

Article

Coupled Systems of Nonlinear Proportional Fractional Differential Equations of the Hilfer-Type with Multi-Point and Integro-Multi-Strip Boundary Conditions

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Abstract: In this paper, we study a coupled system of nonlinear proportional fractional differential equations of the Hilfer-type with a new kind of multi-point and integro-multi-strip boundary conditions. Results on the existence and uniqueness of the solutions are achieved by using Banach's contraction principle, the Leray–Schauder alternative and the well-known fixed-point theorem of Krasnosel'skiĭ. Finally, the main results are illustrated by constructing numerical examples.

Keywords: coupled system; Hilfer fractional proportional derivative; multi-point and multi-strip; nonlocal boundary conditions; fixed-point theorems

MSC: 26A33; 34A08; 34B15



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1. Introduction

Fractional-order differential equations arise in the mathematical modeling of several engineering and scientific phenomena. Examples include physics, chemistry, robotics, signal and image processing, control theory and viscoelasticity (see the monographs in [1–5]). In particular, nonlinear coupled systems of fractional-order differential equations appear often in investigations connected with anomalous diffusion [6], disease models [7] and ecological models [8]. Unlike the classical derivative operator, one can find a variety of its fractional counterparts, such as the Riemann–Liouville, Caputo, Hadamard, Erdelyi–Kober, Hilfer and Caputo–Hadamard counterparts. Recently, a new class of fractional proportional derivative operators was introduced and discussed in [9–11]. The concept of Hilfer-type generalized proportional fractional derivative operators was proposed in [12]. For the detailed advantages of the Hilfer derivative, see [13] and a recent application in calcium diffusion in [14].

Many researchers studied initial and boundary value problems for differential equations and inclusions, including different kinds of fractional derivative operators (for examples, see [15–20]). In [21], the authors studied a nonlocal initial value problem of an order within $(0, 1)$ involving a $\bar{\psi}_*$ Hilfer generalized proportional fractional derivative of a function with respect to another function. Recently, in [22], the authors investigated the existence and uniqueness of solutions for a nonlocal mixed boundary value problem for Hilfer fractional $\bar{\psi}_*$ -proportional-type differential equations and inclusions of an order within $(1, 2]$. In [23], the authors discussed the existence of solutions for a nonlinear coupled system of (k, ψ) Hilfer fractional differential equations of different orders within

(1, 2], complemented with coupled (k, ψ) Riemann–Liouville fractional integral boundary conditions given by

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*} \pi(z) = \Pi(z, \pi(z)) \text{ or } \in \mathbb{H}(z, \pi(z)), & z \in [a_1, b_1], \quad b_1 > a_1 \geq 0, \\ \pi(a_1) = 0, \\ \pi(b_1) = \sum_{j=1}^m \epsilon_j \pi(\zeta_j) + \sum_{i=1}^n \zeta_i \mathbb{I}_{a_1+}^{\phi_i, \vartheta_*, \bar{\psi}_*} \pi(\theta_i) + \sum_{k=1}^r \lambda_k \mathbb{D}_{a_1+}^{\delta_k, \varphi, \vartheta_*, \bar{\psi}_*} \pi(\mu_k). \end{cases}$$

Here, $\mathbb{D}_{a_1+}^{\omega, \varphi, \vartheta_*, \bar{\psi}_*}$ is the $\bar{\psi}_*$ Hilfer fractional proportional derivative operator of the order $\omega \in \{\rho, \delta_k\}$, $\rho, \delta_k \in (1, 2]$ and type $\varphi \in [0, 1]$, $\vartheta_* \in (0, 1]$, $\epsilon_j, \zeta_i, \lambda_k \in \mathbb{R}$, $H : [a_1, b_1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function (or $\mathbb{H} : [a_1, b_1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multi-valued map), $\mathbb{I}_{a_1+}^{\phi_i, \vartheta_*, \bar{\psi}_*}$ is the fractional integral operator of the order $\phi_i > 0$ and $\zeta_j, \theta_i, \mu_k \in (a_1, b_1)$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots, r$. Very recently, in [24], the authors considered a new boundary value problem consisting of a Hilfer fractional $\bar{\psi}_*$ -proportional differential equation and nonlocal integro-multi-strip and multi-point boundary conditions of the form

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*} \sigma(z) = \Psi(z, \sigma(z)), & z \in [a_1, b_1], \\ \sigma(a_1) = 0, \int_{a_1}^{b_1} \bar{\psi}'_*(s) \sigma(s) ds = \sum_{i=1}^n \varphi_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \sigma(s) ds + \sum_{j=1}^m \theta_j \sigma(\zeta_j), \end{cases}$$

where $\mathbb{D}_{a_1+}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*}$ denotes the $\bar{\psi}_*$ Hilfer fractional proportional derivative operator of the order $\rho \in (1, 2]$ and type $\varphi \in [0, 1]$, $\vartheta_* \in (0, 1]$, $a_1 < \zeta_j < \xi_i < \eta_i < b_1$, $\varphi_i, \theta_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $\bar{\psi}_* : [a_1, b_1] \rightarrow \mathbb{R}$ is an increasing function with $\bar{\psi}'_*(z) \neq 0$ for all $z \in [a_1, b_1]$ and $\Psi : [a_1, b_1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Motivated by the foregoing work on boundary value problems involving Hilfer-type fractional $\bar{\psi}_*$ -proportional derivative operators, in this paper, we aim to establish existence and uniqueness results for a class of coupled systems of nonlinear Hilfer-type fractional $\bar{\psi}_*$ -proportional differential equations equipped with nonlocal multi-point and integro-multi-strip coupled boundary conditions. To be precise, we investigate the following problem:

