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Extended Newton-like Midpoint Method for Solving Equations in Banach Space

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Abstract: In this study, we present a convergence analysis of a Newton-like midpoint method for solving nonlinear equations in a Banach space setting. The semilocal convergence is analyzed in two different ways. The first one is shown by replacing the existing conditions with weaker and tighter continuity conditions, thereby enhancing its applicability. The second one uses more general ω-continuity conditions and the majorizing principle. This approach includes only the first order Fréchet derivative and is applicable for problems that were otherwise hard to solve by using approaches seen in the literature. Moreover, the local convergence is established along with the existence and uniqueness region of the solution. The method is useful for solving Engineering and Applied Science problems. The paper ends with numerical examples that show the applicability of our convergence theorems in cases not covered in earlier studies.

Keywords: nonlinear equations; Newton's method; local and semilocal convergence; Banach space; Fréchet derivative; majorizing sequences

MSC: 47J25; 49M15; 65J15; 65H10; 65G99

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1. Introduction

One of the most challenging problems in Engineering and Applied Sciences is to determine a locally unique solution x^* of a nonlinear equation

$$F(x) = 0, \tag{1}$$

where the operator F is defined on the Banach space B_1 with values in a Banach space B_2. As an example, engineering problems reduce to solving differential or integral equations, which in turn are set up as (1). A solution x^* of the Equation (1) is difficult to find in closed form. That forces researchers to develop iterative methods, which generate iterations convergent to x^*, provided that certain initial conditions hold.

A popular iterative method is defined for each $m = 0, 1, 2, \dots$ by

$$x_0 \in \Omega \subseteq B_1, \quad x_{m+1} = x_m - F'(x_m)^{-1}F(x_m). \tag{2}$$

This is the so-called Newton's method (NM), which is only quadratically convergent [1–4]. In order to increase the order of convergence as well as the efficiency, a plethora of iterative methods have been developed (see, e.g., [5–9] and references therein). Among those, special attention has been given to the Newton-like midpoint method (NLMM) defined by

$$
\begin{aligned}
x_0 \in \Omega, \quad y_0 &= x_0 - F'(x_0)^{-1}F(x_0) \\
x_{m+1} &= x_m - F'\left(\frac{x_m + y_m}{2}\right)^{-1} F(x_m), \quad m = 0,1,2,\ldots \\
y_m &= x_m - F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1} F(x_m), \quad m = 1,2,3,\ldots.
\end{aligned} \tag{3}
$$

NLMM requires per iteration one operator evaluation and one of the inverse of F'. The efficiency index according to Ostrowski is shown to be approximately 1.5537 . . . [10]. This index is higher than NM (1.4142 . . .), as well as the one given in [11] (1.3160 . . .). The construction of this method was essentially given in [12] when $B_1 = B_2$ but with no formal proof of convergence. That is why the semilocal convergence is developed in [10] under Kantorovich's hypotheses. Moreover, favorable comparisons are given to methods using similar information.

Motivation for writing this study: The following concerns arise with the applicability of this method in general.

(1) The convergence region in [10] is not large.
(2) The upper bounds on the distances $\|x_n - x^*\|$ and $\|x_{n+1} - x_n\|$ are not tight enough.
(3) The uniqueness region of the solution x^* is not large.
(4) A Lipschitz condition on the second derivative is assumed (see the condition (H_3) in Section 3). However, the second derivative does not appear on the method and may not exist (see the numerical example in Section 4). Additionally, the method may converge.
(5) The local convergence analysis is not studied in [10].

Novelty: Due to the importance of this method, the items (1)–(5) are positively addressed. The current study includes two procedures for analyzing the semilocal convergence of NLMM. The first analysis replaces the conditions used in [10] with weaker and tighter conditions, thereby enlarging the uniqueness region. In the second semilocal convergence, the convergence conditions used in the earlier section have been replaced by more generalized ω-continuity conditions using majorizing sequences [1,4,5,11–17]. The main advantage of this approach is that it uses only the first derivative, which actually appears in NLMM, for proving the convergence result instead of the second derivative used in [10], thereby enhancing its applicability. Thus, our work improves the results derived in [10] under more stringent conditions and generates finer majorizing sequences. The innovation of the study lies in the fact that extensions are achieved under weaker conditions (see also Remarks throughout the paper). The local convergence of NLMM is also established, along with the existence and uniqueness region of the solution. Moreover, the new error analysis is finer, requiring fewer iterates to achieve a predetermined error tolerance. Furthermore, more precise information is provided on the uniqueness domain of the solution. Finally, the technique can be used on other methods utilizing the inverse of an operator.

The rest of the paper is structured as follows: Section 2 includes mathematical background. In Section 3, we develop the first kind of semilocal convergence theorem based on weaker conditions. The generalized ω-continuity conditions are applied to prove the second type of semilocal convergence theorem in Section 4. The local convergence, along with the uniqueness results of NLMM, is studied in Section 5. In Section 6, numerical examples are given to illustrate the theoretical results. Concluding remarks are reported in Section 7.

2. Mathematical Background

The study of the behavior of a certain cubic polynomial and the corresponding scalar Newton function play a role in the semilocal convergence of NLMM. Let $L > 0$, $M \geq 0$ and $d \geq 0$ be given parameters. Define the cubic polynomial

$$q(L, M, d)(t) = q(t) = \frac{L}{6}t^3 + \frac{M}{2}t^2 - t + d, \tag{4}$$

the Newton iteration function

$$N_q(t) = t - q'(t)^{-1}q(t) \tag{5}$$

and the scalar sequences $\{u_m\}$, $\{v_m\}$ for

$$u_0 = 0, \quad v_0 = u_0 - q'(u_0)^{-1}q(u_0),$$
$$v_m = u_m - q'\left(\frac{u_{m-1} + v_{m-1}}{2}\right)^{-1} q(u_m)$$

and

$$u_{m+1} = u_m - q'\left(\frac{u_m + v_m}{2}\right)^{-1} q(u_m). \tag{6}$$

The proof of the following auxiliary result containing some properties of q, N_q, $\{v_m\}$, and $\{u_m\}$ can be found in [10].

