) and (3), respectively.

(C_5) $U[x_0, \bar{\xi}] \subset \Omega$, where $\bar{\xi} = \xi$ or $\bar{\xi} = \xi_1$ depending on which method is used.

Next, we are developing the semi-local convergence theorem for the method (2).

Theorem 3. *Under the conditions (C_1)–(C_5), the sequence $\{x_n\}$ generated by the method (2) is convergent to a solution $x^* \in U[x_0, \xi^*]$ of the given equation $F(x) = 0$.*

Proof. As in Theorem 1, mathematical induction and the following calculations lead in turn to

$$
\begin{aligned}
z_n - y_n &= -(F'(x_n) + F'(y_n))^{-1}(2F'(x_n) - (F'(x_n) + F'(y_n)))F'(x_n)^{-1}F(x_n), \\
\|z_n - y_n\| &\leq \frac{\omega(\|y_n - x_n\|)\|y_n - x_n\|}{(1 - p(\|x_n - x_0\|))(1 - \omega_0(\|x_n - x_0\|))} \\
&\leq \frac{\omega(s_n - t_n)(s_n - t_n)}{(1 - p(t_n))(1 - \omega_0(t_n))} = u_n - s_n, \\
\|x_{n+1} - z_n\| &\leq \frac{1}{2}(3(\frac{\omega(\|y_n - x_n\|)}{1 - \omega_0(\|x_n - x_0\|)})^2 + 2(\frac{\omega(\|y_n - x_n\|)}{1 - \omega_0(\|x_n - x_0\|)}) + 2)\frac{a_n}{1 - \omega_0(\|x_n - x_0\|)} \\
&\leq \frac{1}{2}(3(\frac{\omega(s_n - t_n)}{1 - \omega_0(t_n)})^2 + 2(\frac{\omega(s_n - t_n)}{1 - \omega_0(t_n)}) + 2)\frac{a_n}{1 - \omega_0(t_n)} \\
&= t_{n+1} - u_n, \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - y_n),
\end{aligned}
$$

$$
\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \|\textstyle\int_0^1 F'(x_0)^{-1}(F'(x_n + \vartheta(x_{n+1} + x_n))d\vartheta - F'(x_n))(x_{n+1} - x_n) + \|F'(x_0)^{-1}(F'(x_n) - F'(x_0) - F'(x_0))\| \\
&\leq \textstyle\int_0^1 \omega((1 - \vartheta)\|x_{n+1} - x_n\|)d\vartheta\|x_{n+1} - x_n\| + (1 + \omega_0(\|x_n - x_0\|))\|x_{n+1} - y_n\| \\
&\leq \int_0^1 \omega((1 - \vartheta)(t_{n+1} - t_n))d\vartheta(t_{n+1} - t_n) + (1 + \omega_0(t_n))(t_{n+1} - s_n) = \mu_{n+1},
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\
&\leq \frac{\mu_{n+1}}{1 - \omega_0(\|x_{n+1} - x_0\|)} \leq \frac{\mu_{n+1}}{1 - \omega_0(t_{n+1})} = s_{n+1} - t_{n+1}, \\
\|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \\
&\leq u_n - s_n + s_n - t_0 = u_n \leq \xi^*, \\
\|x_{n+1} - x_0\| &\leq \|x_{n+1} - z_n\| + \|z_n - x_0\| \\
&\leq t_{n+1} - u_n + u_n - t_0 = t_{n+1} \leq \xi^*, \\
\|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \\
&\leq s_{n+1} - t_{n+1} + t_{n+1} - t_0 = s_{n+1} \leq \xi^*,
\end{aligned}
$$

since

$$\begin{aligned} F(z_n) &= F(z_n) - F(x_n) + F(x_n) \\ &= \int_0^1 (F'(x_n + \vartheta(z_n - x_n))d\vartheta - F'(x_0) + F'(x_0))(z_n - x_n) + (F'(x_n) - F'(x_0) + F'(x_0))(y_n - x_n), \end{aligned}$$

so $$\begin{aligned} \|F'(x_0)^{-1}F(z_n)\| &\le (1 + \int_0^1 \omega_0(\|x_n - x_0\| + \vartheta\|z_n - x_n\|)d\vartheta)\|z_n - x_n\| + (1 + \omega_0(\|x_n - x_0\|))\|y_n - x_n\| = \bar{a}_n \\ &\le (1 + \int_0^1 \omega_0(t_n + \vartheta(u_n - t_n))d\vartheta)(u_n - t_n) + (1 + \omega_0(t_n))(s_n - t_n) = a_n. \end{aligned}$$