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho_1, \varphi_1, \vartheta_*, \bar{\psi}_*} \sigma(z) = \Psi_1(z, \sigma(z), \tau(z)), & z \in [a_1, b_1], \\ \mathbb{D}_{a_1+}^{\rho_2, \varphi_2, \vartheta_*, \bar{\psi}_*} \tau(z) = \Psi_2(z, \sigma(z), \tau(z)), & z \in [a_1, b_1], \\ \sigma(a_1) = 0, \int_{a_1}^{b_1} \bar{\psi}'_*(s) \sigma(s) ds = \sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \tau(s) ds + \sum_{j=1}^m \theta_j \tau(\zeta_j), \\ \tau(a_1) = 0, \int_{a_1}^{b_1} \bar{\psi}'_*(s) \tau(s) ds = \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \sigma(s) ds + \sum_{j=1}^m \vartheta_j \sigma(z_j), \end{cases} \tag{1}$$

where $\mathbb{D}_{a_1+}^{\rho_\kappa, \varphi_\kappa, \vartheta_*, \bar{\psi}_*}$, $\kappa = 1, 2$, denote the Hilfer fractional $\bar{\psi}_*$ -proportional derivative operator of the order $\rho_\kappa \in (1, 2]$ and type $\varphi_\kappa \in [0, 1]$, $\vartheta_* \in (0, 1]$, $a_1 < \zeta_j < \xi_i < \eta_i < b_1$, $a_1 < \delta_j < z_i < \epsilon_i < b_1$, $\kappa_i, \theta_j, \phi_i, \vartheta_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $\bar{\psi}_* : [a_1, b_1] \rightarrow \mathbb{R}$ is an increasing function with $\bar{\psi}'_*(z) \neq 0$ for all $z \in [a_1, b_1]$ and $\Psi_1, \Psi_2 : [a_1, b_1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Here we emphasize that system (1) is novel, and its investigation will enhance the scope of the literature on nonlocal Hilfer-type fractional $\bar{\psi}_*$ -proportional boundary value problems. It is worthwhile to mention that the Hilfer fractional $\bar{\psi}_*$ -proportional derivative operators are of a more general nature and reduce to the Hilfer generalized proportional fractional derivative operators [12] when $k = 1$ and $\bar{\psi}_*(t) = t$, which unify the classical Riemann–Liouville and Caputo fractional derivative operators. Our strategy to deal with system (1) is as follows. First of all, we solve a linear variant of system (1) in Lemma 3, which plays a pivotal role in converting the nonlinear problem in system (1) into a fixed-

point problem. Afterward, under certain assumptions, we apply different fixed-point theorems to show that the fixed-point operator related to the problem at hand possesses fixed points. The first result (Theorem 1) shows the existence of a unique solution to system (1) by means of Banach’s contraction mapping principle. In the second result (Theorem 2), the existence of at least one solution to system (1) is established via the Leray–Schauder alternative. The last result (Theorem 3), relying on Krasnosel’skiĭ’s fixed-point theorem, deals with the existence of at least one solution to system (1) under a different hypothesis. We illustrate all the obtained results with the aid of examples in Section 4. In the last section, we describe the scope and utility of the present work by indicating that several new results follow as special cases by fixing the parameters involved in system (1).

The rest of this paper is organized as follows. In the following section, some necessary definitions and preliminary results related to our study are outlined. Section 3 contains the main results for system (1), while numerical examples illustrating these results are presented in Section 4. The paper concludes with some interesting observations.

2. Preliminaries

Let us begin this section with some basic definitions.

Definition 1 ([10,11]). For $\vartheta_* \in (0, 1]$ and $\rho \in \mathbb{R}^+$, the fractional proportional integral of $\hat{h} \in L^1([a_1, b_1], \mathbb{R})$ with respect to $\bar{\psi}_*$ of an order ρ is given by

$$(\mathbb{I}_{a_1+}^{\rho, \vartheta_*, \bar{\psi}_*} \hat{h})(z) = \frac{1}{\vartheta_*^\rho \Gamma(\rho)} \int_{a_1+}^z e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(s))} (\bar{\psi}_*(z) - \bar{\psi}_*(s))^{\rho-1} \bar{\psi}'_*(s) \hat{h}(s) ds, \quad z > a_1. \quad (2)$$

Definition 2 ([10,11]). Let $\bar{\psi}_* \in C([a_1, b_1], \mathbb{R})$ with $\bar{\psi}'_*(z) > 0$, $\vartheta_* \in (0, 1]$ and $\rho \in \mathbb{R}^+$. The fractional proportional derivative for $\hat{h} \in C([a_1, b_1], \mathbb{R})$ with respect to $\bar{\psi}_*$ of an order ρ , is given by

$$(\mathbb{D}_{a_1+}^{\rho, \vartheta_*, \bar{\psi}_*} \hat{h})(z) = \frac{\mathbb{D}^{n, \vartheta_*, \bar{\psi}_*}}{\vartheta_*^{n-\rho} \Gamma(n-\rho)} \int_{a_1+}^z e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(s))} (\bar{\psi}_*(z) - \bar{\psi}_*(s))^{n-\rho-1} \bar{\psi}'_*(s) \hat{h}(s) ds, \quad z > a_1, \quad (3)$$

where $n = [\rho] + 1$ and $[\rho]$ denotes the integer part of the real number ρ .

Definition 3 ([21]). Let $\bar{\psi}_*$ be positive and strictly increasing with $\bar{\psi}'_*(z) \neq 0$ for all $z \in [a_1, b_1]$ and $\hat{h}, \bar{\psi}_* \in C^m([a_1, b_1], \mathbb{R})$. The $\bar{\psi}_*$ Hilfer fractional proportional derivative for \hat{h} with respect to another function $\bar{\psi}_*$ of an order ρ and type φ , is defined by

$$(\mathbb{D}_{a_1+}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*} \hat{h})(z) = (\mathbb{I}_{a_1+}^{\varphi(n-\rho), \vartheta_*, \bar{\psi}_*} (\mathbb{D}^{n, \vartheta_*, \bar{\psi}_*} \mathbb{I}_{a_1+}^{(1-\varphi)(n-\rho), \vartheta_*, \bar{\psi}_*} \hat{h}))(z), \quad (4)$$

where $n - 1 < \rho < n$, $0 \leq \varphi \leq 1$, $n \in \mathbb{N}$ and $\vartheta_* \in (0, 1]$. In addition, $\mathbb{D}^{\vartheta_*, \bar{\psi}_*} \hat{h}(z) = (1 - \vartheta_*) \hat{h}(z) + \vartheta_* \frac{\hat{h}'(z)}{\bar{\psi}'_*(z)}$, and $\mathbb{I}_{a_1+}^{(\cdot)}$ is the fractional proportional integral operator defined in Equation (2).