Lemma 1. *Suppose:*

$$6dM^3 + 9d^2L^2 + 18dML - 3M^2 - 8L \leq 0 \tag{7}$$

or

$$d \leq \frac{4L + M^2 - M\sqrt{M^2 + 2L}}{3L(M + \sqrt{M^2 + 2L})} = \lambda(L, M). \tag{8}$$

Then, the following assertions hold:

(i) *The polynomial q given by the Formula (4) has two zeros u^*, u^{**} with $0 < u^* \leq u^{**}$.*

(ii) *q is decreasing in the interval $[0, u^*]$.*

(iii) *q' is increasing and q is convex in $[0, u^*]$.*

(iv) *q'' is increasing in $[0, u^*]$.*

(v) *N_q is increasing in $[0, u^*]$, $N_q(u^*) = u^*$ and $N_q'(u^*) = 0$.*

(vi) *The function*

$$h_q(t) = \frac{q(t)q''(t)}{q'(t)^2} \tag{9}$$

is positive in $[0, u^*)$ and $h_q(u^*) = 0$.

(vii) $0 \leq u_m \leq v_m \leq u_{m+1} < u^*$ *and* $\lim_{m\to\infty} u_m = u^*$.

3. Semilocal Convergence I

The following conditions relating the parameters L, M, d to NLMM have been used in the semilocal convergence.

Suppose:

(H_1) There exist an initial guess $x_0 \in D$ and a parameter $d \geq 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq d$.

(H_2) $\|F'(x_0)^{-1}F''(x_0)\| \le M$.

(H_3) $\|F'(x_0)^{-1}(F''(y) - F''(x))\| \le K_1\|x - y\|$ for some parameter $K_1 > 0$ and each $x, y \in D$.

(H_4) $d \le \lambda(K_1, M)$.
and

(H_5) $U[x_0, u^*] \subset D$.

The following semilocal convergence result was shown in [2,10].

Theorem 1. *Suppose that the conditions* (H_1)–(H_5) *and*

(H_6) $U[x_0, u^*] \subset \Omega$

for $L = K_1$ *hold. Then, the following assertions hold:*

(1) $\|F'(x_0)^{-1}F(x_0)\| \le -q'(0)^{-1}q(0)$.

(2) $\|F'(x_0)^{-1}F''(x)\| \le q''(u)$ *for each* $x \in \Omega$ *such that* $\|x - x_0\| \le u \le u^*$.

(3) $\|F'(x_0)^{-1}(F''(y) - F''(x))\| \le |q''(v) - q''(u)|$ *for each* $u, v \in [0, u^*]$ *such that* $\|y - x\| \le |v - u|$.

(4) $\|F'(x_0)^{-1}F(x_m)\| \le q(u_m)$.

(5) $\|F'(x_m)^{-1}F'(x_0)\| \le q'(t_m)^{-1}$.

(6) $\|y_m - x_m\| \le v_m - u_m$.

(7) $\|F'\left(\frac{x_m+y_m}{2}\right)^{-1}F'(x_0)\| \le -q'\left(\frac{u_m+s_m}{2}\right)^{-1}$.

(8) $\|x_{m+1} - x_m\| \le v_{m+1} - v_m$.

(9) $\|x_{m+1} - y_m\| \le v_{m+1} - u_m$.

(10) *The sequence* $\{x_m\}$ *generated by NLMM is well defined in the ball* $U[x_0, u^*]$, *remains in* $U[x_0, u^*]$ *and converges to the only solution* x^* *of the equation* $F(x) = 0$ *in* $U[x_0, u^*]$.

(11) *Moreover, the following error estimates hold:*

$$\|x^* - x_m\| \le u^* - u_m$$

and

$$\lim_{m\to+\infty} \frac{\|x_{m+1} - x^*\|}{\|x_m - x^*\|^{1+\sqrt{2}}} = \left\|\frac{1}{2}F'(x^*)^{-1}F''(x^*)\right\|^{\frac{2(1+\sqrt{2})}{2+\sqrt{2}}}.$$

Next, the preceding results are extended without additional conditions. Suppose:

(H_3') $\|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \le K_0\|x - x_0\|$ for some $K_0 > 0$ and each $x \in D$.

Define the parameter

$$r = \frac{2}{M + \sqrt{M^2 + 2K_0}} \tag{10}$$

and the region

$$D_0 = U(x_0, r) \cap D. \tag{11}$$

(H_3'') $\|F'(x_0)^{-1}(F''(y) - F''(x))\| \le K\|y - x\|$ for some parameter $K > 0$ and each $x, y \in D_0$.

Clearly, we have

$$K_0 \le K_1 \tag{12}$$

and

$$K \le K_1, \tag{13}$$

since
$$D_0 \subseteq D. \tag{14}$$

It is assumed without loss of generality
$$K_0 \leq K. \tag{15}$$

Otherwise, the results that follow hold with K_0 replacing K. Notice also that the computation of the parameter K_1 requires the computation of K_0, K and $K_1 = K_1(D)$, $K_0 = K_0(D)$, but $K = K(D_0, D)$. Hence, no additional conditions are required to develop the results that follow.

Let us consider the cubic polynomials
$$q_0(t) = q(K_0, L, d) = \frac{K_0}{6}t^3 + \frac{M}{2}t^2 - t + d \tag{16}$$
and
$$q_1(t) = q(K, L, d) = \frac{K}{6}t^3 - \frac{M}{2}t^2 - t + d. \tag{17}$$

It follows by (4), (12), (13), (16) and (17) that
$$q_0(t) \leq q(t), \tag{18}$$
$$q_0'(t) \leq q'(t), \tag{19}$$
$$q_1(t) \leq q(t), \tag{20}$$
$$q_1'(t) \leq q'(t), \tag{21}$$
$$q_0(t) \leq q_1(t), \tag{22}$$
and
$$q_0'(t) \leq q_1'(t) \tag{23}$$
for each $t \geq 0$.

Suppose that

(H_4') $d \leq \lambda(K, M)$.

The following auxiliary result is needed.

Lemma 2. *Suppose that the condition (H_4') holds. Then, the conclusions of Lemma 1 with K, q_1 replace L and q, respectively.*

Proof. Simply replace L, q, u^*, u^{**} by $K, q_1, \overline{u}^*, \overline{u}^{**}$, respectively, in the proof of Lemma 1. □

Remark 1. *It follows by (i) of Lemma 1 and (H_4') that the polynomial q_1 has two zeros $\overline{u}^*, \overline{u}^{**}$ with $0 < \overline{u}^* \leq \overline{u}^{**}$. Moreover, if (H_4) holds, then by (13) and (20),*
$$\overline{u}^* \leq u^* \tag{24}$$
and
$$\overline{u}^{**} \leq u^{**}, \tag{25}$$
since $q_1(u^*) \leq q(u^*) = 0$, $q_1(0) = d > 0$, and $q_1(u^{**}) \leq q(u^{**}) = 0$.