Notice also that $\|y_0 - x_0\| = \lambda = s_0 - t_0 \le \xi^*$, so $y_0 \in U[x_0, \xi]$ initiating the induction. Thus, the sequence $\{x_n\}$ is fundamental in a Banach space X (since $\{t_n\}$ is fundamental as convergent by the condition (C_4)). By letting $n \to \infty$ in (18) and using the continuity of the operator F, we conclude that $F(x^*) = 0$. □

Proposition 2. *Assume:*

(i) $\exists$ a solution $y^* \in U(x_0, \rho_3)$ of (1) for some $\rho_3 > 0$.
(ii) The condition (C_2) holds on the ball $U(x_0, \rho_3)$.
There exists $\rho_4 \ge \rho_3$ such that

$$\int_0^1 \omega_0((1-\theta)\rho_3 + \theta\rho_4)d\theta < 1. \tag{19}$$

Set $\Omega_2 = U[x_0, \rho_4] \cap \Omega$.
Then, the only solution of the equation $F(x) = 0$ in the region Ω_2 is y^*.

Proof. Define the linear operator $G = \int_0^1 F'(y^* + \theta(\bar{y}^* - y^*))d\theta$ provided $\bar{y}^* \in \Omega$ and $F(\bar{y}^*) = 0$. It then follows that

$$\begin{aligned} \|F'(x_0)^{-1}(G - F'(x_0))\| &\le \int_0^1 \omega_0((1-\theta)\|y^* - x_0\| + \theta\|\bar{y}^* - x_0\|)d\theta \\ &\le \int_0^1 \omega_0((1-\theta)\rho_3 + \theta\rho_4)d\theta < 1. \end{aligned}$$

Hence, we deduce that $\bar{y}^* = y^*$. □

Remark 3.

(1) The parameter ρ_0 can replace ξ^* or ξ_1^* in the Theorem 3.
(2) Under conditions of Theorem 3, set $\rho_3 = \xi^*$ or $\rho_3 = \xi_1^*$ in the Proposition 2.

Similarly, for the method (3), we have in turn the estimates

$$\begin{aligned} \|z_n - y_n\| &\le 4\|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\ &\le \frac{4\int_0^1 \omega((1-\vartheta)\|y_n - x_n\|)d\vartheta\|y_n - x_n\|}{1 - \psi(\|x_n - x_0\|)} \\ &\le u_n - s_n, \\ \|x_{n+1} - z_n\| &\le 4\|B_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\ &\le \frac{4b_n}{1 - \psi_1(\|x_n - x_0\|)} \le t_{n+1} - u_n, \end{aligned}$$

and $$\begin{aligned} \|y_{n+1} - x_{n+1}\| &\le \frac{\mu_{n+1}}{1 - \omega_0(\|x_{n+1} - x_0\|)} \\ &\le s_{n+1} - t_{n+1}. \end{aligned}$$

Thus, the conclusions of Theorem 3 and Proposition 2 hold for the method (3) with (14), (16) replacing (15) and (17), respectively.

Theorem 4. *Under the conditions (C_1)–(C_5), the sequence $\{x_n\}$ provided by the method (3) is convergent to a solution $x^* \in U[x_0, \xi_1^*]$ of (1).*

Proof. See Theorem 3 under the preceding changes. □

4. Numerical Examples

Example 1. *Let $X = Y = \mathbb{R}$. Define the function F on $\Omega = [-1, 1]$ by*

$$F(x) = e^x - 1.$$

We obtain $x^* = 0$ as a root of $F(x)$. The conditions (H_1)–(H_4) are satisfied for $\omega_0(t) = (e-1)t, \rho_0 = 0.581977, \Omega_0 = U(x^*, \rho_0) \cap \Omega$ and $\omega(t) = et$. Then, the radii obtained are as given in Table 1.

Table 1. Radii for Examples 1 and 2.

Example 1		Example 2	
Method (2)	**Method (3)**	**Method (2)**	**Method (3)**
$r_1 = 0.324947$	$r_1 = 0.324947$	$r_1 = 0.666667$	$r_1 = 0.666667$
$r_2 = 0.201201$	$r_2 = 0.196671$	$r_2 = 0.390253$	$r_2 = 0.444428$
$r_3 = 0.123369$	$r_3 = 0.16627$	$r_3 = 0.240567$	$r_3 = 0.373689$
$r = 0.123369$	$\bar{r} = 0.16627$	$r = 0.240567$	$\bar{r} = 0.373689$

Example 2. *We define the function $F(x) = \sin x$ on Ω, where $X = Y = \Omega = \mathbb{R}$. We have $F'(x) = \cos x$ and also $x^* = 0$ is the solution of $F(x) = 0$. Now, the conditions (H_1)–(H_4) are validated for $\omega_0(t) = \omega(t) = t$. Then, the RC are as given in Table 1.*