Now, we recall some known results.

Lemma 1 ([21]). The $\bar{\psi}_*$ Hilfer fractional proportional derivative can be expressed as

$$(\mathbb{D}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*} \hat{h})(z) = (\mathbb{I}_{a_1+}^{\varphi(n-\rho), \vartheta_*, \bar{\psi}_*} (\mathbb{D}^{n, \vartheta_*, \bar{\psi}_*} (\mathbb{I}_{a_1+}^{(1-\varphi)(n-\rho), \vartheta_*, \bar{\psi}_*} \hat{h}))(z) = (\mathbb{I}_{a_1+}^{\varphi(n-\rho), \vartheta_*, \bar{\psi}_*} \mathbb{D}_{a_1+}^{\gamma_1, \vartheta_*, \bar{\psi}_*} \hat{h})(z),$$

where $\gamma_1 = \rho + \varphi(n - \rho)$.

Remark 1 ([21]). The following relations hold:

$$\gamma_1 = \rho + \varphi(n - \rho), \quad n - 1 < \rho, \quad \gamma_1 \leq n, \quad 0 \leq \varphi \leq 1,$$

and

$$\gamma_1 \geq \rho, \quad \gamma_1 > \varphi, \quad n - \gamma_1 < n - \varphi(n - \rho).$$

Lemma 2 ([21]). Let $n - 1 < \rho < n, n \in \mathbb{N}, \vartheta_* \in (0, 1], 0 \leq \varphi \leq 1$ and $\gamma_1 = \rho + \varphi(n - \rho)$ be such that $n - 1 < \gamma_1 < n$. If $\hat{h} \in C([a_1, b_1], \mathbb{R})$ and $\mathbb{I}_{a_1+}^{n-\gamma_1, \vartheta_*, \bar{\psi}_*} f \in C^n([a_1, b_1], \mathbb{R})$, then

$$\mathbb{I}_{a_1+}^{\rho, \vartheta_*, \bar{\psi}_*} \mathbb{D}_{a_1+}^{\rho, \varphi, \vartheta_*, \bar{\psi}_*} \hat{h}(z) = \hat{h}(z) - \sum_{k=1}^n \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-k}}{\vartheta_*^{\gamma_1-k} \Gamma(\gamma_1 - k + 1)} (\mathbb{I}_{a_1+}^{k-\gamma_1, \vartheta_*, \bar{\psi}_*} \hat{h})(a).$$

3. Main Results

Before proceeding for the existence and uniqueness results for the system (1), we consider the following lemma associated with the linear variant of the coupled system of Hilfer-type fractional $\bar{\psi}_*$ -proportional differential equations considered in system (1).

Lemma 3. Let $h_1, h_2 \in C([a_1, b_1], \mathbb{R})$ and $\mathbb{L} \neq 0$. Then, (σ, τ) is a solution to the following coupled, linear, nonlocal integro-multi-strip and multi-point, $\bar{\psi}_*$ Hilfer generalized proportional fractional system:

$$\left\{ \begin{array}{l} \mathbb{D}_{a_1+}^{\rho_1, \varphi_1, \vartheta_*, \bar{\psi}_*} \sigma(z) = h_1(z), \quad z \in [a_1, b_1], \\ \mathbb{D}_{a_1+}^{\rho_2, \varphi_2, \vartheta_*, \bar{\psi}_*} \tau(z) = h_2(z), \quad z \in [a_1, b_1], \\ \sigma(a_1) = 0, \quad \int_{a_1}^{b_1} \bar{\psi}'_*(s) \sigma(s) ds = \sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \tau(s) ds + \sum_{j=1}^m \theta_j \tau(\zeta_j), \\ \tau(a_1) = 0, \quad \int_{a_1}^{b_1} \bar{\psi}'_*(s) \tau(s) ds = \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \sigma(s) ds + \sum_{j=1}^m \vartheta_j \sigma(z_j), \end{array} \right. \quad (5)$$

if and only if

$$\begin{aligned} \sigma(z) = & \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left\{ \mathbb{A}_2 \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s) ds \right. \right. \\ & + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(\zeta_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s) ds \Big) + (\mathbb{B}_1 + \mathbb{C}_1) \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_2(s) ds \right. \\ & \left. \left. + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s) ds \right) \right\}, \quad z \in [a_1, b_1], \end{aligned} \quad (6)$$

and

$$\begin{aligned} \tau(z) = & \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left\{ \mathbb{A}_1 \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s) ds \right. \right. \\ & + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s) ds \Big) + (\mathbb{B}_2 + \mathbb{C}_2) \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s) ds \right. \\ & \left. \left. + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(\zeta_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s) ds \right) \right\}, \quad z \in [a_1, b_1], \end{aligned} \quad (7)$$

where

$$\begin{aligned} \mathbb{A}_1 &= \int_{a_1}^{b_1} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_1-1} \bar{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} ds, \\ \mathbb{B}_1 &= \sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_2-1} \bar{\psi}'_*(s)}{\vartheta_*^{\gamma_2-1} \Gamma(\gamma_2)} ds, \end{aligned}$$

$$\begin{aligned}
 \mathbb{C}_1 &= \sum_{j=1}^m \theta_j \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z_j)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z_j)-\bar{\psi}_*(a_1))^{\gamma_2-1}}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)}, \\
 \mathbb{A}_2 &= \int_{a_1}^{b_1} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))^{\gamma_2-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)} ds, \\
 \mathbb{B}_2 &= \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))^{\gamma_1-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} ds, \\
 \mathbb{C}_2 &= \sum_{j=1}^m \theta_j \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(\zeta_j)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(\zeta_j)-\bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}
 \end{aligned} \tag{8}$$

and

$$\mathbb{L} = \mathbb{A}_1\mathbb{A}_2 - (\mathbb{B}_1 + \mathbb{C}_1)(\mathbb{B}_2 + \mathbb{C}_2).$$