Notice also that:

(a) The parameter r is the unique positive zero of the equation
$$q_0'(t) - 1 = 0 \tag{26}$$

and

$$q_0'(t) = q_1'(t) \quad \text{for each} \quad t \in [0,r]. \tag{27}$$

(b)

$$(H_4) \Rightarrow (H_4'), \tag{28}$$

but not necessarily vice versa unless if $K_1 = K$.

Hence, we arrived at the following extension of the Theorem 1.

Theorem 2. *Suppose that the conditions* (H_1), (H_2), (H_3'), (H_3''), (H_4') *and*

(H_5') $U[x_0, \bar{u}^*] \subset D$

hold. Then, the assertions (1)–(11) *of Theorem 1 hold with* K, q_1, $\bar{u}^*$, $\{\bar{u}_m\}$, $\{\bar{v}_m\}$ *replacing* K_1, q, u^*, $\{u_m\}$, $\{v_m\}$, *respectively, where*

$$\bar{u}_0 = 0, \bar{v}_0 = \bar{u}_0 - q_1'(\bar{u}_0)^{-1} q_1(u_0),$$
$$\bar{v}_m = \bar{u}_m - q_1'\left(\frac{u_{m-1} + v_{m-1}}{2}\right)^{-1} q(\bar{u}_m)$$

and

$$\bar{u}_{m+1} = \bar{u}_m - q_1'\left(\frac{u_m + v_m}{2}\right)^{-1} q_1(\bar{u}_m).$$

Proof. The assertions (1)–(6) follow with the above changes. Concerning the assertion (7), set $z_0 = \frac{x_0 + y_0}{2}$, $\mu_0 = \frac{\bar{u}_0 + \bar{v}_0}{2}$ and $F_1(x) = F'(x_0)^{-1} F(x)$ for each $x \in D$. Then, we have in turn that

$$F_1'(z_0) = F_1'(x_0) + \int_0^1 F_1''(x_0 + \theta(z_0 - x_0))(z_0 - x_0, \cdot) d\theta,$$

leading to

$$\begin{aligned}
\|F_1'(z_0) - I\| &\le \left\| \int_0^1 F_1''(x_0 + \theta(z_0 - x_0))(z_0 - x_0, \cdot) d\theta \right\| \\
&\le \left\| F_1''(x_0)(z_0 - x_0) + \int_0^1 [F_1''(x_0 + \theta(z_0 - x_0)) - F_1''(x_0)](z_0 - x_0, \cdot) d\theta \right\| \\
&\le \|F_1''(x_0)\| \|z_0 - x_0\| + \int_0^1 K\theta(\mu_0 - \bar{u}_0) \|z_0 - x_0\| d\theta.
\end{aligned} \tag{29}$$

Notice that the last inequality in (29) follows from (H_3''), $F_1''(x) = F'(x_0)^{-1} F''(x)$ and $q''(t) = Kt + M$.

Then, from (H_2) and the definition of y_0 and $\bar{u}_0$

$$\begin{aligned}
\|F_1'(z_0) - I\| &\le q_0'(\bar{u}_0)(\mu_0 - \bar{u}_0) + \frac{K}{2}(\mu_0 - \bar{u}_0)^2 \\
&\le q_0'(\mu_0) - q_0'(\bar{u}_0) \\
&= 1 + q_0'(\mu_0) < 1.
\end{aligned} \tag{30}$$

The condition (H_5') in Theorem 2 can be replaced by

(H_5') $U[x_0, r] \subset D$,

where r is given by (10).

Moreover, the uniqueness ball can be enlarged from $U(x_0, u^*)$ given in Theorem 1 to $U(x_0, r)$. This can be seen using the weaker condition (H_3') instead of (H_3) used in

Theorem 1 in [10] or Theorem 2 used by us. Indeed, in Theorem 1, the estimate was obtained for $y^* \in U(x_0, u^*)$,

$$\left\| \int_0^1 F'(x_0)^{-1} F'(y^* + \theta(x^* - y^*)) d\theta - I \right\| \leq \int_0^1 (q'(\|y^* + \theta(x^* - y^*) - x_0\|) - q'(0)) d\theta < 1,$$

since $q'(t) < 0$ for all $t \in [0,1]$, leading to

$$0 = F(y^*) - F(x^*) = \int_{x^*}^{y^*} F(v) dv = \int_0^1 F'(y^* + t(x^* - y^*)) dt (y^* - x^*),$$

and consequently, $y^* = x^*$. However, the same estimate is obtained using the tighter condition (H_3') with $q_1, \overline{u}^*$ replacing q, u^*, respectively.
(c) The Lipschitz constant K can be replaced by an at least as small.

It follows by the Banach lemma on linear operators with inverses [3,5] and (30) that

$$\begin{aligned} \|F'(z_0)^{-1} F'(x_0)\| &\leq \frac{1}{1 - \|F'(z_0) - I\|} \\ &\leq -\frac{1}{q_0'(\mu_0)} \leq -\frac{1}{q_1'(\mu_0)}, \end{aligned} \tag{31}$$

showing the assertion (7). Notice that in Theorem 1, the less tight estimate than (31) is shown under the stronger and not actually needed condition (H_3), which is

$$\|F'(z_0)^{-1} F'(x_0)\| \leq -\frac{1}{q'(\mu_0)}. \tag{32}$$

In view of the estimate (31), the rest of the proof follows as in [10]. □

Remark 2. (a) *In view of (24) and (28), Theorem 2 extends Theorem 1 with advantages already stated.*

Define the ball $U(y_0, r-d)$ for $d < r$. Notice that $U(y_0, r-d) \subset U(x_0, r)$. Then, suppose $(H_3''')\|F'(x_0)^{-1}(F''(y) - F''(x))\| \leq K_2\|y - x\|$ for some $K_2 > 0$ and each $y, x \in D_1 = U(y_0, r-d) \cap D$.

It follows that (H_3'''), K_2 can replace (H_3''), K, respectively, in our results and

$$K_2 \leq K.$$

The iterates $\{x_m\} \subset U(y_0, r-d)$.

4. Semilocal Convergence II

The convergence conditions of the previous section may not hold, even if the method (3) converges. As a motivational example, consider the function f defined on the interval $D_2 = [-0.5, 1.5]$ by

$$f(t) = \begin{cases} t^2 \log t + t^4 - t^3, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Then, clearly the function f'' is unbounded on D_2. Hence, the results of the previous section cannot guarantee the convergence of method (3) to the solution $x^* = 1 \in D_2$. That is why we drop the conditions (H_2)-(H_5), (H_3'), (H_3''), (H_4') and (H_5') and utilize the more general ω-continuity conditions, the first derivative that actually only appears on the method (3) and majorizing sequences to present another semilocal convergence result under weaker conditions.