Example 3. *Consider the system of differential equations with*

$$F_1'(v_1) = e^{v_1}, \quad F_2'(v_2) = (e-1)v_2 + 1, \quad F_3'(v_3) = 1$$

subject to the initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $X = Y = \mathbb{R}^3$ and $\Omega = U[0, 1]$. Then, $x^* = (0, 0, 0)^T$ solves (1). Define the function F on Ω for $v = (v_1, v_2, v_3)^T$ as

$$F(v) = (e^{v_1} - 1, \frac{e-1}{2}v_2^2 + v_2, v_3)^T.$$

Then, the Fréchet derivative is given by

$$F'(v) = \begin{bmatrix} e^{v_1} & 0 & 0 \\ 0 & (e-1)v_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, by the definition of F we have $F'(x^*) = 1$. Then, conditions (H_1)–(H_4) are satisfied if $\omega_0(t) = (e-1)t$, $\rho_0 = 0.581977$, $\Omega_0 = U(x^*, \rho_0) \cap \Omega$ and $\omega(t) = e^{\frac{1}{e-1}}t$. Then, the radii are listed in Table 2.

Table 2. Radii for Example 3.

Method 2	Method 3
$r_1 = 0.382692$	$r_1 = 0.382692$
$r_2 = 0.224974$	$r_2 = 0.242274$
$r_3 = 0.140272$	$r_3 = 0.205931$
$r = 0.140272$	$\bar{r} = 0.205931$

Example 4. *Let $X = Y = \mathbb{R}$. Let the function F on Ω for $\Omega = U(x_0, 1-\alpha)$ for some $\alpha \in [0,1)$ be*

$$F(x) = x^3 - \alpha$$

Fix $x_0 = 1$. Then, the conditions (C_1)–(C_3) are satisfied for $\lambda = \frac{1-\alpha}{3}$, $\omega_0(t) = (3-\alpha)t$, $\Omega_1 = (x_0, \frac{1}{3-\alpha})$, $\omega(t) = 2(1+\frac{1}{3-\alpha})t$. Choose $\bar{\xi} = \frac{1}{3-\alpha}$. The conditions of Lemmas 1 and 2 are verified in Tables 3 and 4, respectively.

Table 3. Estimates for method (2).

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\omega_0(t_n)$	0	0.0274273	0.060139	0.147707	0.1489201	0.1498202	0.1498211	0.1498211
$p(t_n)$	0.0153889	0.0325636	0.0728308	0.182755	0.1847021	0.1848412	0.1848423	0.1848423
t_n	0	0.013713	0.030068	0.0738499	0.0740089	0.07412632	0.0742212	0.0742212

$\xi^* = 0.0742212$.

Table 4. Estimates for method (3).

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\omega_0(t_n)$	0	0.0556283	0.0934655	0.114921	0.123201	0.124746	0.124812	0.124813	0.124813
$\psi(t_n)$	0.0170833	0.0403661	0.0547196	0.0610351	0.0623482	0.0624062	0.0624063	0.0624063	0.0624063
$\psi_1(t_n)$	0.0188072	0.0691335	0.101844	0.118574	0.123952	0.124779	0.124813	0.124813	0.124813
t_n	0	0.0271357	0.0455929	0.0560588	0.0600982	0.0608519	0.0608841	0.0608842	0.0608842

$\xi_1^* = 0.0608842$.

Hence, we can observe that conditions (C_4) and (C_5) hold for both the methods (2) and (3). Thus, the conclusions of Theorem (3) and Theorem (4) hold for (2) and (3), respectively, i.e., the sequence $\{x_n\}$ produced by the method (2) (or (3)) converges to $x^* \in U[x_0, \xi^*]$ (or $U[x_0, \xi_1^*]$).

Example 5. *Let $X = Y = \mathbb{R}^5, \Omega = U[0,1]$ and consider the system of 5 equations defined by*

$$\sum_{j=1, j\neq i}^{5} x_j - e^{x_i} = 0, \quad 1 \leq i \leq 5,$$

where $x^* = (0.20388835470224016\ldots, 0.20388835470224016\ldots, 0.20388835470224016\ldots, 0.20388835470224016\ldots, 0.20388835470224016\ldots)^T$ is a root.

Choose $x_0 = (0.3, 0.3, 0.3, 0.3, 0.3)^T$. Then, the errors are in Table 5.

Table 5. Error estimates for Example (5).

Methods	$\|x_0 - x^*\|$	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$
Method (2)	9.61116×10^{-2}	5.38484×10^{-4}	1.7643×10^{-17}	2.18261×10^{-98}
Method (3)	9.61116×10^{-2}	4.29875×10^{-5}	4.45051×10^{-25}	5.4804×10^{-145}

Therefore, we can say that methods (2) and (3) converge to x^*.