Proof. From Lemma 2 with $n = 2$, we have

$$\begin{aligned}
 \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \mathbb{D}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \sigma(z) &= \sigma(z) - \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} (\mathbb{I}_{a_1+}^{1-\gamma_1, \vartheta_*, \bar{\psi}_*} \sigma)(a_1) \\
 &\quad - \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-2}}{\vartheta_*^{\gamma_1-2}\Gamma(\gamma_1-1)} (\mathbb{I}_{a_1+}^{2-\gamma_1, \vartheta_*, \bar{\psi}_*} \sigma)(a_1),
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \mathbb{D}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \tau(z) &= \tau(z) - \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} (\mathbb{I}_{a_1+}^{1-\gamma_2, \vartheta_*, \bar{\psi}_*} \tau)(a_1) \\
 &\quad - \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_2-2}}{\vartheta_*^{\gamma_2-2}\Gamma(\gamma_2-1)} (\mathbb{I}_{a_1+}^{2-\gamma_2, \vartheta_*, \bar{\psi}_*} \tau)(a_1),
 \end{aligned}$$

which yields

$$\begin{aligned}
 \sigma(z) &= \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z) + c_0 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
 &\quad + c_1 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-2}}{\vartheta_*^{\gamma_1-2}\Gamma(\gamma_1-1)},
 \end{aligned} \tag{9}$$

and

$$\begin{aligned}
 \tau(z) &= \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(z) + d_0 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_2-1}}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)} \\
 &\quad + d_1 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_2-2}}{\vartheta_*^{\gamma_2-2}\Gamma(\gamma_2-1)},
 \end{aligned} \tag{10}$$

where $c_0 = (\mathbb{I}_{a_1+}^{1-\gamma_1, \vartheta_*, \bar{\psi}_*} \sigma)(a_1)$, $c_1 = (\mathbb{I}_{a_1+}^{2-\gamma_1, \vartheta_*, \bar{\psi}_*} \sigma)(a_1)$, $d_0 = (\mathbb{I}_{a_1+}^{1-\gamma_2, \vartheta_*, \bar{\psi}_*} \tau)(a_1)$ and $d_1 = (\mathbb{I}_{a_1+}^{2-\gamma_2, \vartheta_*, \bar{\psi}_*} \tau)(a_1)$. Using Equations (9) and (10) in the conditions $\sigma(a_1) = 0$ and $\tau(a_1) = 0$, we obtain $c_1 = 0$ and $d_1 = 0$, since $\gamma_1 \in [\rho_1, 2]$ and $\gamma_2 \in [\rho_2, 2]$. Hence, Equations (9) and (10) take the forms

$$\sigma(z) = \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z) + c_0 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}, \tag{11}$$

and

$$\tau(z) = \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(z) + d_0 \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_2-1}}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)}. \tag{12}$$

By inserting Equations (11) and (12) into the conditions $\int_{a_1}^{b_1} \bar{\psi}'_*(s)\sigma(s)ds = \sum_{i=1}^n \kappa_i \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s)\tau(s)ds + \sum_{j=1}^m \theta_j \tau(\zeta_j)$ and $\int_{a_1}^{b_1} \bar{\psi}'_*(s)\tau(s)ds = \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s)\sigma(s)ds + \sum_{j=1}^m \vartheta_j \sigma(z_j)$, we obtain

$$\begin{aligned} & \int_{a_1}^{b_1} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s)ds + c_0 \int_{a_1}^{b_1} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_1-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} ds \\ = & \sum_{i=1}^n \kappa_i \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s)ds + d_0 \sum_{i=1}^n \kappa_i \int_{\xi_i}^{\eta_i} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_2-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)} ds \\ & + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(\zeta_j) + d_0 \sum_{j=1}^m \theta_j \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(\zeta_j)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\gamma_2-1}}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} & \int_{a_1}^{b_1} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s)ds + d_0 \int_{a_1}^{b_1} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_2-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_2-1}\Gamma(\gamma_2)} ds \\ = & \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s)ds + c_0 \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(s)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(s) - \bar{\psi}_*(a_1))^{\gamma_1-1}\bar{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} ds \\ & + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z_j) + c_0 \sum_{j=1}^m \vartheta_j \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z_j)-\bar{\psi}_*(a_1))}(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}. \end{aligned} \tag{14}$$

In light of the notation (8), we can express Equations (13) and (14) in the form of the following system:

$$\begin{aligned} \mathbb{A}_1 c_0 - (\mathbb{B}_1 + \mathbb{C}_1) d_0 &= P, \\ -(\mathbb{B}_2 + \mathbb{C}_2) c_0 + \mathbb{A}_2 d_0 &= Q, \end{aligned} \tag{15}$$

where

$$\begin{aligned} P &= \sum_{i=1}^n \kappa_i \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s)ds + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(\zeta_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s)ds, \\ Q &= \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(s)ds + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} h_1(z_j) - \int_{a_1}^{b_1} \bar{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} h_2(s)ds. \end{aligned}$$

By solving the system (15) for c_0 and d_0 , we find that

$$c_0 = \frac{1}{\mathbb{L}} [\mathbb{A}_2 P + (\mathbb{B}_1 + \mathbb{C}_1) Q], \quad d_0 = \frac{1}{\mathbb{L}} [\mathbb{A}_1 Q + (\mathbb{B}_2 + \mathbb{C}_2) P].$$

Substituting the above values of c_0 and d_0 in Equations (11) and (12) leads to the solutions in Equations (6) and (7), respectively. The converse of the lemma can be established by direct computation. \square

We denote the Banach space of all continuous functions from $[a_1, b_1]$ to \mathbb{R} endowed with the norm $\|\sigma\| := \max_{z \in [a_1, b_1]} |\sigma(z)|$ as $\mathbb{X} = C([a_1, b_1], \mathbb{R})$. Obviously, the space $\mathbb{X} \times \mathbb{X}$ endowed with the norm $\|(\sigma, \tau)\| = \|\sigma\| + \|\tau\|$ is a Banach space.