Let $h_0 : [0, +\infty) \to \mathbb{R}$ be a continuous and nondecreasing function. Suppose that the equation

$$h_0(t) - 1 = 0$$

has a smallest positive solution s. Moreover, suppose that there exists a function $h : [0, s) \to \mathbb{R}$, which is continuous and nondecreasing. Let also parameters t_0, s_0, t_1 be such that $t_0 = 0$, $s_0 > 0$ and $s_0 < t_1$. Define the sequences $\{t_m\}$, $\{s_m\}$ by

$$\begin{aligned} p_m &= h_0\left(\frac{t_m + s_m}{2}\right), \\ t_{m+1} &= s_m + \frac{h\left(\frac{s_m - s_{m-1} + t_m - t_{m-1}}{2}\right)}{1 - p_m}(s_m - t_m), \\ \alpha_{m+1} &= \int_0^1 h\left(\frac{s_m - t_m}{2} + \theta(t_{m+1} - t_m)\right) d\theta (t_{m+1} - t_m) \\ s_{m+1} &= t_{m+1} + \frac{\alpha_{m+1}}{1 - p_m}. \end{aligned} \tag{33}$$

This sequence shall be shown to be majorizing for the method (3). However, a convergence result is developed first.

Lemma 3. *Suppose that for each $m = 0, 1, 2, \ldots$ and some $\beta > 0$*

$$p_m < 1 \quad \text{and} \quad t_m \leq \beta \tag{34}$$

Then, the following items hold:

$$0 \leq t_m \leq s_m \leq t_{m+1} \leq \beta \tag{35}$$

and

$$\lim_{m \to \infty} t_m = \alpha \leq \beta. \tag{36}$$

Proof. It follows from the definition of the sequence $\{t_m\}$ given by Formula (33) and the condition (34) that items (35) and (36) hold, the $\lim_{m \to \infty} t_m$ exists, satisfying (36), where α is the unique least upper bound of this sequence. $\square$

Remark 3. *A possible choice for β is any number in the interval $(0, s]$.*

The conditions connecting the "h "functions to the operators F and F' are:

(A_1) There exists an initial guess $x_0 \in D$, $s_0 \geq 0$, $t_1 \geq s_0$ such that $F'(x_0)^{-1} \in \mathcal{L}(B_2, B_1)$, $\|F'(x_0)^{-1}F(x_0)\| \leq s_0$, and for $y_0 = x_0 - F'(x_0)^{-1}F(x_0)$, $F'\left(\frac{x_0+y_0}{2}\right)^{-1} \in \mathcal{L}(B_2, B_1)$ with

$$\left\| F'\left(\frac{x_0 + y_0}{2}\right)^{-1} F(x_0) - F'(x_0)^{-1}F(x_0) \right\| \leq t_1 - s_0.$$

(A_2) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq h_0(\|x - x_0\|)$ for each $x \in D$.
Set

$$D_3 = U(x_0, s) \cap D.$$

(A_3) $\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq h(\|y - x\|)$ for each $x, y \in D_3$.

(A_4) The condition (34) holds
and

(A_5) $U[x_0, \alpha] \subset D$.

An Ostrowski-like representation [17] is needed for the iterates $\{x_m\}$ and $\{y_m\}$.

Lemma 4. *Suppose that the iterates $\{x_m\}$, $\{y_m\}$ exist for each $m = 1, 2, \ldots$. Then, the following items hold*

$$x_{m+1} - y_m = F'\left(\frac{x_m + y_m}{2}\right)^{-1}\left(F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right) - F'\left(\frac{x_m + y_m}{2}\right)\right)(y_m - x_m) \quad (37)$$

and

$$F(x_{m+1}) = \left[\int_0^1 F'(x_m + \theta(x_{m+1} - x_m))d\theta - F'\left(\frac{x_m + y_m}{2}\right)\right](x_{m+1} - x_m). \quad (38)$$

Proof. By subtracting the first substep of the method from the second substep, we obtain in turn that

$$\begin{aligned} x_{m+1} - y_m =& \left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1} - F'\left(\frac{x_m + y_m}{2}\right)^{-1}\right]F(x_m) \\ =& -\left[F'\left(\frac{x_m + y_m}{2}\right)^{-1} - F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1}\right]F(x_m) \\ =& -F'\left(\frac{x_m + y_m}{2}\right)^{-1}\left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right) - F'\left(\frac{x_m + y_m}{2}\right)\right]F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1}F(x_m) \\ =& F'\left(\frac{x_m + y_m}{2}\right)^{-1}\left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right) - F'\left(\frac{x_m + y_m}{2}\right)\right](y_m - x_m) \end{aligned}$$

showing the estimate (34). The estimate (38) follows from the identity

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - F'\left(\frac{x_m + y_m}{2}\right)(x_{m+1} - x_m),$$

which is obtained by the first substep of NLMM. □

Next, the semilocal convergence is developed for the method (3).

Theorem 3. *Suppose that the conditions (A_1)–(A_5) hold. Then, the sequence $\{x_m\}$ converges to a solution $x^* \in U[x_0, \alpha]$ such that*

$$\|x_m - x^*\| \leq \alpha - t_m.$$

Proof. Mathematical induction is applied to show

$$\|y_m - x_m\| \leq s_m - t_m < \alpha \quad (39)$$

and

$$\|x_{m+1} - y_m\| \leq t_{m+1} - s_m. \quad (40)$$

The condition (A_1) and the Formula (4) (for $m = 0$) imply that the estimate (39) holds for $m = 0$. Then, we also have the iterate $y_0 \in U(x_0, \alpha)$. Moreover, $\left\|\frac{x_0 + y_0}{2} - x_0\right\| = \frac{1}{2}\|y_0 - x_0\| < \alpha$, so $\frac{x_0 + y_0}{2} \in U(x_0, \alpha)$. Let $v \in U(x_0, \alpha)$ be an arbitrary point. Then, the condition (A_2) gives

$$\|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq h_0(\|v - x_0\|) \leq h_0(\alpha) < 1.$$

That is, $F'(v)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|F'(v)^{-1}F'(x_0)\| \leq \frac{1}{1 - h_0(\|v - x_0\|)}. \quad (41)$$

