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A Comparison Result for the Nabla Fractional Difference Operator

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Abstract: This article establishes a comparison principle for the nabla fractional difference operator $\nabla^{\nu}_{\rho(a)}$, $1 < \nu < 2$. For this purpose, we consider a two-point nabla fractional boundary value problem with separated boundary conditions and derive the corresponding Green's function. I prove that this Green's function satisfies a positivity property. Then, I deduce a relatively general comparison result for the considered boundary value problem.

Keywords: nabla fractional boundary value problem; separated boundary conditions; Green's function; positivity property; comparison principle

MSC: 39A12; 39A22; 39A27

1. Introduction

The theory of fractional differential equations is a growing area of research that has widespread applications in science and engineering. Indeed, it has been realized that fractional differential equations describe many nonlinear phenomena in different fields such as physics, chemistry, biology, viscoelasticity, control hypothesis, speculation, fluid dynamics, hydrodynamics, aerodynamics, information processing, system networking, and picture processing. Besides, fractional differential equations provide marvellous tools for depicting the memory and inherited properties of many materials and processes. For this purpose, we refer to [1–9] and the references cited therein.

Nabla fractional calculus is an integrated theory of arbitrary order sums and differences in the backward sense. The concept of nabla fractional difference has been intensively studied in the last two decades. For a detailed introduction to the evolution of nabla fractional calculus, we refer to a recent monograph [10] and the references therein.

During the past decade, there has been an increasing interest in analyzing nabla fractional boundary value problems. Gholami et al. [11,12] initiated the study of two-point nabla fractional boundary value problems. Their analysis relied on the nonlinear alternative of Leray–Shauder and the Krasnosel'skii–Zabreiko fixed point theorem. In [13–15], the authors established sufficient conditions on the existence and uniqueness of solutions for different classes of two-point Riemann–Liouville nabla fractional boundary value problems associated with various types of boundary conditions. Ikram [16] established the uniqueness of solutions to boundary value problems involving the nabla Caputo fractional difference under two-point boundary conditions and explicitly expressed Green's functions for these problems. Ahrendt et al. [17] considered a discrete self-adjoint fractional operator involving the nabla Caputo fractional difference, which can be thought of as an analogue to the self-adjoint differential operator, and showed that solutions to difference equations involving this operator had expected properties, such as the form of solutions to homogeneous and nonhomogeneous equations. Chen et al. [18] obtained some existence and uniqueness theorems for solutions of discrete fractional Caputo equations using the Banach fixed point theorem. Atici et al. [19] proved the existence of solutions for an eigenvalue problem in fractional h-discrete calculus.

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One of the exciting aspects of fractional calculus (continuous and discrete) is based on the specific classical results and their statements in the fractional case, which are the same as or different from their statements in the integer-order case. In some instances, well-known and crucially essential properties in the integer-order case fail in specific fractional problems [20,21]. On the other hand, even if a given property remains true, it may have to be formulated differently. This formulation may yield insight into a fractional problem that would only be possible with the given property. With these thoughts in mind, Goodrich [22] obtained a relatively general comparison principle for the delta fractional difference operator.

In this article, we are concerned with establishing whether or not the nabla fractional difference operator satisfies a kind of comparison principle. The comparison principle that I prove here is well-known in the integer-order case but, so far as the author knows, it has yet to be established in the nabla fractional case.

To produce a suitable scheme to deduce this comparison result, we consider a very general nabla fractional boundary problem of the type

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ \alpha u(a) - \beta\big(\nabla u\big)(a+1) = 0, \\ \gamma u(b) + \delta\big(\nabla u\big)(b) = 0, \end{cases} \tag{1}$$

where a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_3 = \{3, 4, 5, \cdots\}$; $\mathbb{N}^{b}_{a+2} = \{a+2, a+3, \cdots, b\}$; $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$; $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$; $1 < \nu < 2$; and $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla fractional difference of u based at $\rho(a) = a - 1$. The Green's function changes its sign on its domain for the Caputo version of (1). So, we consider the Riemann–Liouville case only. We point out that (1) is a nabla fractional version of Hill's equation, which has many applications in astronomy, cyclotrons, electrical circuits, and the electric conductivity of metals. We also note that the boundary conditions cover the Dirichlet, the Neumann, and the mixed ones.

In particular, the results of this work provide the following generalizations and contributions.

1. It is shown that the Green's function associated with (1) is nonnegative. As mentioned above, this generalizes some of the results in [13–15]. Further, the nonnegativity property of the Green's function is an important tool to establish sufficient conditions under which (1) will have at least one positive solution. While that analysis is not carried out in this work, the positivity of the Green's function provides an initial step in that direction. Of course, such an analysis is well-known in the integer-order case.
2. A comparison-type theorem for the operator $\nabla^{\nu}_{\rho(a)}$, $1 < \nu < 2$ is deduced, which is an obvious generalization of the well-known result in the case of $\nu = 2$.
3. Some consequences of the comparison principle are provided. In particular, I explain how it implies a concavity-type interpretation for the nabla fractional difference.

The present article is organized as follows. Section 2 contains some preliminaries on nabla fractional calculus. In Section 3, I construct an associated Green's function for the boundary value problem (1) and show that this Green's function satisfies a positivity property. I also obtain a few essential properties of the Green's function. In Section 4, I deduce a comparison-type theorem for the operator $\nabla^{\nu}_{\rho(a)}$ with $1 < \nu < 2$, and also observe that this result is an obvious generalization of the well-known result in the case of $\nu = 2$. I give some consequences of the comparison principle in Section 5. In Section 6, I outline the future scope of the current work.

2. Preliminaries

In this paper, I use the fundamentals of discrete calculus [23] and discrete fractional calculus [10]. Denote by $\mathbb{N}_c = \{c, c+1, c+2, \ldots\}$ and $\mathbb{N}^{d}_{c} = \{c, c+1, c+2, \ldots, d\}$ for any real numbers c, d such that $d - c \in \mathbb{N}_1$.

Definition 1 ([23])**.** *The backward jump operator* $\rho : \mathbb{N}_{c+1} \to \mathbb{N}_c$ *is defined by*

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{c+1}.$$

Definition 2 ([10])**.** *The* μ^{th}*-order nabla fractional Taylor monomial is defined by*

$$H_\mu(t,a) = \frac{1}{\Gamma(\mu+1)}\left[(t-a)^{\overline{\mu}}\right] = \frac{1}{\Gamma(\mu+1)}\left[\frac{\Gamma(t-a+\mu)}{\Gamma(t-a)}\right], \ \mu \in \mathbb{R} \setminus \{\ldots,-2,-1\},$$

provided that the right-hand side exists. Here, $\Gamma(\cdot)$ *denotes the Euler gamma function.*

Definition 3 ([23])**.** *Let* $u : \mathbb{N}_a \to \mathbb{R}$ *and* $N \in \mathbb{N}_1$*. The first-order nabla difference of* u *is defined by*

$$(\nabla u)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}*-order nabla difference of* u *is defined recursively by*

$$(\nabla^N u)(t) = \left(\nabla(\nabla^{N-1} u)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 4 ([10])**.** *Let* $u : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $\nu > 0$*. The* ν^{th}*-order nabla fractional sum of* u *based at* a *is given by*

$$(\nabla_a^{-\nu} u)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $(\nabla_a^{-\nu} u)(a) = 0$*.*

Definition 5 ([10])**.** *Let* $u : \mathbb{N}_{a+1} \to \mathbb{R}$*,* $\nu > 0$ *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu \leq N$*. The* ν^{th}*-order Riemann–Liouville nabla fractional difference of* u *based at* a *is given by*

$$(\nabla_a^\nu u)(t) = \left(\nabla^N (\nabla_a^{-(N-\nu)} u)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Theorem 1 ([17])**.** *Let* $u : \mathbb{N}_a \to \mathbb{R}$*,* $\nu > 0$ *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu < N$*. The* ν^{th}*-order Riemann–Liouville nabla fractional difference of* u *based at* a *is given by*

$$(\nabla_a^\nu u)(t) = \sum_{s=a+1}^{t} H_{-\nu-1}(t,\rho(s))u(s), \quad t \in \mathbb{N}_{a+1}.$$

In the subsequent lemmas, I present some properties of nabla fractional Taylor monomials, which will be used in the main results.