In light of Lemma 3, we define an operator $\mathbb{K} : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$ as

$$\mathbb{K}(\sigma, \tau)(z) = \begin{pmatrix} \mathbb{K}_1(\sigma, \tau)(z) \\ \mathbb{K}_2(\sigma, \tau)(z) \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned} \mathbb{K}_1(\sigma, \tau)(z) = & \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z, \sigma(z), \tau(z)) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z) - \bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\ & \times \left\{ \mathbb{A}_2 \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right. \right. \\ & + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \Big) \\ & + (\mathbb{B}_1 + \mathbb{C}_1) \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right. \\ & \left. \left. + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z_j, \sigma(z_j), \tau(z_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right) \right\}, \quad z \in [a_1, b_1], \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mathbb{K}_2(\sigma, \tau)(z) = & \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(z, \sigma(z), \tau(z)) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z) - \bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\ & \times \left\{ \mathbb{A}_1 \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right. \right. \\ & + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z_j, \sigma(z_j), \tau(z_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \Big) \\ & + (\mathbb{B}_2 + \mathbb{C}_2) \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right. \\ & \left. \left. + \sum_{j=1}^m \vartheta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right) \right\}, \quad z \in [a_1, b_1]. \end{aligned} \tag{18}$$

For convenience, in the sequel, the following notations are used:

$$\begin{aligned} \mathbb{Q}_1 = & \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left[|\mathbb{A}_2| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1+1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\ & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\ & \left. \left. + \sum_{j=1}^m |\vartheta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right) \right], \\ \mathbb{Q}_2 = & \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left[|\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2+1} - (\bar{\psi}_*(\zeta_i) - \bar{\psi}_*(a_1))^{\rho_2+1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} + |\mathbb{B}_1 + \mathbb{C}_1| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)}, \tag{19} \\
 \mathbb{Q}_3 & = \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left[(|\mathbb{A}_1| \sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\
 & \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + |\mathbb{B}_2 + \mathbb{C}_2| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right], \\
 \mathbb{Q}_4 & = \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left[|\mathbb{A}_1| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right. \\
 & \left. + |\mathbb{B}_2 + \mathbb{C}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2+1} - (\bar{\psi}_*(\zeta_i) - \bar{\psi}_*(a_1))^{\rho_2+1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right) \right. \\
 & \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \right)].
 \end{aligned}$$

Existence of a Unique Solution

In what follows, we prove the uniqueness of the solutions to the system (1) by applying Banach’s contraction mapping principle [25].

Theorem 1. Assume that $\Psi_1, \Psi_2 : [a_1, b_1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy the following conditions:

(G₁) There exist constants $\mu_i, \nu_i, i = 1, 2$ such that for all $z \in [a_1, b_1]$ and $o_i, \bar{o}_i \in \mathbb{R}, i = 1, 2$, we have

$$|\Psi_1(z, o_1, o_2) - \Psi_1(z, \bar{o}_1, \bar{o}_2)| \leq \mu_1 |o_1 - \bar{o}_1| + \mu_2 |o_2 - \bar{o}_2|$$

and

$$|\Psi_2(z, o_1, o_2) - \Psi_2(z, \bar{o}_1, \bar{o}_2)| \leq \nu_1 |o_1 - \bar{o}_1| + \nu_2 |o_2 - \bar{o}_2|.$$

In addition, we suppose that

$$(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(\nu_1 + \nu_2) < 1, \tag{20}$$

where $\mathbb{Q}_i, i = 1, 2, 3, 4$, are given in Equation (19). Then, the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1) has a unique solution on $[a_1, b_1]$.

Proof. We define $\sup_{\theta \in [a_1, b_1]} \Psi_1(z, 0, 0) = N_1 < \infty$ and $\sup_{\theta \in [a_1, b_1]} \Psi_2(z, 0, 0) = N_2 < \infty$ and consider the set $\mathbb{B}_r = \{(\sigma, \tau) \in \mathbb{X} \times \mathbb{X} : \|(\sigma, \tau)\| \leq r\}$ with

$$r \geq \frac{(\mathbb{Q}_1 + \mathbb{Q}_3)N_1 + (\mathbb{Q}_2 + \mathbb{Q}_4)N_2}{1 - [(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(\nu_1 + \nu_2)]}. \tag{21}$$

In the first step, it will be shown that $\mathbb{K}\mathbb{B}_r \subset \mathbb{B}_r$, where the operator \mathbb{K} is given by Equation (16).

For $(\sigma, \tau) \in \mathbb{B}_r$, and using $0 < e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(\cdot) - \bar{\psi}_*(\cdot))} \leq 1$, we have

$$\begin{aligned}
 & |\mathbb{K}_1(\sigma, \tau)(z)| \\
 \leq & \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [\Psi_1(s, \sigma(s), \tau(s)) - \Psi_1(s, 0, 0)] + |\Psi_1(s, 0, 0)| \\
 & + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z) - \bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [\Psi_2(s, \sigma(s), \tau(s)) - \Psi_2(s, 0, 0)] + |\Psi_2(s, 0, 0)| \right) ds \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m |\theta_j| \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [|\Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j)) - \Psi_2(\zeta_j, 0, 0)| + |\Psi_2(\zeta_j, 0, 0)|] \\
 & + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [|\Psi_1(s, \sigma(s), \tau(s)) - \Psi_1(s, 0, 0)| + |\Psi_1(s, 0, 0)|] ds \\
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [|\Psi_1(s, \sigma(s), \tau(s)) - \Psi_1(s, 0, 0)| + |\Psi_1(s, 0, 0)|] ds \right. \\
 & + \sum_{j=1}^m \|\theta_j\| \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [|\Psi_1(z_j, \sigma(z_j), \tau(z_j)) - \Psi_1(z_j, 0, 0)| + |\Psi_1(z_j, 0, 0)|] \\
 & \left. + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [|\Psi_2(s, \sigma(s), \tau(s)) - \Psi_2(s, 0, 0)| + |\Psi_2(s, 0, 0)|] ds \right) \Big\} \\
 \leq & \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] (b_1) + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] ds \right. \right. \\
 & + \sum_{j=1}^m |\theta_j| \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2](\zeta_j) \\
 & + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] ds \\
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] ds \right. \\
 & + \sum_{j=1}^m \|\theta_j\| \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1](z_j) \\
 & \left. + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] ds \right) \\
 \leq & \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1} - (\bar{\psi}_*(\zeta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right. \right. \\
 & \times [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] \\
 & + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1 + 1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] \Big) \\
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1 + 1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1 + 1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\
 & \times [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] + \sum_{j=1}^m \|\theta_j\| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] \\
 & \left. + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2 + 1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] \right) \Big\} \\
 = & \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1} \Gamma(\gamma_1)} \left(|\mathbb{A}_2| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1 + 1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left\{ \sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\
 & + \left. \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right\} [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] \\
 & + \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left(|\mathbb{A}_2| \sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2+1} - (\bar{\psi}_*(\xi_i) - \bar{\psi}_*(a_1))^{\rho_2+1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right. \right. \\
 & + \left. \left. \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} + |\mathbb{B}_1 + \mathbb{C}_1| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right) \right\} [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] \\
 & = \mathbb{Q}_1 [\mu_1 \|\sigma\| + \mu_2 \|\tau\| + N_1] + \mathbb{Q}_2 [v_1 \|\sigma\| + v_2 \|\tau\| + N_2] \\
 & = (\mathbb{Q}_1 \mu_1 + \mathbb{Q}_2 v_1) \|\sigma\| + (\mathbb{Q}_1 \mu_2 + \mathbb{Q}_2 v_2) \|\tau\| + \mathbb{Q}_1 N_1 + \mathbb{Q}_2 N_2 \\
 & \leq (\mathbb{Q}_1 \mu_1 + \mathbb{Q}_2 v_1 + \mathbb{Q}_1 \mu_2 + \mathbb{Q}_2 v_2) r + \mathbb{Q}_1 N_1 + \mathbb{Q}_2 N_2.
 \end{aligned}$$