In particular, if $v = \dfrac{x_0 + y_0}{2}$, then the iterate y_1 and x_1 are well defined by the method (3). Moreover, the last condition in (A_1) and (33) give

$$\|x_1 - y_0\| = \left\| F'\left(\frac{x_0 + y_0}{2}\right)^{-1} F(x_0) - F'(x_0)^{-1} F(x_0) \right\| \le t_1 - s_0.$$

Thus, the assertion (40) holds for $m = 0$. Suppose that the assertions (39) and (40) held for all integers smaller than $n - 1$. Then, we obtain from

$$\begin{aligned}\|F'(x_0)^{-1}\left(F'\left(\frac{x_m + y_m}{2}\right) - F'(x_0)\right)\| &\le h_0\left(\left\|\frac{x_m + y_m}{2} - x_0\right\|\right) \\ &\le h_0\left(\frac{\|x_m - x_0\| + \|y_m - x_0\|}{2}\right) \le p_m < 1,\end{aligned}$$

thus,

$$\left\| F'\left(\frac{x_m + y_m}{2}\right)^{-1} F'(x_0) \right\| \le \frac{1}{1 - p_m}. \tag{42}$$

Then, by (37), we obtain in turn that

$$\begin{aligned} x_{m+1} - y_m &= \left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1} - F'\left(\frac{x_m + y_m}{2}\right)^{-1}\right] F(x_m) \\ &= -F'\left(\frac{x_m + y_m}{2}\right)^{-1}\left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right) - F'\left(\frac{x_m + y_m}{2}\right)\right] F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right)^{-1} F(x_m) \\ &= F'\left(\frac{x_m + y_m}{2}\right)^{-1}\left[F'\left(\frac{x_{m-1} + y_{m-1}}{2}\right) - F'\left(\frac{x_m + y_m}{2}\right)\right](y_m - x_m). \end{aligned} \tag{43}$$

It follows from (33), (A_3), (42), (43) and the induction hypotheses that

$$\begin{aligned} \|x_{m+1} - y_m\| &\le \frac{h\left(\frac{\|y_m - y_{m-1}\| + \|x_m - x_{m-1}\|}{2}\right)\|y_m - x_m\|}{1 - p_m} \\ &\le \frac{h\left(\frac{s_m - s_{m-1} + t_m - t_{m-1}}{2}\right)(s_m - t_m)}{1 - p_m} = t_{m+1} - s_m \end{aligned} \tag{44}$$

and

$$\|x_{m+1} - x_0\| \le \|x_{m+1} - y_m\| + \|y_m - x_0\| \le t_{m+1} - s_m + s_m - t_0 = t_{m+1} < \alpha.$$

Hence, (40) holds, and the iterate $x_{m+1} \in U(x_0, \alpha)$. Furthermore, by (38) and the second substep of the method (3), we have in turn that

$$\begin{aligned} &\|y_{m+1} - x_{m+1}\| \le \|F'(x_{m+1})^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(x_{m+1})\| \\ &\le \frac{\int_0^1 h\left(\|x_m + \theta(x_{m+1} - x_m) - \frac{x_m + y_m}{2}\|\right) d\theta \|x_{m+1} - x_m\|}{1 - p_m} \\ &\le \frac{\int_0^1 h\left(\frac{s_m - t_m}{2} + \theta(t_{m+1} - t_m)\right) d\theta (t_{m+1} - t_m)}{1 - p_m} \\ &= s_{m+1} - t_{m+1} \end{aligned} \tag{45}$$

and

$$\|y_{m+1} - x_0\| \le \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\| \le s_{m+1} - t_{m+1} + t_{m+1} - t_0 = s_{m+1} < \alpha.$$

Thus, the induction for the estimates (39) and (40) is terminated.

However, the sequence $\{t_m\}$ is Cauchy as convergent by Lemma 3. Therefore, the sequence $\{x_m\}$ is also Cauchy in a Banach space B_1, and as such, it converges to some $x^* \in U[x_0, \alpha]$, since this set is closed. Furthermore, by using the continuity of F and letting $n \to +\infty$ in the calculation,

$$\|F'(x_0)^{-1}F(x_{m+1})\| \leq \alpha_{m+1}, \tag{46}$$

we conclude that $F(x^*) = 0$. $\square$

Concerning the uniqueness of the solution in a neighborhood about the point x_0, we have:

Proposition 1. *Suppose:*

(a) *There exists a solution $z^* \in U(x_0, \vartheta)$ of the equation $F(x) = 0$ for some $\vartheta > 0$.*

(b) *The condition (A_2) holds on the ball $U(x_0, \vartheta)$*

and

(c) *There exists $\overline{\vartheta} \geq \vartheta$ such that*

$$\int_0^1 h_0(\theta\overline{\vartheta})d\theta < 1. \tag{47}$$

Set $D_4 = U[x_0, \overline{\vartheta}] \cap D$. Then, the equation $F(x) = 0$ is uniquely solvable by z^* in the region D_4.

Proof. Let $v^* \in D_4$ with $F(v^*) = 0$. Define the linear operator $T = \int_0^1 F'(z^* + \theta(v^* - z^*))d\theta$. Then, it follows by condition (A_2) and (47) that

$$\begin{aligned}\|F'(x_0)^{-1}(T - F'(x_0))\| &\leq \int_0^1 h_0(\|z^* - x_0 + \theta(z^* - v^*)\|d\theta \\ &\int_0^1 h_0((1-\theta)\vartheta + \theta\overline{\vartheta})d\theta < 1,\end{aligned} \tag{48}$$

which implies $v^* = z^*$. $\square$

Remark 4. (1) *If all the conditions of Theorem 3 hold, then we can set $\vartheta = \alpha$.*

(2) *The condition (A_5) can be replaced by*

(A_5') *$U[x_0, \alpha] \subset D$.*

(3) *Suppose that $d < s$. Define the set $D_5 = U(y_0, s - d) \cap D$. Notice that $D_5 \subset U(x_0, s)$. Then, a tighter function h is obtained if D_5 replaces D_3 in the condition (A_2).*

5. Local Convergence

We shall introduce some scalar functions and some parameters to show the local convergence analysis of NLMM.