Lemma 1 ([10])**.** *The following properties hold, provided that the expressions are well-defined:*

1. $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$;
2. $H_\mu(t,a) - H_{\mu-1}(t,a) = H_\mu(t,a+1)$;
3. $\sum_{s=a+1}^{t} H_\mu(s,a) = H_{\mu+1}(t,a)$;
4. $\sum_{s=a+1}^{t} H_\mu(t,\rho(s)) = H_{\mu+1}(t,a)$.

Lemma 2 ([16])**.** *Let* $s \in \mathbb{N}_a$ *and* $\mu > -1$*. Then, the following properties hold:*

(a) $H_\mu(t,\rho(s)) \geq 0$ *for* $t \in \mathbb{N}_{\rho(s)}$;
(b) $H_\mu(t,\rho(s)) > 0$ *for* $t \in \mathbb{N}_s$;
(c) $H_\mu(t,\rho(s))$ *is a decreasing function of* s *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu > 0$;
(d) $H_\mu(t,\rho(s))$ *is an increasing function of* s *for* $t \in \mathbb{N}_s$ *and* $-1 < \mu < 0$;

(e) $H_\mu(t,\rho(s))$ *is a nondecreasing function of* t *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \geq 0$;
(f) $H_\mu(t,\rho(s))$ *is an increasing function of* t *for* $t \in \mathbb{N}_s$ *and* $\mu > 0$;
(g) $H_\mu(t,\rho(s))$ *is a decreasing function of* t *for* $t \in \mathbb{N}_{s+1}$ *and* $-1 < \mu < 0$.

Lemma 3 ([13])**.** *Let* $\mu > -1$, $s \in \mathbb{N}_a$ *and* $t \in \mathbb{N}_s$. *Denote by*

$$h_\mu(t,s) = \frac{H_\mu(t,\rho(s))}{H_\mu(t,\rho(a))}.$$

Then, the following properties hold:

(i) $0 < h_\mu(t,s)$;
(ii) $h_\mu(t,s) \leq 1$ *for* $\mu > 0$, *and* $h_\mu(t,s) \geq 1$ *for* $-1 < \mu < 0$. *In particular,* $h_0(t,s) = 1$;
(iii) $h_\mu(t,s)$ *is a nondecreasing function of* t *for* $\mu > 0$;
(iv) $h_\mu(t,s)$ *is a nonincreasing function of* t *for* $-1 < \mu < 0$.

I use the following composition rule of the nabla fractional sum in the next section.

Lemma 4 ([10])**.** *Let* $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $k \in \mathbb{N}_0$, $\mu > 0$ *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \mu \leq N$. *Then,*

$$\Big(\nabla^k \big(\nabla_a^{-\mu} u\big)\Big)(t) = \big(\nabla_a^{k-\mu} u\big)(t), \quad t \in \mathbb{N}_{a+k}.$$

3. Construction of Green's Function

In this section, I construct the Green's function for the linear boundary value problem

$$\begin{cases} -\big(\nabla_{\rho(a)}^{\nu} u\big)(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b}, \\ \alpha u(a) - \beta\big(\nabla u\big)(a+1) = 0, \\ \gamma u(b) + \delta\big(\nabla u\big)(b) = 0, \end{cases} \tag{2}$$

associated with (1). Here $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$, $1 < \nu < 2$ and $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$. Introduce the notations:

$$\begin{aligned} A_1 &= \alpha + \beta(1-\nu), \\ A_2 &= \alpha + \beta(2-\nu) = A_1 + \beta, \\ \phi(r) &= \gamma H_{\nu-1}(b,\rho(r)) + \delta H_{\nu-2}(b,\rho(r)), \quad r \in \mathbb{N}_a^b, \\ \omega(r) &= A_2 H_{\nu-1}(r,\rho(a)) - A_1 H_{\nu-2}(r,\rho(a)), \quad r \in \mathbb{N}_a^b, \\ \Omega &= \gamma H_{\nu-2}(b,\rho(a)) + \delta H_{\nu-3}(b,\rho(a)), \\ \Lambda &= A_2 \phi(a) - A_1 \Omega. \end{aligned}$$

Theorem 2 ([14])**.** *Assume that* $1 < \nu < 2$ *and* $h : \mathbb{N}_{a+2} \to \mathbb{R}$. *The general solution of the nonhomogeneous nabla fractional difference equation*

$$-\big(\nabla_{\rho(a)}^{\nu} u\big)(t) = h(t), \quad t \in \mathbb{N}_{a+2}$$

is given by

$$u(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)) - \sum_{s=a+2}^{t} H_{\nu-1}(t,\rho(s))h(s)$$

for $t \in \mathbb{N}_a$. *Here,* C_1 *and* C_2 *are arbitrary constants.*

Theorem 3. *Assume that* $\Lambda \neq 0$. *The linear boundary value problem* (2) *has a unique solution given in the form*

$$u(t) = \sum_{s=a+2}^{b} \mathcal{G}(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3}$$

where

$$\mathcal{G}(t,s) = \begin{cases} \mathcal{G}_1(t,s), & t \in \mathbb{N}_a^{\rho(s)}, \\ \mathcal{G}_2(t,s), & t \in \mathbb{N}_s^b, \end{cases}$$

with

$$\mathcal{G}_1(t,s) = \frac{\omega(t)}{\Lambda}\phi(s),$$

and

$$\mathcal{G}_2(t,s) = \mathcal{G}_1(t,s) - H_{\nu-1}(t,\rho(s)).$$

Proof. From Theorem 2, the general solution of the nonhomogeneous nabla fractional difference equation in (2) is given by

$$u(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)) - \left(\nabla_{a+1}^{-\nu}h\right)(t), \quad t \in \mathbb{N}_a^b, \tag{4}$$

where C_1 and C_2 are arbitrary constants. Now, upon applying ∇ to both sides of equality (4), apply Lemma 1 (1) and Lemma 4 and obtain

$$(\nabla u)(t) = C_1 H_{\nu-2}(t,\rho(a)) + C_2 H_{\nu-3}(t,\rho(a)) - \left(\nabla_{a+1}^{1-\nu}h\right)(t), \quad t \in \mathbb{N}_{a+1}^b. \tag{5}$$