In a similar manner, we can obtain

$$|\mathbb{K}_2(\sigma, \tau)(z)| \leq (\mathbb{Q}_3 \mu_1 + \mathbb{Q}_4 v_1 + \mathbb{Q}_3 \mu_2 + \mathbb{Q}_4 v_2) r + \mathbb{Q}_3 N_1 + \mathbb{Q}_4 N_2.$$

In light of the foregoing inequalities, we have

$$\begin{aligned}
 \|\mathbb{K}(\sigma, \tau)\| & = \|\mathbb{K}_1(\sigma, \tau)\| + \|\mathbb{K}_2(\sigma, \tau)\| \\
 & \leq [(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(v_1 + v_2)] r \\
 & \quad + (\mathbb{Q}_1 + \mathbb{Q}_3) N_1 + (\mathbb{Q}_2 + \mathbb{Q}_4) N_2 \leq r,
 \end{aligned}$$

which implies that $\mathbb{K}B_r \subset B_r$.

Now, for $(\sigma_2, \tau_2), (\sigma_1, \tau_1) \in \mathbb{X} \times \mathbb{X}$, and for any $z \in [a_1, b_1]$, we obtain

$$\begin{aligned}
 & |\mathbb{K}_1(\sigma_2, \tau_2)(z) - \mathbb{K}_1(\sigma_1, \tau_1)(z)| \\
 \leq & \quad k_{\mathbb{I}^{\bar{\alpha}; \varphi}} |\Psi_1(z, \sigma_2(z), \tau_2(z)) - \Psi_1(z, \sigma_1(z), \tau_1(z))| \\
 & + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z) - \bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} |\Psi_2(z, \sigma_2(z), \tau_2(z)) - \Psi_2(z, \sigma_1(z), \tau_1(z))| ds \right. \right. \\
 & + \sum_{j=1}^m |\theta_j| \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} |\Psi_2(\zeta_j, \sigma_2(\zeta_j), \tau_2(\zeta_j)) - \Psi_2(\zeta_j, \sigma_1(\zeta_j), \tau_1(\zeta_j))| \\
 & + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} |\Psi_1(z, \sigma_2(z), \tau_2(z)) - \Psi_1(z, \sigma_1(z), \tau_1(z))| ds \\
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} |\Psi_1(z, \sigma_2(z), \tau_2(z)) - \Psi_1(z, \sigma_1(z), \tau_1(z))| ds \right. \\
 & + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} |\Psi_1(z_j, \sigma_2(z_j), \tau_2(z_j)) - \Psi_1(z_j, \sigma_1(z_j), \tau_1(z_j))| \\
 & \left. \left. + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} |\Psi_2(z, \sigma_2(z), \tau_2(z)) - \Psi_2(z, \sigma_1(z), \tau_1(z))| ds \right) \right\} \\
 \leq & \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left[|\mathbb{A}_2| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1+1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \right. \\
 & \left. \left. + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m |\vartheta_j| \left(\frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right) \Big] \Big\} (\mu_1 \|\sigma_2 - \sigma_1\| + \mu_2 \|\tau_2 - \tau_1\|) \\
 & + \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1} \Gamma(\gamma_1)} \left[|\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1} - (\bar{\psi}_*(\zeta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right) \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} + |\mathbb{B}_1 + \mathbb{C}_1| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2 + 1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right] \right\} (\nu_1 \|\sigma_2 - \sigma_1\| + \nu_2 \|\tau_2 - \tau_1\|) \\
 = & \mathbb{Q}_1 (\mu_1 \|\sigma_2 - \sigma_1\| + \mu_2 \|\tau_2 - \tau_1\|) + \mathbb{Q}_2 (\nu_1 \|\sigma_2 - \sigma_1\| + \nu_2 \|\tau_2 - \tau_1\|) \\
 = & (\mathbb{Q}_1 \mu_1 + \mathbb{Q}_2 \nu_1) \|\sigma_2 - \sigma_1\| + (\mathbb{Q}_1 \mu_2 + \mathbb{Q}_2 \nu_2) \|\tau_2 - \tau_1\|,
 \end{aligned}$$

Consequently, we obtain

$$\|\mathbb{K}_1(\sigma_2, \tau_2) - \mathbb{K}_1(\sigma_1, \tau_1)\| \leq (\mathbb{Q}_1 \mu_1 + \mathbb{Q}_2 \nu_1 + \mathbb{Q}_1 \mu_2 + \mathbb{Q}_2 \nu_2) [\|\sigma_2 - \sigma_1\| + \|\tau_2 - \tau_1\|]. \tag{22}$$

Similarly, it can be established that

$$\|\mathbb{K}_2(\sigma_2, \tau_2) - \mathbb{K}_2(\sigma_1, \tau_1)\| \leq (\mathbb{Q}_3 \mu_1 + \mathbb{Q}_4 \nu_1 + \mathbb{Q}_3 \mu_2 + \mathbb{Q}_4 \nu_2) [\|\sigma_2 - \sigma_1\| + \|\tau_2 - \tau_1\|]. \tag{23}$$