Suppose:

(i) There exists a function $\psi_1 : [0, +\infty) \to \mathbb{R}$, which is nondecreasing and continuous and a parameter $M_2 \geq 0$ such that the function $\psi_2(t) - 1$ has a smallest zero $\rho_2 \in (0, +\infty)$, where $\psi_2 : [0, +\infty) \to \mathbb{R}$ is defined by

$$\psi_2(t) = \left(\int_0^1 \psi_1(\theta t)d\theta + M_2\right)t.$$

(ii) The function $\psi_3 : [0, \rho_2) \to \mathbb{R}$ is such that

$$\psi_3(t) - 1$$

has a smallest zero $r_3 \in (0, \rho_2)$, where

$$\psi_3(t) = \frac{\left[\int_0^1 \int_0^1 \psi_1(((1-\tau)+\theta\tau)t)\left(\frac{1}{2}+|\frac{1}{2}-\theta|\right)d\tau d\theta + \frac{1}{2}M_2\right]t}{1-\psi_2(t)}$$

The parameter r_3 shall be shown to be a radius of convergence for NLMM.

The convergence conditions are:

(l_1) There exists a simple solution $x^* \in \Omega$ of the equation $F(x) = 0$ and a parameter $M_2 \geq 0$ such that $\|F'(x^*)^{-1}F''(x^*)\| \leq M_2$.

(l_2) $\|F'(x^*)^{-1}(F''(x) - F''(x^*))\| \leq \psi_1(\|x - x^*\|)$ for each $x \in \Omega$.

(l_3) $U[x^*, r_3] \in \Omega$.

Next, the local convergence is given for NLMM.

Theorem 4. *Suppose that the conditions $(l_1) - (l_3)$ hold. Then, the sequence $\{x_m\}$ generated by NLMM converges to x^* provided that $x_0 \in U(x^*, r_3)$.*

Proof. Let $z \in U(x^*, r_3)$. It follows by the conditions (l_1), (l_2) and the definition of the radius r_3 in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}(F'(z) - F'(x^*))\| \\
&= \left\| \int_0^1 F'(x^*)^{-1}F''(x^* + \theta(z - x^*))d\theta(z - x^*) \right\| \\
&\leq \left\| \int_0^1 F'(x^*)^{-1}(F''(x^* + \theta(z - x^*) - F''(x^*))d\theta(z - x^*) \right\| + \|F'(x^*)^{-1}F''(x^*)(z - x^*)\| \\
&\leq \int_0^1 \psi_1(\theta\|z - x^*\|)d\theta\|z - x^*\| + M_2\|z - x^*\| = \psi_2(\|z - x^*\|) < 1;
\end{aligned}$$

thus, $F'(z)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|F'(z)^{-1}F'(x^*)\| \leq \frac{1}{1 - \psi_2(\|z - x^*\|)}. \tag{49}$$

By hypothesis $x_0 \in U(x^*, r_3)$ and (49) for $z = x_0$, we have that $F'(x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ and the iterate y_0 is well defined by the first substep of NLMM.
Moreover, we can write for $y_\theta = x^* + \theta(x_0 - x^*)$

$$\begin{aligned}
y_0 - x^* &= -F'(x_0)^{-1}\left[\int_0^1 F'(y_\theta)d\theta - F'(x_0)\right](y_\theta - x^*) \\
&= -F'(x_0)^{-1}\int_0^1\int_0^1 F''(x_0 + \tau(y_\theta - x_0))d\theta d\tau(y_\theta - x^*) \\
&= -F'(x_0)^{-1}\int_0^1\int_0^1 [F''(x_0 + \tau(y_\theta - x_0)) - F''(x^*)]d\theta d\tau(y_\theta - x^*) \\
&- F'(x_0)^{-1}\int_0^1\int_0^1 F''(x^*)d\theta d\tau(y_\theta - x^*).
\end{aligned} \tag{50}$$

By applying the condition (l_2) and (49) (for $z = x_0$) on (50), we obtain in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \frac{\int_0^1\int_0^1 \psi_1(((1-\tau)+\theta\tau)\|x_0 - x^*\|)\theta d\theta d\tau\|x_0 - x^*\| + \dfrac{1}{2}M_2\|x_0 - x^*\|}{1 - \psi_2(\|x_0 - x^*\|)} \\
&\leq \psi_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r_3,
\end{aligned} \tag{51}$$

where we used that

$$x_0 - x^* + \tau(y_\theta - x_0) = ((1-\tau) + \theta\tau)(x_0 - x^*)$$

and $\theta \leq \frac{1}{2} + \left|\frac{1}{2} - \theta\right|$. Thus, the iterate $y_0 \in U(x^*, r_3)$. Then, notice that

$$\left\|\frac{x_0 + y_0}{2} - x^*\right\| \leq \frac{1}{2}(\|x_0 - x^*\| + \|y_0 - x^*\|) < r_3,$$

so the point $z_0 = \frac{x_0+y_0}{2} \in U(x^*, r_3)$.

Suppose that $z_m = \frac{x_m+y_m}{2} \in U(x^*, r_3)$. Then, we also have

$$\|z_m - x^*\| \leq \frac{1}{2}(\|x_m - x^*\| + \|y_m - x^*\|) < r_3,$$

so the iterates x_{m+1} and y_m are well defined by NLMM. Then, we can write for $y_\theta^m = x^* + \theta(x_m - x^*)$ and $S_m = F'\left(\frac{x_m+y_m}{2}\right)$

$$x_{m+1} - x^* = -S_m^{-1}\left[\int_0^1 F'(y_\theta)d\theta - S_m\right](x_m - x^*) \tag{52}$$

However, the expression in the bracket can be written as

$$\begin{aligned}\int_0^1 F'(y_\theta)d\theta - S_m &= \int_0^1\int_0^1 F''(z_m + \tau(y_\theta^m - z_m))d\theta d\tau(y_\theta^m - x^*) \\ &= \int_0^1\int_0^1 \left[F''(z_m + \tau(y_\theta^m - z_m)) - F''(x^*)\right]d\theta d\tau(y_\theta^m - x^*) \\ &+ \int_0^1\int_0^1 F''(x^*)d\theta d\tau(y_\theta^m - x^*).\end{aligned} \tag{53}$$