From the first boundary condition $\alpha u(a) - \beta(\nabla u)(a+1) = 0$ in (4)–(5), we obtain

$$A_1 C_1 + A_2 C_2 = 0. \tag{6}$$

From the second boundary condition $\gamma u(b) + \delta(\nabla u)(b) = 0$ in (4)–(5), we obtain

$$C_1\phi(a) + C_2\Omega = \sum_{s=a+2}^{b} \phi(s)h(s), \tag{7}$$

From (6) and (7), we have

$$C_1 = \frac{A_2}{\Lambda}\sum_{s=a+2}^{b} \phi(s)h(s) \tag{8}$$

and

$$C_2 = -\frac{A_1}{\Lambda}\sum_{s=a+2}^{b} \phi(s)h(s). \tag{9}$$

Substitute the equalities (8) and (9) in (4), and obtain (3). □

Example 1. *Consider the linear boundary value problem*

$$\begin{cases} -\left(\nabla_{\rho(a)}^{\nu}u\right)(t) = h(t), & t \in \mathbb{N}_{a+2}^b, \\ u(a) = 0, \quad u(b) = 0. \end{cases} \tag{10}$$

Here, $1 < \nu < 2$ *and* $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$. *Clearly,* $\Lambda \neq 0$. *The linear boundary value problem* (10) *has a unique solution given in the form*

$$u(t) = \sum_{s=a+2}^{b} \mathcal{G}_D(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{11}$$

where

$$\mathcal{G}_D(t,s) = \begin{cases} \mathcal{G}_{1D}(t,s), & t \in \mathbb{N}_a^{\rho(s)}, \\ \mathcal{G}_{2D}(t,s), & t \in \mathbb{N}_s^b, \end{cases}$$

with

$$\mathcal{G}_{1D}(t,s) = \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} H_{\nu-1}(b,\rho(s)),$$

and

$$\mathcal{G}_{2D}(t,s) = \mathcal{G}_{1D}(t,s) - H_{\nu-1}(t,\rho(s)).$$

Example 2. *Consider the linear boundary value problem*

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)} u\right)(t) = h(t), & t \in \mathbb{N}_{a+2}^b, \\ u(a) = 0, \quad (\nabla u)(b) = 0. \end{cases} \tag{12}$$

Here, $1 < \nu < 2$ *and* $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$. *Clearly,* $\Lambda \neq 0$. *The linear boundary value problem* (12) *has a unique solution given in the form*

$$u(t) = \sum_{s=a+2}^{b} \mathcal{G}_R(t,s) h(s), \quad t \in \mathbb{N}_a^b, \tag{13}$$

where

$$\mathcal{G}_R(t,s) = \begin{cases} \mathcal{G}_{1R}(t,s), & t \in \mathbb{N}_a^{\rho(s)}, \\ \mathcal{G}_{2R}(t,s), & t \in \mathbb{N}_s^b, \end{cases}$$

with

$$\mathcal{G}_{1R}(t,s) = \frac{H_{\nu-1}(t,a)}{H_{\nu-2}(b,a)} H_{\nu-2}(b,\rho(s)),$$

and

$$\mathcal{G}_{2R}(t,s) = \mathcal{G}_{1R}(t,s) - H_{\nu-1}(t,\rho(s)).$$

4. Positivity & Other Properties of the Green's Function

In this section, I prove that the Green's function derived in Section 3 is positive on its domain. This important result will allow us in Section 5 to deduce a relatively general comparison theorem. I also obtain a few important properties of the Green's function. I begin with the following lemma.

Lemma 5. *Let* α, β, γ, *and* δ *be nonnegative real numbers such that* $\alpha \geq \beta$. *Then, the following properties hold:*

(I) $A_1 > 0$, $A_2 > 0$ *and* $\phi(r) > 0$ *for* $r \in \mathbb{N}_a^b$;
(II) $\phi(a) - \Omega > 0$;
(III) $\Lambda > 0$;
(IV) $\omega(r) \geq 0$ *for* $r \in \mathbb{N}_a^b$.
(V) $(\nabla \omega)(r) > 0$ *for* $r \in \mathbb{N}_{a+1}^b$.

Proof. The proof of (I) follows from Lemma 2 (b). To prove (II), consider

$$\begin{aligned} \phi(a) - \Omega &= \gamma[H_{\nu-1}(b,\rho(a)) - H_{\nu-2}(b,\rho(a))] \\ &\quad + \delta[H_{\nu-2}(b,\rho(a)) - H_{\nu-3}(b,\rho(a))] \\ &= \gamma H_{\nu-1}(b,a) + \delta H_{\nu-2}(b,a) \quad \text{(By Lemma 1 (2))} \\ &> 0. \quad \text{(By Hypothesis and Lemma 2 (b))} \end{aligned}$$

To prove (III), consider

$$\begin{aligned}\Lambda = A_2\phi(a) - A_1\Omega &= (A_1+\beta)\phi(a) - A_1\Omega \\ &= A_1[\phi(a)-\Omega] + \beta\phi(a) \\ &> 0. \quad (\text{By} \quad \text{I} \quad \text{and} \quad \text{II})\end{aligned}$$

To prove (IV), for $r \in \mathbb{N}_a^b$, we consider

$$\begin{aligned}\omega(r) &= A_2H_{\nu-1}(r,\rho(a)) - A_1H_{\nu-2}(r,\rho(a)) \\ &= (A_1+\beta)H_{\nu-1}(r,\rho(a)) - A_1H_{\nu-2}(r,\rho(a)) \\ &= A_1[H_{\nu-1}(r,\rho(a)) - H_{\nu-2}(r,\rho(a))] + \beta H_{\nu-1}(r,\rho(a)) \\ &= A_1H_{\nu-1}(r,a) + \beta H_{\nu-1}(r,\rho(a)) \quad (\text{By} \quad \text{Lemma 1 (2)}) \\ &\geq 0. \quad (\text{By} \quad \text{I} \quad \text{and} \quad \text{Lemma 2 (a, b)})\end{aligned}$$

To prove (V), for $r \in \mathbb{N}_{a+1}^b$, we consider

$$\begin{aligned}(\nabla\omega)(r) &= \nabla[A_2H_{\nu-1}(r,\rho(a)) - A_1H_{\nu-2}(r,\rho(a))] \\ &= A_2H_{\nu-2}(r,\rho(a)) - A_1H_{\nu-3}(r,\rho(a)) \quad (\text{By} \quad \text{Lemma 1 (1)}) \\ &= (A_1+\beta)H_{\nu-2}(r,\rho(a)) - A_1H_{\nu-3}(r,\rho(a)) \\ &= A_1[H_{\nu-2}(r,\rho(a)) - H_{\nu-3}(r,\rho(a))] + \beta H_{\nu-2}(r,\rho(a)) \\ &= A_1H_{\nu-2}(r,a) + \beta H_{\nu-2}(r,\rho(a)) \quad (\text{By} \quad \text{Lemma 1 (2)}) \\ &> 0. \quad (\text{By} \quad \text{Hypothesis,} \quad \text{I} \quad \text{and} \quad \text{Lemma 2 (b)})\end{aligned}$$

The proof is complete. □

Lemma 6. *Assume that α, β, γ, and δ are nonnegative real numbers such that $\alpha \geq \beta$. Then, the Green's function $\mathcal{G}(t,s)$, defined by (4), is nonnegative for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.*