It follows from Equations (22) and (23) that

$$\|\mathbb{K}(\sigma_2, \tau_2) - \mathbb{K}(\sigma_1, \tau_1)\| \leq [(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(\nu_1 + \nu_2)] (\|\sigma_2 - \sigma_1\| + \|\tau_2 - \tau_1\|).$$

Since $(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(\nu_1 + \nu_2) < 1$ under the condition in Equation (20), the operator \mathbb{K} is a contraction. Therefore, the conclusion of Banach’s contraction mapping principle applies, and hence the operator \mathbb{K} has a unique fixed point. As a consequence, there exists a unique solution to the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1). \square

The following result is based on the Leray–Schauder alternative [26]:

Theorem 2. Let $\Psi_1, \Psi_2 : [a_1, b_1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous functions such that the following condition holds:

(G₂) There exist $\pi_i, \omega_i \geq 0$ for $i = 1, 2$ and $\pi_0, \omega_0 > 0$ such that for any $\sigma, \tau \in \mathbb{R}$, we have

$$\begin{aligned}
 |\Psi_1(z, \sigma, \tau)| & \leq \pi_0 + \pi_1 |\sigma| + \pi_2 |\tau|, \\
 |\Psi_2(z, \sigma, \tau)| & \leq \omega_0 + \omega_1 |\sigma| + \omega_2 |\tau|.
 \end{aligned}$$

If $(\mathbb{Q}_1 + \mathbb{Q}_3)\pi_1 + (\mathbb{Q}_2 + \mathbb{Q}_4)\omega_1 < 1$ and $(\mathbb{Q}_1 + \mathbb{Q}_3)\pi_2 + (\mathbb{Q}_2 + \mathbb{Q}_4)\omega_2 < 1$, where $\mathbb{Q}_i, i = 1, 2, 3, 4$ are given in Equation (19), then the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1) has at least one solution on $[a_1, b_1]$.

Proof. Observe that the operator \mathbb{K} defined in Equation (16) is continuous, owing to the continuity of functions Ψ_1 and Ψ_2 on $[a_1, b_1] \times \mathbb{R}^2$. Next, we show that the operator \mathbb{K} is complete continuous. We define $\mathbb{B}_\varepsilon = \{(\sigma, \tau) \in \mathbb{X} \times \mathbb{Y} : \|(\sigma, \tau)\| \leq \varepsilon\}$. Then, for all $(\sigma, \tau) \in \mathbb{B}_\varepsilon$, there exist $\mathbb{D}_1, \mathbb{D}_2 > 0$ such that $|\Psi_1(z, \sigma(z), \tau(z))| \leq \mathbb{D}_1$ and $|\Psi_2(z, \sigma(z), \tau(z))| \leq \mathbb{D}_2$. Therefore, for all $(\sigma, \tau) \in \mathbb{B}_\varepsilon$, we have

$$\begin{aligned}
 & |\mathbb{K}_1(\sigma, \tau)(z)| \\
 \leq & \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \mathbb{D}_1 + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1} - (\bar{\psi}_*(\zeta_i) - \bar{\psi}_*(a_1))^{\rho_2 + 1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right) \mathbb{D}_2 \right. \\
 & \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \mathbb{D}_2 + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1 + 1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \mathbb{D}_1 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \\
 & \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \mathbb{D}_1 + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \mathbb{D}_2 \right) \Big\} \\
 = & \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left(|\mathbb{A}_2| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1+1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \right. \\
 & \left. \left. + |\mathbb{B}_1 + \mathbb{C}_1| \sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right) \Big\} \mathbb{D}_1 \\
 & + \left\{ \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \left(|\mathbb{A}_2| \sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2+1} - (\bar{\psi}_*(\xi_i) - \bar{\psi}_*(a_1))^{\rho_2+1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} + |\mathbb{B}_1 + \mathbb{C}_1| \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \right) \Big\} \mathbb{D}_2 \\
 = & \mathbb{Q}_1 \mathbb{D}_1 + \mathbb{Q}_2 \mathbb{D}_2,
 \end{aligned}$$

which implies that

$$\|\mathbb{K}_1(\sigma, \tau)\| \leq \mathbb{Q}_1 \mathbb{D}_1 + \mathbb{Q}_2 \mathbb{D}_2.$$

Similarly, we can obtain

$$\|\mathbb{K}_2(\sigma, \tau)\| \leq \mathbb{Q}_3 \mathbb{D}_1 + \mathbb{Q}_4 \mathbb{D}_2.$$

Consequently, we have

$$\|\mathbb{K}(\sigma, \tau)\| \leq (\mathbb{Q}_1 + \mathbb{Q}_3) \mathbb{D}_1 + (\mathbb{Q}_2 + \mathbb{Q}_4) \mathbb{D}_2,$$

Thus, we deduce that the operator \mathbb{K} is uniformly bounded.

Now, we establish that the operator \mathbb{K} is equicontinuous. Let $z_1, z_2 \in [a_1, b_1]$ with $z_1 < z_2$. Then, we have