In view of (49) (for $z = z_m$), (52), (53), we obtain

$$\begin{aligned}\|x_{m+1} - x^*\| &\leq \frac{q_m\|x_m - x^*\|^2}{1 - \psi_2(\|x_m - x^*\|)} \\ &\leq \psi_3(\|x_m - x^*\|)\|x_m - x^*\| \leq \|x_m - x^*\|,\end{aligned} \tag{54}$$

where, for the numerator, we obtain

$$\begin{aligned}&\int_0^1\int_0^1 \psi_1(z_m - x^* + \tau(y_\theta^m - z_m))(y_\theta^m - z_m)d\theta d\tau\|x_m - x^*\| + M_2\int_0^1 \|y_\theta^m - z_m\|d\theta\|x_m - x^*\| \\ &\leq \int_0^1\int_0^1 \psi_1\left(\frac{1}{2}(1-\tau)(\|x_m - x^*\| + \|y_m - x^*\|) + \tau\theta\|x_m - x^*\|\right) \\ &* \left(\frac{\|y_m - x^*\|}{2} + \left|\frac{1}{2} - \theta\right|\|x_m - x^*\|\right)d\theta d\tau\|x_m - x^*\| + \frac{1}{2}M_2\int_0^1\left(\frac{1}{2} + \left|\frac{1}{2} - \theta\right|\right)d\theta\|x_m - x^*\| \\ &\leq \int_0^1\int_0^1 \psi_1(((1-\tau) + \tau\theta)\|x_m - x^*\|)\left(\frac{1}{2} + \left|\frac{1}{2} - \theta\right|\right)d\theta d\tau\|x_m - x^*\|^2 \\ &+ \frac{1}{2}M_2\|x_m - x^*\|^2 = q_m\|x_m - x^*\|^2,\end{aligned} \tag{55}$$

where we also used $\|y_m - x^*\| \leq \|x_m - x^*\|$.

Similarly, the estimate

$$y_m - x^* = -S_{m-1}^{-1}\left[\int_0^1 F'(y_\theta^{m-1}) - S_{m-1}\right](x_m - x^*),$$

we obtain as in (54)

$$
\begin{aligned}
\|y_m - x^*\| \leq & \frac{q_{m-1}\|x_{m-1} - x^*\|\|x_m - x^*\|}{1 - \psi_2(\|x_{m-1} - x^*\|)} \\
& \leq \psi_3(\|x_{m-1} - x^*\|)\|x_m - x^*\| \leq \|x_m - x^*\|.
\end{aligned}
$$

Hence, the iterates x_{m+1}, $y_m \in U(x^*, r_3)$ for each $m = 0, 1, 2, \ldots$. Then, from the estimates

$$
\|x_{m+1} - x^*\| \leq \psi_3(\|x_0 - x^*\|)\|x_m - x^*\|
$$

and

$$
\|y_m - x^*\| \leq \psi_3(\|x_0 - x^*\|)\|x_m - x^*\|,
$$

we deduce that $\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = x^*$ since $\psi_3(\|x_0 - x^*\|) \in [0, 1)$. □

The uniqueness of the solution ball is determined in the next result.

Proposition 2. *Suppose:*

(*i*) *The condition* (l_1) *holds.*
(*ii*) *There exists a solution* $z^* \in U(x^*, \rho_3)$ *for some* $\rho_3 > 0$.
(*iii*) *The condition* (l_3) *holds on the ball* $U(x^*, \rho_3)$.
and
(*iv*) *There exists* $\rho_4 \geq \rho_3$ *such that*

$$
\int_0^1 \int_0^1 \psi_1(\tau\theta\rho_4)\theta\rho_4 d\theta d\rho + \frac{1}{2}M_2\rho_4 < 1. \tag{56}
$$

Set $D_6 = U[x^*, \rho_4] \cap \Omega$. *Then, the only solution of the equation* $F(x) = 0$ *in the region* D_6 *is* x^*.

Proof. Let $z^* \in D_6$ with $F(z^*) = 0$. Set $z_\theta^* = x^* + \theta(z^* - x^*)$ and $T = \int_0^1 F'(z_\theta^*)d\theta$. Then, we can write

$$
\begin{aligned}
\int_0^1 (F'(z_\theta^*) - F'(x^*))d\theta = & \int_0^1 \int_0^1 F''(x^* + \tau(z_\theta^* - x^*))(z_\theta^* - x^*)d\theta d\tau \\
= & \int_0^1 \int_0^1 [F''(x^* + \tau(z_\theta^* - x^*)) - F'(x^*)]\theta(z_\theta^* - x^*)d\theta d\tau \\
+ & \int_0^1 \int_0^1 F''(x^*)\theta(z_\theta^* - x^*)d\theta d\tau.
\end{aligned} \tag{57}
$$

By composing (57) with $F'(x^*)^{-1}$, using (l_2) and (56), we obtain in turn that

$$
\begin{aligned}
\|F'(x^*)^{-1}(T - F'(x^*)\| \leq & \int_0^1 \int_0^1 \psi_1(\tau\theta\|z^* - x^*\|)\theta\|z^* - x^*\|d\theta d\tau + \frac{1}{2}M_2\|z^* - x^*\| \\
\leq & \int_0^1 \int_0^1 \psi_1(\tau\theta\rho_4)\theta\rho_4 d\theta d\rho + \frac{1}{2}M_2\rho_4 < 1;
\end{aligned}
$$

thus, we conclude again that $z^* = x^*$. □

Remark 5. *We can certainly choose* $\rho_3 = r_3$.

6. Numerical Examples

In this section, some numerical examples are solved in order to corroborate the theoretical results obtained and the efficacy of our approach.

Example 1. *Let $B_1 = B_2$ and $\Omega = U(x_0, 1-\gamma)$ for some parameter $\gamma \in (0,1)$. Define the polynomial F on the interval Ω by*

$$F(v) = \frac{v^4}{4} + \varphi - \gamma v \quad \text{for some} \quad \varphi \in \mathbb{R}. \tag{58}$$

Choose $x_0 = 1$. Then, if we substitute F on the "h "conditions, we see that the conditions

$$h_0(t) = \frac{(\gamma^2 - 5\gamma + 7)}{1-\gamma} t \quad \text{and} \quad h(t) = \frac{3(1+s)^2}{1-\gamma} t$$

are verified provided that

$$d = \frac{|\frac{1}{4} - \gamma + \varphi|}{1-\gamma}, \quad K_0 = \frac{3(3-\gamma)}{1-\gamma}, \quad K_1 = \frac{6(2-\gamma)}{1-\gamma}, \quad K = \frac{6(1+r)}{1-\gamma}$$

for

$$r = \frac{2(1-\gamma)}{3 + \sqrt{9 + 6(3-\gamma)(1-\gamma)}}, \quad \text{and} \quad M = \frac{3}{1-\gamma}.$$

Notice that $K_0 < K_1$ and $K < K_1$. Moreover, $\lambda(K, M) < \lambda(K_1, M)$.