Proof. For $t \in \mathbb{N}_a^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^b$, we define $\mathcal{G}_1(t,s) = \frac{\omega(t)}{\Lambda}\phi(s)$. From Lemma 5, it follows that $\omega(t) \geq 0$, $\phi(s) > 0$, $\Lambda > 0$ and, thus,

$$\mathcal{G}_1(t,s) \geq 0 \text{ for } t \in \mathbb{N}_a^{\rho(s)} \text{ and } s \in \mathbb{N}_{a+2}^b. \tag{14}$$

For $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$, we consider

$$\begin{aligned}\mathcal{G}_2(t,s) &= \mathcal{G}_1(t,s) - H_{\nu-1}(t,\rho(s)) \\ &= \frac{\omega(t)}{\Lambda}\phi(s) - H_{\nu-1}(t,\rho(s)) \\ &= \frac{1}{\Lambda}[\omega(t)\phi(s) - \Lambda H_{\nu-1}(t,\rho(s))] \\ &= \frac{1}{\Lambda}[A_1\gamma E_1 + A_1\delta E_2 + \beta\gamma E_3 + \beta\delta E_4],\end{aligned} \tag{15}$$

where

$$\begin{aligned}E_1 &= H_{\nu-1}(b,\rho(s))H_{\nu-1}(t,a) - H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,a), \\ E_2 &= H_{\nu-2}(b,\rho(s))H_{\nu-1}(t,a) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,a), \\ E_3 &= H_{\nu-1}(b,\rho(s))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,\rho(a)), \\ E_4 &= H_{\nu-2}(b,\rho(s))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,\rho(a)).\end{aligned}$$

Therefore,

$$\begin{aligned} E_1 &= H_{\nu-1}(b,\rho(s))H_{\nu-1}(t,a) - H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,a) \\ &= H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,a)\left[\frac{H_{\nu-1}(b,\rho(s))}{H_{\nu-1}(b,a)}\frac{H_{\nu-1}(t,a)}{H_{\nu-1}(t,\rho(s))} - 1\right] \\ &= H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,a)\left[\frac{h_{\nu-1}(b,s)}{h_{\nu-1}(t,s)} - 1\right]. \end{aligned}$$

From Lemma 2 (b), it follows that $H_{\nu-1}(t,\rho(s)) > 0$ and $H_{\nu-1}(b,a) > 0$. Furthermore, from Lemma 3, we have $h_{\nu-1}(t,s) \le h_{\nu-1}(b,s)$, thus implying that

$$E_1 \ge 0. \tag{16}$$

From Lemma 2 (c, d), we have $H_{\nu-1}(t,\rho(s)) < H_{\nu-1}(t,a)$, $H_{\nu-2}(b,a) < H_{\nu-2}(b,\rho(s))$, and, thus,

$$\begin{aligned} E_2 &= H_{\nu-2}(b,\rho(s))H_{\nu-1}(t,a) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,a) \\ &> H_{\nu-2}(b,a)H_{\nu-1}(t,\rho(s)) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,a) \\ &= 0. \end{aligned} \tag{17}$$

From the definition of E_3, Lemma 2 (b) and Lemma 3, we obtain

$$\begin{aligned} E_3 &= H_{\nu-1}(b,\rho(s))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,\rho(a)) \\ &= H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,\rho(a))\left[\frac{H_{\nu-1}(b,\rho(s))}{H_{\nu-1}(b,\rho(a))}\frac{H_{\nu-1}(t,\rho(a))}{H_{\nu-1}(t,\rho(s))} - 1\right] \\ &= H_{\nu-1}(t,\rho(s))H_{\nu-1}(b,\rho(a))\left[\frac{h_{\nu-1}(b,s)}{h_{\nu-1}(t,s)} - 1\right], \end{aligned}$$

or

$$E_3 \ge 0. \tag{18}$$

From Lemma 2 (c, d), we have $H_{\nu-1}(t,\rho(s)) < H_{\nu-1}(t,\rho(a))$, $H_{\nu-2}(b,\rho(a)) < H_{\nu-2}(b,\rho(s))$, and, thus,

$$\begin{aligned} E_4 &= H_{\nu-2}(b,\rho(s))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,\rho(a)) \\ &> H_{\nu-2}(b,\rho(a))H_{\nu-1}(t,\rho(s)) - H_{\nu-1}(t,\rho(s))H_{\nu-2}(b,\rho(a)) \\ &= 0. \end{aligned} \tag{19}$$

From definition (15) of $\mathcal{G}_2(t,s)$, as well as inequalities $\Lambda > 0$, $A_1 > 0$, $\beta \ge 0$, $\gamma \ge 0$, $\delta \ge 0$, and (16)–(19), it follows that that

$$\mathcal{G}_2(t,s) > 0 \text{ for } t \in \mathbb{N}_s^b \text{ and } s \in \mathbb{N}_{a+2}^b. \tag{20}$$

Thus, from (14) and (20), we obtain that $\mathcal{G}(t,s) \ge 0$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. The proof is complete. □

Lemma 7. *Assume that α, β, γ, and δ are nonnegative real numbers such that $\alpha \ge \beta$. The Green's function $\mathcal{G}(t,s)$ defined in (4) satisfies the following property:*

$$\max_{t \in \mathbb{N}_a^b} \mathcal{G}(t,s) = \mathcal{G}(s-1,s), \quad s \in \mathbb{N}_{a+2}^b.$$

Proof. Note that the operator ∇ denotes the first order nabla difference operator with respect to t. For $t \in \mathbb{N}_{a+1}^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^{b}$, we consider the function $\mathcal{G}_1(t,s)$ defined by (4) and

$$\nabla\mathcal{G}_1(t,s) = \nabla\left[\frac{\omega(t)}{\Lambda}\phi(s)\right] = \frac{(\nabla\omega)(t)}{\Lambda}\phi(s).$$

From Lemma 5, it follows that $(\nabla\omega)(t) > 0$, $\phi(s) > 0$ and $\Lambda > 0$, thus implying that $\nabla\mathcal{G}_1(t,s) > 0$ for $t \in \mathbb{N}_{a+1}^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^{b}$. That is, $\mathcal{G}_1(t,s)$ is an increasing function of t for $t \in \mathbb{N}_{a}^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^{b}$. For $t \in \mathbb{N}_{s}^{b}$ and $s \in \mathbb{N}_{a+2}^{b}$, consider the function $\mathcal{G}_2(t,s)$ defined by (4), and, thus,

$$\begin{aligned}
\nabla\mathcal{G}_2(t,s) &= \nabla[\mathcal{G}_1(t,s) - H_{\nu-1}(t,\rho(s))] \\
&= \frac{(\nabla\omega)(t)}{\Lambda}\phi(s) - H_{\nu-2}(t,\rho(s)) \quad \text{(By Lemma 1 (1))} \\
&= \frac{1}{\Lambda}\big[(\nabla\omega)(t)\phi(s) - \Lambda H_{\nu-2}(t,\rho(s))\big] \\
&= \frac{1}{\Lambda}[A_1\gamma F_1 + A_1\delta F_2 + \beta\gamma F_3 + \beta\delta F_4],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
F_1 &= H_{\nu-1}(b,\rho(s))H_{\nu-2}(t,a) - H_{\nu-2}(t,\rho(s))H_{\nu-1}(b,a), \\
F_2 &= H_{\nu-2}(b,\rho(s))H_{\nu-2}(t,a) - H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,a), \\
F_3 &= H_{\nu-1}(b,\rho(s))H_{\nu-2}(t,\rho(a)) - H_{\nu-2}(t,\rho(s))H_{\nu-1}(b,\rho(a)), \\
F_4 &= H_{\nu-2}(b,\rho(s))H_{\nu-2}(t,\rho(a)) - H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,\rho(a)).
\end{aligned}$$