5. Conclusions

The LCA and SLA for the methods (2) and (3) are validated by applying a generalized condition of Lipschitz to the first derivative only. A comparison is made between the two convergence balls, which are very similar in terms of their efficiency. This study derives estimates of convergence balls, measurement of error distances, and existence-uniqueness regions of the solution. Finally, the proposed theoretical results are checked for application problems. The process of this article shall be applied on other high convergence order methods using inverses of operators that are linear in our future research [1–8].

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathscr{L}(X,Y)$	Set of Linear operators from X to Y
$\{t_n\}$	Scalar sequence

References

1. Cordero, A.; Gómez, E.; Torregrosa, J.R. Efficient high-order iterative methods for solving nonlinear systems and their application on heat conduction problems. *Complexity* **2017**, *2017*, 6457532. [CrossRef]
2. Babajee, D.K.R. On the Kung-Traub conjecture for iterative methods for solving quadratic equations. *Algorithms* **2016**, *9*, 1. [CrossRef]
3. Sharma, J.R.; Sharma, R.; Kalra, N. A novel family of composite Newton–Traub methods for solving systems of nonlinear equations. *Appl. Math. Comput.* **2015**, *269*, 520–535. [CrossRef]
4. Kung, H.; Traub, J.F. Optimal order of one-point and multipoint iteration. *J. ACM* **1974**, *21*, 643–651. [CrossRef]
5. Noor, M.A.; Noor, K.I.; Al-Said, E.; Waseem, M. Some new iterative methods for nonlinear equations. *Math. Probl. Eng.* **2010**, *2010*, 198943. [CrossRef]
6. Herceg, D.; Herceg, D. A family of methods for solving nonlinear equations. *Appl. Math. Comput.* **2015**, *259*, 882–895. [CrossRef]
7. Lotfi, T.; Bakhtiari, P.; Cordero, A.; Mahdiani, K.; Torregrosa, J.R. Some new efficient multipoint iterative methods for solving nonlinear systems of equations. *Int. J. Comput. Math.* **2015**, *92*, 1921–1934. [CrossRef]
8. Waseem, M.; Noor, M.A.; Noor, K.I. Efficient method for solving a system of nonlinear equations. *Appl. Math. Comput.* **2016**, *275*, 134–146. [CrossRef]
9. Argyros, I.K.; Sharma, D.; Argyros, C.I.; Parhi, S.K.; Sunanda, S.K. Extended iterative schemes based on decomposition for nonlinear models. *J. Appl. Math. Comput.* **2022**, *68*, 1485–1504. [CrossRef]
10. Argyros, C.I.; Argyros, I.K.; Regmi, S.; John, J.A.; Jayaraman, J. Semi-Local Convergence of a Seventh Order Method with One Parameter for Solving Non-Linear Equations. *Foundations* **2022**, *2*, 827–838. [CrossRef]
11. Argyros, I.K.; Regmi, S.; Shakhno, S.; Yarmola, H. Perturbed Newton Methods for Solving Nonlinear Equations with Applications. *Symmetry* **2022**, *14*, 2206. [CrossRef]
12. Argyros, I.K. *The Theory and Applications of Iteration Methods*, 2nd ed.; CRC Press/Taylor and Francis Publishing Group Inc.: Boca Raton, FL, USA, 2022.
13. John, J.A.; Jayaraman, J.; Argyros, I.K. Local Convergence of an Optimal Method of Order Four for Solving Non-Linear System. *Int. J. Appl. Comput. Math.* **2022**, *8*, 194. [CrossRef]
14. Hammad, H.; Cholamjiak, W.; Yambangwai, D.; Dutta, H. A modified shrinking projection methods for numerical reckoning fixed points of G-nonexpansive mappings in Hilbert spaces with graphs. *Miskolc Math. Notes* **2019**, *20*, 941–956. [CrossRef]
15. Hammad, H.A.; Rehman, H.U.; De la Sen, M. Shrinking projection methods for accelerating relaxed inertial Tseng-type algorithm with applications. *Math. Probl. Eng.* **2020**, *2020*, 7487383. [CrossRef]
16. Tuyen, T.M.; Hammad, H.A. Effect of shrinking projection and CQ-methods on two inertial forward–backward algorithms for solving variational inclusion problems. In *Rendiconti del Circolo Matematico di Palermo Series 2*; Springer: Cham, Switzerland, 2021; pp. 1–15.

17. Abro, H.A.; Shaikh, M.M. A new time-efficient and convergent nonlinear solver. *Appl. Math. Comput.* **2019**, *355*, 516–536. [CrossRef]
18. Kantorovich, L.V.; Akilov, G.P. *Functional Analysis in Normed Spaces*; Pergamon Press: Oxford, UK, 1964.