For $\varphi = -\frac{1}{10}$ and $\gamma \in (0.0175194\ldots, 0.0301594\ldots) \cup (0.230014\ldots, 0.233713\ldots) \subset (0,1)$,

$$d > \lambda(K_1, M).$$

and

$$d < \lambda(K, M).$$

Thus, it is clear that our new condition (H_4') holds true, but the condition (H_4) used in [10] does not hold. By taking $\gamma = \frac{1}{10}$ and $\varphi = -\frac{1}{10}$, we obtain the following

$$\begin{array}{llll}
\beta = 0.138249\ldots, & s_0 = 0.055555\ldots, & t_1 = 0.061053\ldots, & p_0 = 0.200926\ldots, \\
\alpha_1 = 0.015373\ldots, & s_1 = 0.080291\ldots, & t_2 = 0.087582\ldots, & p_1 = 0.511197\ldots, \\
\alpha_2 = 0.002621\ldots, & s_2 = 0.092946\ldots, & t_3 = 0.094254\ldots, & p_2 = 0.652915\ldots, \\
\alpha_3 = 0.000173\ldots, & s_3 = 0.094754\ldots, & t_4 = 0.094782\ldots, & p_3 = 0.683581\ldots, \\
\alpha_4 = 1.17314e-6, & s_4 = 0.094786\ldots, & t_5 = 0.094786\ldots, & p_4 = 0.685610\ldots, \\
\alpha_5 = 5.97079e-11, & s_5 = 0.094786\ldots, & t_6 = 0.094786\ldots, & p_5 = 0.685623\ldots, \\
\ldots\ldots\ldots\ldots\ldots\ldots & & &
\end{array}$$

which shows that conditions (A_1)-(A_4) are satisfied. Hence, by Theorem 3, the sequence $\{x_m\}$ converges to a unique solution $x^* \in U(x_0, \alpha)$ where, $\alpha = 0.094786\ldots$. Thus, this example can be solved using the weaker condition used in our study but not using the earlier one [10].

Example 2. *Let $B_1 = B_2 = \mathbb{R}^2$. Define the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$F(v) = (2\vartheta_1 - \frac{1}{9}\vartheta_1^2 - \vartheta_2, -\vartheta_1 + 2\vartheta_2 - \frac{1}{9}\vartheta_2^2)^T,$$

where $v = (\vartheta_1, \vartheta_2)^T$. We shall find a solution to the equation $F(v) = 0$. The first and second-order Fréchet derivatives are calculated to be

$$F'(v) = \begin{pmatrix} 2 - \frac{2}{9}\vartheta_1 & -1 \\ -1 & 2 - \frac{2}{9}\vartheta_2 \end{pmatrix}$$

and

$$F''(v) = \left(\begin{array}{cc|cc} \frac{2}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{9} \end{array} \right).$$

Pick $x_0 = (11.4, 11.4)^T$. Then, it follows that $d = 1.9826$, $M = 0.47618$, and $K \geq 0$ can be arbitrary. By setting $K = 10^{-3}$, we see that $d < \lambda(K, M)$. Moreover, the solution $x^* = (9,9)^T$ is obtained after $m = 3$ iterations. Notice that it takes $m = 5$ iterations for NM but only three for NLMM to reach x^*. Thus, this method requires fewer computations than that of Newton's method.

Example 3. *Let $B_1 = B_2 = \mathbb{R}$ and $D = U[0,1]$. Define a function F on D by*

$$F(v) = e^v - 1.$$

Clearly, we have $x^* = 0$. Then, the conditions (l_1) and (l_2) hold if $M_2 = 1$ and $\psi_1(t) = (e-1)t$. Then, $\psi_2(t) = t\left(1 - \frac{t}{2} + \frac{et}{2}\right)$ and $\psi_3(t) = -\frac{(9(e-1)t+8)t}{8((e-1)t^2+2t-2)}$.

The parameter $\rho_2 = 0.64385$ and radius of convergence $r_3 = 0.435659$.

Example 4. *Let $B_1 = B_2 = \mathbb{R}^3$ and $D = U[0,1]$. Define a mapping F on D by*

$$F(v) = \left(\frac{e-1}{6}\vartheta_1^3 + \vartheta_1, e^{\vartheta_2} - 1, \frac{\vartheta_3^3}{6} + \vartheta_3 \right)^T,$$

where $v = (\vartheta_1, \vartheta_2, \vartheta_3)^T$. Clearly, the solution is $x^* = (0,0,0)^T$. It follows by the definition of the mapping F that the first two Fréchet derivatives are

$$F'(v) = \begin{pmatrix} \frac{e-1}{2}\vartheta_1^2 + 1 & 0 & 0 \\ 0 & e^{\vartheta_2} & 0 \\ 0 & 0 & \frac{\vartheta_3^2}{2} + 1 \end{pmatrix}$$

and

$$F''(v) = \left(\begin{array}{ccc|ccc|ccc} (e-1)\vartheta_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\vartheta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vartheta_3 \end{array} \right).$$

Notice that $F'(x^*) = F'(x^*)^{-1} = diag(1,1,1)$. Therefore, the conditions (l_1) and (l_2) are verified for $M_2 = 1$ and $\psi_1(t) = (e-1)t$. Then, $\psi_2(t) = t\left(1 - \frac{t}{2} + \frac{et}{2}\right)$ and $\psi_3(t) = -\frac{(9(e-1)t+8)t}{8((e-1)t^2+2t-2)}$.

The parameter $\rho_2 = 0.64385$ and radius of convergence $r_3 = 0.435659$.

The numerical examples were simulated by using Mathematica 8 on Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz 1.80 GHz, with 8 GB of RAM running on Windows 10 Pro version 2017. This kind of local and semilocal convergence demonstrates that the guarantee the existence and uniqueness of the solution are especially valuable in processes where it is difficult to prove the existence of solutions.

7. Conclusions

The present study deals with new local and semilocal convergence results for the Newton-like midpoint method under improved initial conditions. In the first type of semilocal convergence, the previous results are extended without using any additional postulates. The estimate obtained in [10] is less tight and uses stronger conditions in comparison to our results. The second semilocal convergence utilizes more general ω-continuity conditions and can be applied to the problems where earlier conditions fail. Notice also that the condition on F'' is dropped (see also the Example 1). Both semilocal convergence results are computationally verifiable and improve the previous study [10] in several directions, which are of practical importance. The local convergence theorem not given in [10] is established for the existence-uniqueness of the solution. We present varied numerical examples to show the applicability of our results. The innovation demonstrated that NLMM can also be used to extend the applicability of other methods requiring the inversion of a linear operator in an analogous way since our technique is method-free. This is the future area of research.

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