From Lemma 2 (c, d), we have $H_{\nu-1}(b,\rho(s)) < H_{\nu-1}(b,a)$ and $H_{\nu-2}(t,a) < H_{\nu-2}(t,\rho(s))$, thus implying that

$$\begin{aligned}
F_1 &= H_{\nu-1}(b,\rho(s))H_{\nu-2}(t,a) - H_{\nu-2}(t,\rho(s))H_{\nu-1}(b,a) \\
&< H_{\nu-1}(b,a)H_{\nu-2}(t,\rho(s)) - H_{\nu-2}(t,\rho(s))H_{\nu-1}(b,a) = 0,
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
F_3 &= H_{\nu-1}(b,\rho(s))H_{\nu-2}(t,\rho(a)) - H_{\nu-2}(t,\rho(s))H_{\nu-1}(b,\rho(a)) \\
&< H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(s)) - H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(s)) = 0.
\end{aligned}$$

Consider,

$$\begin{aligned}
F_2 &= H_{\nu-2}(b,\rho(s))H_{\nu-2}(t,a) - H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,a) \\
&= H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,a)\left[\frac{H_{\nu-2}(b,\rho(s))}{H_{\nu-2}(b,a)}\frac{H_{\nu-2}(t,a)}{H_{\nu-2}(t,\rho(s))} - 1\right] \\
&= H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,a)\left[\frac{h_{\nu-2}(b,s)}{h_{\nu-2}(t,s)} - 1\right],
\end{aligned}$$

and

$$\begin{aligned}
F_4 &= H_{\nu-2}(b,\rho(s))H_{\nu-2}(t,\rho(a)) - H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,\rho(a)) \\
&= H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,\rho(a))\left[\frac{H_{\nu-2}(b,\rho(s))}{H_{\nu-2}(b,\rho(a))}\frac{H_{\nu-2}(t,\rho(a))}{H_{\nu-2}(t,\rho(s))} - 1\right] \\
&= H_{\nu-2}(t,\rho(s))H_{\nu-2}(b,\rho(a))\left[\frac{h_{\nu-2}(b,s)}{h_{\nu-2}(t,s)} - 1\right].
\end{aligned}$$

It follows from Lemma 2 (b) that $H_{\nu-2}(t,\rho(s)) > 0$ and $H_{\nu-2}(b,a) > 0$. Furthermore, from Lemma 3, we have $h_{\nu-2}(t,s) \geq h_{\nu-2}(b,s)$, thus implying that

$$F_2 \leq 0, \tag{23}$$

and

$$F_4 \leq 0. \tag{24}$$

Since $\Lambda > 0$, $A_1 > 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$, it follows from (21) that $\nabla\mathcal{G}_2(t,s) < 0$ for $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$. That is, $\mathcal{G}_2(t,s)$ is a decreasing function of t for $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$. Thus,

$$\max_{t\in\mathbb{N}_a^b} \mathcal{G}(t,s) = \max\{\mathcal{G}_1(s-1,s), \mathcal{G}_2(s,s)\}, \quad s \in \mathbb{N}_{a+2}^b. \tag{25}$$

We have

$$\mathcal{G}_1(s-1,s) = \frac{\omega(s-1)}{\Lambda}\phi(s), \quad s \in \mathbb{N}_{a+2}^b, \tag{26}$$

and

$$\mathcal{G}_2(s,s) = \frac{\omega(s)}{\Lambda}\phi(s) - H_{\nu-1}(s,\rho(s)), \quad s \in \mathbb{N}_{a+2}^b. \tag{27}$$

Now, consider

$$\begin{aligned}
\mathcal{G}_2(s,s) - \mathcal{G}_1(s-1,s) &= \frac{\omega(s)}{\Lambda}\phi(s) - H_{\nu-1}(s,\rho(s)) - \frac{\omega(s-1)}{\Lambda}\phi(s) \\
&= \frac{1}{\Lambda}\phi(s)[\omega(s) - \omega(s-1)] - H_{\nu-1}(s,\rho(s)) \\
&= \frac{(\nabla\omega)(s)}{\Lambda}\phi(s) - H_{\nu-2}(s,\rho(s)) \\
&= \nabla\mathcal{G}_2(s,s) < 0,
\end{aligned}$$

thus implying that

$$\max_{t\in\mathbb{N}_a^b} \mathcal{G}(t,s) = \mathcal{G}_1(s-1,s), \quad s \in \mathbb{N}_{a+2}^b.$$

The proof is complete. □

Lemma 8. *Assume that α, β, γ, and δ are nonnegative real numbers such that $\alpha \geq \beta$. The Green's function $\mathcal{G}(t,s)$ defined in (4) satisfies the following property:*

$$\min_{t\in\mathbb{N}_a^b} \mathcal{G}(t,s) \geq \Theta\mathcal{G}(s-1,s), \quad s \in \mathbb{N}_{a+2}^b,$$

where

$$\Theta = \frac{1}{\omega(b-1)} \min\left\{\omega(a), \omega(b) - \Lambda\left[\frac{1}{\gamma + \delta\left(\frac{\nu-1}{b-a+\nu-3}\right)}\right]\right\}.$$

Proof. From Lemma 7, we have

$$\mathcal{G}_1(a,s) \leq \mathcal{G}_1(t,s) \leq \mathcal{G}_1(s-1,s), \quad t \in \mathbb{N}_a^{\rho(s)}, \quad s \in \mathbb{N}_{a+2}^b, \tag{28}$$

and

$$\mathcal{G}_2(b,s) \leq \mathcal{G}_2(t,s) \leq \mathcal{G}_2(s,s), \quad t \in \mathbb{N}_s^b, \quad s \in \mathbb{N}_{a+2}^b. \tag{29}$$

Consider

$$\frac{\mathcal{G}(t,s)}{\mathcal{G}(s-1,s)} = \begin{cases} \frac{\mathcal{G}_1(t,s)}{\mathcal{G}_1(s-1,s)}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{\mathcal{G}_2(t,s)}{\mathcal{G}_1(s-1,s)}, & t \in \mathbb{N}_s^b, \end{cases}$$

$$\geq \begin{cases} \frac{\mathcal{G}_1(a,s)}{\mathcal{G}_1(s-1,s)}, & t \in \mathbb{N}_a^{\rho(s)}, \quad (\text{By } (28)) \\ \frac{\mathcal{G}_2(b,s)}{\mathcal{G}_1(s-1,s)}, & t \in \mathbb{N}_s^b, \quad (\text{By } (29)) \end{cases}$$

$$= \begin{cases} \frac{\omega(a)}{\omega(s-1)}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{\omega(b)}{\omega(s-1)} - \frac{\Lambda H_{\nu-1}(b,\rho(s))}{\omega(s-1)\phi(s)}, & t \in \mathbb{N}_s^b, \end{cases}$$

$$= \frac{1}{\omega(s-1)} \begin{cases} \omega(a), & t \in \mathbb{N}_a^{\rho(s)}, \\ \omega(b) - \Lambda\left[\frac{H_{\nu-1}(b,\rho(s))}{\phi(s)}\right], & t \in \mathbb{N}_s^b. \end{cases} \tag{30}$$

Now, consider

$$\begin{aligned} \frac{H_{\nu-1}(b,\rho(s))}{\phi(s)} &= \frac{H_{\nu-1}(b,\rho(s))}{\gamma H_{\nu-1}(b,\rho(s)) + \delta H_{\nu-2}(b,\rho(s))} \\ &= \frac{1}{\gamma + \delta\left[\frac{H_{\nu-2}(b,\rho(s))}{H_{\nu-1}(b,\rho(s))}\right]} \\ &= \frac{1}{\gamma + \delta\left(\frac{\nu-1}{b-s+\nu-1}\right)} \leq \frac{1}{\gamma + \delta\left(\frac{\nu-1}{b-a+\nu-3}\right)} \end{aligned} \tag{31}$$

for $s \in \mathbb{N}_{a+2}^b$. Using (31) in (30), we obtain

$$\frac{\mathcal{G}(t,s)}{\mathcal{G}(s-1,s)} \geq \frac{1}{\omega(s-1)} \begin{cases} \omega(a), & t \in \mathbb{N}_a^{\rho(s)}, \\ \omega(b) - \Lambda\left[\frac{1}{\gamma+\delta\left(\frac{\nu-1}{b-a+\nu-3}\right)}\right], & t \in \mathbb{N}_s^b. \end{cases} \tag{32}$$

It follows from Lemma 5 (V) that $\omega(r)$ is an increasing function of r for each $r \in \mathbb{N}_a^b$. Then, we have

$$\omega(a+1) \leq \omega(s-1) \leq \omega(b-1), \quad s \in \mathbb{N}_{a+2}^b. \tag{33}$$

Using (33) in (32), we obtain

$$\frac{\mathcal{G}(t,s)}{\mathcal{G}(s-1,s)} \geq \frac{1}{\omega(b-1)} \begin{cases} \omega(a), & t \in \mathbb{N}_a^{\rho(s)}, \\ \omega(b) - \Lambda\left[\frac{1}{\gamma+\delta\left(\frac{\nu-1}{b-a+\nu-3}\right)}\right], & t \in \mathbb{N}_s^b. \end{cases} \tag{34}$$

The proof is complete. □

5. A General Comparison Result

I prove a general comparison result for the boundary value problem (1). For this purpose, we consider the following nabla fractional boundary value problem with nonhomogeneous boundary conditions corresponding to (2):

$$\begin{cases} -\left(\nabla_{\rho(a)}^{\nu} u\right)(t) = h(t), & t \in \mathbb{N}_{a+2}^b, \\ \alpha u(a) - \beta(\nabla u)(a+1) = A, \\ \gamma u(b) + \delta(\nabla u)(b) = B, \end{cases} \tag{35}$$

where $A, B \in \mathbb{R}$.

Lemma 9. *The unique solution of the boundary value problem*

$$\begin{cases} -\left(\nabla_{\rho(a)}^{\nu} w\right)(t) = 0, & t \in \mathbb{N}_{a+2}^b, \\ \alpha w(a) - \beta(\nabla w)(a+1) = A, \\ \gamma w(b) + \delta(\nabla w)(b) = B \end{cases} \tag{36}$$

is given by

$$w(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)), \quad t \in \mathbb{N}_a^b, \tag{37}$$

where

$$C_1 = \frac{1}{\Lambda}[A_2 B - A\Omega], \tag{38}$$

and

$$C_2 = \frac{1}{\Lambda}[A\phi(a) - A_1 B]. \tag{39}$$

Proof. Apply the operator $\nabla_{a+1}^{-\nu}$ to both sides of the nabla fractional difference equation in (36) and obtain

$$w(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)), \quad t \in \mathbb{N}_a, \tag{40}$$

where C_1 and C_2 are arbitrary constants.

Upon applying the operator ∇ to both sides of (40), apply Lemma 1 (1) and obtain

$$\big(\nabla w\big)(t) = C_1 H_{\nu-2}(t,\rho(a)) + C_2 H_{\nu-3}(t,\rho(a)), \quad t \in \mathbb{N}_{a+1}. \tag{41}$$

From the first boundary condition $\alpha w(a) - \beta\big(\nabla w\big)(a+1) = A$ in (40)–(41), we obtain

$$A_1 C_1 + A_2 C_2 = A. \tag{42}$$

From the second boundary condition $\gamma w(b) + \delta\big(\nabla w\big)(b) = B$ in (40)–(41) we have

$$C_1 \phi(a) + C_2 \Omega = B. \tag{43}$$

From equalities (42) and (43), we obtain (38) and (39). □

Now, we will obtain the explicit unique solution of the boundary value problem (35).

Theorem 4. *Assume that α, β, γ, and δ are nonnegative real numbers such that $\alpha \geq \beta$. The unique solution of the boundary value problem (35) is given by*

$$u(t) = w(t) + \sum_{s=a+2}^{b} \mathcal{G}(t,s) h(s), \quad t \in \mathbb{N}_a^b, \tag{44}$$

where the Green's function $\mathcal{G}(t,s)$ is defined by (4), and w is given by (37).

Now, let us take $L = \nabla_{\rho(a)}^{\nu}$. We use Lemma 9 together with Lemma 6 to prove a comparison theorem regarding the nabla fractional difference operator L. In order to accomplish this, we make certain assumptions regarding the numbers A and B appearing in the boundary value problem (36). These assumptions yield a few diverse situations under which our comparison theorem will hold. In light of this, we have the following lemmas.

Lemma 10. *Assume that α, β, and γ are nonnegative real numbers such that $\alpha \geq \beta$. If $A = 0$, $B \geq 0$, and $\delta = 0$, it follows that*

$$w(t) \geq 0, \quad t \in \mathbb{N}_a^b.$$

Proof. By taking $A = 0$, it is clear from (37), (38), and (39) that

$$w(t) = \frac{1}{\Lambda}[A_2 B H_{\nu-1}(t,\rho(a)) - A_1 B H_{\nu-2}(t,\rho(a))] = \frac{B}{\Lambda}\omega(t), \quad t \in \mathbb{N}_a^b.$$

Consider

$$\Lambda = A_2 \phi(a) - A_1 \Omega$$

$$
\begin{aligned}
&= (A_1+\beta)\gamma H_{\nu-1}(b,\rho(a)) - A_1\gamma H_{\nu-2}(b,\rho(a))\\
&= \gamma[A_1(H_{\nu-1}(b,\rho(a)) - H_{\nu-2}(b,\rho(a))) + \beta H_{\nu-1}(b,\rho(a))]\\
&= \gamma[A_1 H_{\nu-1}(b,a) + \beta H_{\nu-1}(b,\rho(a))] \quad \text{(By Lemma 1 (2))}\\
&\geq 0. \quad \text{(By Hypothesis and Lemma 2 (b))}
\end{aligned}
$$

It follows from Lemma 5 that $\omega(t) \geq 0$ for $t \in \mathbb{N}_a^b$, thus implying that $w(t) \geq 0$ for $t \in \mathbb{N}_a^b$. □

Lemma 11. *Assume that α and δ are nonnegative real numbers. If A, $B \geq 0$ and $\beta = \gamma = 0$, it follows that*

$$
w(t) \geq 0, \quad t \in \mathbb{N}_a^b.
$$

Proof. From (37), (38), and (39), we have

$$
\begin{aligned}
w(t) &= \frac{1}{\Lambda}[(A_2B - A\Omega)H_{\nu-1}(t,\rho(a)) + (A\phi(a) - A_1B)H_{\nu-2}(t,\rho(a))]\\
&= \frac{B}{\Lambda}\omega(t) - \frac{A}{\Lambda}[\Omega H_{\nu-1}(t,\rho(a)) - \phi(a)H_{\nu-2}(t,\rho(a))], \quad t \in \mathbb{N}_a^b. \qquad (45)
\end{aligned}
$$

Take $\beta = \gamma = 0$ in (3)–(3). Then, by applying Lemma 1 (2), we obtain that

$$
\begin{aligned}
A_1 &= \alpha,\\
A_2 &= A_1,\\
\phi(r) &= \delta H_{\nu-2}(b,\rho(r)), \quad r \in \mathbb{N}_a^b,\\
\omega(r) &= \alpha[H_{\nu-1}(r,\rho(a)) - H_{\nu-2}(r,\rho(a))] = \alpha H_{\nu-1}(r,a), \quad r \in \mathbb{N}_a^b,\\
\Omega &= \delta H_{\nu-3}(b,\rho(a)),\\
\Lambda &= \alpha\delta[H_{\nu-2}(b,\rho(a)) - H_{\nu-3}(b,\rho(a))] = \alpha\delta H_{\nu-2}(b,a).
\end{aligned}
$$

Consequently, from (45), we have

$$
\begin{aligned}
w(t) &= \frac{B}{\alpha\delta H_{\nu-2}(b,a)}[\alpha H_{\nu-1}(t,a)] - \frac{A}{\alpha\delta H_{\nu-2}(b,a)}\\
&\quad \times [\delta H_{\nu-3}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - \delta H_{\nu-2}(b,\rho(a))H_{\nu-2}(t,\rho(a))]\\
&= \frac{BH_{\nu-1}(t,a)}{\delta H_{\nu-2}(b,a)} - \frac{A}{\alpha H_{\nu-2}(b,a)}w_1(t), \quad t \in \mathbb{N}_a^b, \qquad (46)
\end{aligned}
$$

where

$$
w_1(t) = H_{\nu-3}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - H_{\nu-2}(b,\rho(a))H_{\nu-2}(t,\rho(a)), \quad t \in \mathbb{N}_a^b.
$$

Consider

$$
\begin{aligned}
w_1(t) &= H_{\nu-3}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - H_{\nu-2}(b,\rho(a))H_{\nu-2}(t,\rho(a))\\
&= \frac{\Gamma(b-a+\nu-2)}{\Gamma(b-a+1)\Gamma(\nu-2)}\frac{\Gamma(t-a+\nu)}{\Gamma(t-a+1)\Gamma(\nu)}\\
&\quad - \frac{\Gamma(b-a+\nu-1)}{\Gamma(b-a+1)\Gamma(\nu-1)}\frac{\Gamma(t-a+\nu-1)}{\Gamma(t-a+1)\Gamma(\nu-1)}\\
&= \frac{\Gamma(b-a+\nu-2)}{\Gamma(b-a+1)\Gamma(\nu-2)}\frac{\Gamma(t-a+\nu-1)}{\Gamma(t-a+1)\Gamma(\nu-1)}\\
&\quad \times \left[\frac{t-a+\nu-1}{\nu-1} - \frac{b-a+\nu-2}{\nu-2}\right]\\
&= -\frac{\Gamma(b-a+\nu-2)}{\Gamma(b-a+1)\Gamma(\nu-1)}\frac{\Gamma(t-a+\nu-1)}{\Gamma(t-a+1)\Gamma(\nu)}
\end{aligned}
$$

$$\times [(t-a)+(b-t)(\nu-1)].$$

We know that $\Gamma(b-a+\nu-2)>0, \Gamma(b-a+1)>0, \Gamma(\nu-1)>0, \Gamma(t-a+\nu-1)>0$, $\Gamma(t-a+1)>0, \Gamma(\nu)>0, (t-a)\geq 0, (b-t)\geq 0$, and $(\nu-1)>0$ for $t\in\mathbb{N}_a^b$, thus implying that

$$w_1(t)\leq 0, \quad t\in\mathbb{N}_a^b.$$

Consequently, it follows from (46) and Lemma 2 (a, b) that $w(t)\geq 0$ for $t\in\mathbb{N}_a^b$. □

Lemma 12. *Assume that α and γ are nonnegative real numbers. If $A, B\geq 0$ and $\beta=\delta=0$, it follows that*

$$w(t)\geq 0, \quad t\in\mathbb{N}_a^b.$$

Proof. Take $\beta=\delta=0$ in (3)–(3). Then, by applying Lemma 1 (2), we obtain that

$$\begin{aligned}
A_1 &= \alpha,\\
A_2 &= A_1,\\
\phi(r) &= \gamma H_{\nu-1}(b,\rho(r)), \quad r\in\mathbb{N}_a^b,\\
\omega(r) &= \alpha[H_{\nu-1}(r,\rho(a))-H_{\nu-2}(r,\rho(a))] = \alpha H_{\nu-1}(r,a), \quad r\in\mathbb{N}_a^b,\\
\Omega &= \gamma H_{\nu-2}(b,\rho(a)),\\
\Lambda &= \alpha\gamma[H_{\nu-1}(b,\rho(a))-H_{\nu-2}(b,\rho(a))] = \alpha\gamma H_{\nu-1}(b,a).
\end{aligned}$$

Consequently, from (45), we have

$$\begin{aligned}
w(t) = {}& \frac{B}{\alpha\gamma H_{\nu-1}(b,a)}[\alpha H_{\nu-1}(t,a)] - \frac{A}{\alpha\gamma H_{\nu-1}(b,a)}\\
&\times[\gamma H_{\nu-2}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - \gamma H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(a))]\\
={}& \frac{BH_{\nu-1}(t,a)}{\gamma H_{\nu-1}(b,a)} - \frac{A}{\alpha H_{\nu-1}(b,a)}w_2(t), \quad t\in\mathbb{N}_a^b,
\end{aligned} \tag{47}$$

where

$$w_2(t) = H_{\nu-2}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(a)), \quad t\in\mathbb{N}_a^b.$$

From Lemma 2 (c, d) we have $H_{\nu-1}(t,\rho(a)) < H_{\nu-1}(b,\rho(a))$ and $H_{\nu-2}(b,\rho(a)) < H_{\nu-2}(t,\rho(a))$, implying that

$$\begin{aligned}
w_2(t) &= H_{\nu-2}(b,\rho(a))H_{\nu-1}(t,\rho(a)) - H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(a))\\
&< H_{\nu-2}(t,\rho(a))H_{\nu-1}(b,\rho(a)) - H_{\nu-1}(b,\rho(a))H_{\nu-2}(t,\rho(a))\\
&= 0.
\end{aligned}$$

Consequently, it follows from (47) and Lemma 2 (a, b) that $w(t)\geq 0$ for $t\in\mathbb{N}_a^b$. □

Remark 1. *Observe that the boundary conditions implied by Lemma 11 are right focal boundary conditions, whereas the boundary conditions implied by Lemma 12 are Dirichlet boundary conditions.*

I now prove a comparison result for the operator $L=\nabla^\nu_{\rho(a)}$. For convenience, let us take $L_1u=\alpha u(a)-\beta(\nabla u)(a+1)$ and $L_2u=\gamma u(b)+\delta(\nabla u)(b)$. Let us also call hypothesis (H1) the hypothesis of Lemma 10, hypothesis (H2) the hypotheses of Lemma 11, and hypothesis (H3) the hypothesis of Lemma 12. We then obtain the following comparison-type theorem.

Theorem 5. *Assume that α, β, γ, and δ are nonnegative real numbers such that $\alpha \geq \beta$. Suppose that u and v satisfy $Lu \leq Lv$, $L_1 u \geq L_1 v$, and $L_2 u \geq L_2 v$. In addition, suppose that one of the conditions (H1), (H2), or (H3) holds. Then,*

$$u(t) \geq v(t), \quad t \in \mathbb{N}_a^b.$$

Proof. Put $z = u - v$. Then, it follows from Theorem 4 that z is the solution of the problem

$$\begin{cases} Lz = h(t), & t \in \mathbb{N}_{a+2}^b, \\ L_1 z = A, \\ L_2 z = B, \end{cases} \tag{48}$$

where $A = L_1 z = L_1 u - L_1 v \geq 0$, $B = L_2 z = L_2 u - L_2 v \geq 0$, and, for $t \in \mathbb{N}_{a+2}^b$, $h(t) = Lz = Lu - Lv \leq 0$. In particular, from Theorem 4, we know that z has the form

$$z(t) = w(t) - \sum_{s=a+2}^{b} \mathcal{G}(t,s)h(s), \quad t \in \mathbb{N}_a^b. \tag{49}$$

Indeed, from (36) and (49), the definitions of L and $\mathcal{G}$, we find that

$$Lz = Lw - L\nabla_{\rho(a)}^{-\nu} h = 0 - [-h(t)] = h(t), \quad t \in \mathbb{N}_{a+2}^b. \tag{50}$$

However, as one of the conditions (H1), (H2) or (H3) holds, we have from Lemmas 10–12 that

$$w(t) \geq 0, \quad t \in \mathbb{N}_a^b.$$

Moreover, Lemma 6 shows that

$$\mathcal{G}(t,s) \geq 0, \quad (t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b.$$

So, given that

$$h(t) \leq 0, \quad t \in \mathbb{N}_a^b,$$

it follows that

$$z(t) \geq 0, \quad t \in \mathbb{N}_a^b,$$

whence

$$u(t) \geq v(t), \quad t \in \mathbb{N}_a^b.$$

□

Remark 2. *Observe that using Theorem 5 together with condition (H3) implies that the operator L, together with the Dirichlet boundary conditions, satisfy the usual comparison theorem as is well-known in the classical theory of differential Equations [24] and as is also well-known in the more general time scales case [23].*

Remark 3. *In case $\alpha = \gamma = 1$ and $\beta = \delta = 0$, the result of Theorem 5 implies that the ν-th order nabla fractional difference operator satisfies a kind of classical concavity property. In particular, given $1 < \nu < 2$, the result of Theorem 5 can be recast by asserting that, if*

$$\left(\nabla_{\rho(a)}^{\nu} u\right)(t) \leq \left(\nabla_{\rho(a)}^{\nu} v\right)(t), \quad t \in \mathbb{N}_{a+2}^b,$$

and, if both $u(a) \geq v(a)$ and $u(b) \geq v(b)$, then

$$u(t) \geq v(t), \quad t \in \mathbb{N}_a^b.$$

Of course, when $\nu = 2$, this is a well-known result with a clear geometric interpretation. When $\nu \neq 2$, this result implies that the ν-th order nabla fractional sum operator satisfies an abstract *concavity property, which is mathematically interesting. This is made particularly clear by taking $v \equiv 0$; similarly, a convexity result is implied if we take $u \equiv 0$.*

Remark 4. *As stated in the above remark, we established that the ν-th order nabla fractional difference operator satisfies a kind of concavity result. It should be noted that this fact might not automatically be expected. Indeed, as is well known from the existing literature on fractional boundary value problems (particularly in the continuous case), certain very important properties that hold in the integer-order case fail to hold in the fractional case. So, it seems useful to know that this particular property does remain true in the fractional case.*

I will provide two examples illustrating the comparison result of Theorem 5 and Remarks 3 and 4.

Example 3. *Let $z : \mathbb{N}_a^b \to \mathbb{R}$ be a function satisfying*

$$\left(\nabla^{\nu}_{\rho(a)} z\right)(t) \leq 0, \quad t \in \mathbb{N}^b_{a+2}, \tag{51}$$

together with the boundary conditions

$$z(a) \geq 0, \tag{52}$$

and

$$z(b) \geq 0. \tag{53}$$

Then, by Theorem 5 with $v \equiv 0$, from (51)–(53), we deduce that

$$z(t) \geq 0, \quad t \in \mathbb{N}^b_a. \tag{54}$$

In the case of $\nu = 2$, the inequality (54) is just a standard concavity result. However, in the fractional case, by applying Theorem 1 for $t \in \mathbb{N}_a$, we have

$$\begin{aligned}
\left(\nabla^{\nu}_{\rho(a)} z\right)(t) &= \sum_{s=a}^{t} H_{-\nu-1}(t,\rho(s))z(s) \\
&= \sum_{s=a}^{t-2} H_{-\nu-1}(t,\rho(s))z(s) \\
&\quad + H_{-\nu-1}(t,\rho(t-1))z(t-1) + H_{-\nu-1}(t,\rho(t))z(t) \\
&= \sum_{s=a}^{t-2} H_{-\nu-1}(t,\rho(s))z(s) - \nu z(t-1) + z(t).
\end{aligned} \tag{55}$$

Given (51)–(53), it does not follow immediately from (55) that the function z is nonnegative for each admissible t. In the case of $\nu \neq 2$ by applying Theorem 5, the property (54) holds.

Example 4. *Let $z : \mathbb{N}_a^b \to \mathbb{R}$ be a function satisfying*

$$\left(\nabla^{\nu}_{\rho(a)} z\right)(t) \geq 0, \quad t \in \mathbb{N}^b_{a+2}, \tag{56}$$

together with the boundary conditions

$$z(a) \leq 0, \tag{57}$$

and

$$z(b) \leq 0. \tag{58}$$

Then, by applying Theorem 5 with $u \equiv 0$ from (56)–(58), we deduce that

$$z(t) \leq 0, \quad t \in \mathbb{N}^b_a. \tag{59}$$

In the case of $\nu = 2$ *the above inequality is just a standard convexity result. However, in the fractional case of applying Theorem 1 for* $t \in \mathbb{N}_a$*, we have (55). Given (56)–(58), it does not follow immediately from (55) that the function z is nonpositive for each admissible t. In the case of* $\nu \neq 2$ *by applying Theorem 5, the property (59) holds.*

6. Conclusions

In this article, I proved a general comparison result in Theorem 5 for the two-point nabla fractional boundary value problem (1). I also obtained a few essential properties of the Green's function associated with the boundary value problem (1) in Section 4.

The results of Section 4 provide an important step in the direction of a full analysis of the boundary value problem (1). Using an appropriate cone fixed point theorem on a suitable cone, and under relevant conditions on the nonlinear part of the difference equation, one can establish sufficient conditions for the existence of multiple positive solutions to the boundary value problem (1). One can also discuss the existence of a unique bounded solution to the problem (1) by using the Banach fixed point theorem.

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