$$\begin{aligned}
 & |\mathbb{K}_1(\sigma, \tau)(z_2) - \mathbb{K}_1(\sigma, \tau)(z_1)| \\
 \leq & \frac{1}{\vartheta_*^{\rho_1} \Gamma(\rho_1)} \left| \int_{a_1+}^{t_1} [(\bar{\psi}_*(z_2) - \bar{\psi}_*(s))^{\rho_1-1} - (\bar{\psi}_*(z_1) - \bar{\psi}_*(s))^{\rho_1-1}] \bar{\psi}'_*(s) \Psi_1(s, \sigma(s), \tau(s)) ds \right| \\
 & + \frac{1}{\vartheta_*^{\rho_1} \Gamma(\rho)} \left| \int_{t_1}^{t_2} (\bar{\psi}_*(z_2) - \bar{\psi}_*(s))^{\rho_1-1} \bar{\psi}'_*(s) \Psi_1(s, \sigma(s), \tau(s)) ds \right| \\
 & + \frac{(\bar{\psi}_*(z_2) - \bar{\psi}_*(a_1))^{\gamma_1-1} - (\bar{\psi}_*(z_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \int_{\xi_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*} \bar{\psi}_* |\Psi_2(s, \sigma(s), \tau(s))| ds \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*} \bar{\psi}_* |\Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j))| + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*} \bar{\psi}_* |\Psi_1(s, \sigma(s), \tau(s))| ds \right) \right. \\
 & \left. + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*} \bar{\psi}_* |\Psi_1(s, \sigma(s), \tau(s))| ds \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 & \left. + \sum_{j=1}^m |\theta_j| \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} |\Psi_1(z_j, \sigma(z_j), \tau(z_j))| + \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} |\Psi_2(s, \sigma(s), \tau(s))| ds \right\} \\
 \leq & \frac{\mathbb{D}_1}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \left[2(\bar{\psi}_*(z_2) - \bar{\psi}_*(z_1))^{\rho_1} + |(\bar{\psi}_*(z_2) - \bar{\psi}_*(a_1))^{\rho_1} - (\bar{\psi}_*(z_1) - \bar{\psi}_*(a_1))^{\rho_1}| \right] \\
 & + \frac{(\bar{\psi}_*(z_2) - \bar{\psi}_*(a_1))^{\gamma_1-1} - (\bar{\psi}_*(z_1) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}| \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\
 & \times \left\{ |\mathbb{A}_2| \left(\sum_{i=1}^n |\kappa_i| \frac{[(\bar{\psi}_*(\eta_i) - \bar{\psi}_*(a_1))^{\rho_2+1} - (\bar{\psi}_*(\xi_i) - \bar{\psi}_*(a_1))^{\rho_2+1}]}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \mathbb{D}_2 \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(\zeta_j) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \mathbb{D}_2 + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1+1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \mathbb{D}_1 \right) \right. \\
 & \left. + |\mathbb{B}_1 + \mathbb{C}_1| \left(\sum_{i=1}^n |\phi_i| \frac{[(\bar{\psi}_*(\epsilon_i) - \bar{\psi}_*(a_1))^{\rho_1+1} - (\bar{\psi}_*(\delta_i) - \bar{\psi}_*(a_1))^{\rho_1+1}]}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 2)} \mathbb{D}_1 \right. \right. \\
 & \left. \left. + \sum_{j=1}^m |\theta_j| \frac{(\bar{\psi}_*(z_j) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \mathbb{D}_1 + \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2+1}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 2)} \mathbb{D}_2 \right) \right\},
 \end{aligned}$$

which implies that $|\mathbb{K}_1(\sigma, \tau)(z_2) - \mathbb{K}_1(\sigma, \tau)(z_1)| \rightarrow 0$ as $z_1 \rightarrow z_2$ independent of $(\sigma, \tau) \in \mathbb{B}_\epsilon$. Thus, the operator $\mathbb{K}_1 : \mathbb{X} \rightarrow \mathbb{X}$ is completely continuous under the Arzelá–Ascoli theorem. Similarly, it can be shown that

$$|\mathbb{K}_2(\sigma, \tau)(z_2) - \mathbb{K}_2(\sigma, \tau)(z_1)| \rightarrow 0,$$

as $z_1 \rightarrow z_2$ independent of $(\sigma, \tau) \in \mathbb{B}_\epsilon$. Hence, the operator \mathbb{K} is completely continuous.

Lastly, we verify that the set $\mathbb{E} = \{(\sigma, \tau) \in \mathbb{X} \times \mathbb{X} : (\sigma, \tau) = \lambda \mathbb{K}(\sigma, \tau), 0 \leq \lambda \leq 1\}$ is bounded. Let $(\sigma, \tau) \in \mathbb{E}$. Then, $(\sigma, \tau) = \lambda \mathbb{K}(\sigma, \tau)$. Hence, for all $z \in [a_1, b_1]$, we have

$$\sigma(z) = \lambda \mathbb{K}_1(\sigma, \tau)(z), \quad \tau(z) = \lambda \mathbb{K}_2(\sigma, \tau)(z).$$

Under assumption (G_2) , we have

$$\begin{aligned}
 \|\sigma\| & \leq (\pi_0 + \pi_1 \|\sigma\| + \pi_2 \|\tau\|) \mathbb{Q}_1 + (\omega_0 + \omega_1 \|\sigma\| + \omega_2 \|\tau\|) \mathbb{Q}_2, \\
 \|\tau\| & \leq (\pi_0 + \pi_1 \|\sigma\| + \pi_2 \|\tau\|) \mathbb{Q}_3 + (\omega_0 + \omega_1 \|\sigma\| + \omega_2 \|\tau\|) \mathbb{Q}_4,
 \end{aligned}$$

which imply that

$$\begin{aligned}
 \|\sigma\| + \|\tau\| & \leq (\mathbb{Q}_1 + \mathbb{Q}_3) \pi_0 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_0 + [(\mathbb{Q}_1 + \mathbb{Q}_3) \pi_1 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_1] \|\sigma\| \\
 & \quad + [(\mathbb{Q}_1 + \mathbb{Q}_3) \pi_2 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_2] \|\tau\|.
 \end{aligned}$$

Consequently, we have

$$\|(\sigma, \tau)\| \leq \frac{(\mathbb{Q}_1 + \mathbb{Q}_3) \pi_0 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_0}{D^*}, \tag{24}$$

where $D^* = \min\{1 - [(\mathbb{Q}_1 + \mathbb{Q}_3) \pi_1 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_1], 1 - [(\mathbb{Q}_1 + \mathbb{Q}_3) \pi_2 + (\mathbb{Q}_2 + \mathbb{Q}_4) \omega_2]\}$. Hence, the set \mathbb{E} is bounded. Under the Leray–Schauder alternative, the operator \mathbb{K} has at least one fixed point. Therefore, the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1) has at least one solution on $[a_1, b_1]$. \square

Our second existence result is based on Krasnosel’skiĭ’s fixed-point theorem [27]:

Theorem 3. Let $\Psi_1, \Psi_2 : [a_1, b_1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous functions satisfying condition (G_1) . In addition, the following assumption holds:

(G_3) There exist non-negative functions $\Phi_1, \Phi_2 \in C([a_1, b_1], \mathbb{R}^+)$ such that $|h_1(z, r, s)| \leq \Phi_1(z), |h_2(z, r, s)| \leq \Phi_2(z)$ for all $(z, r, s) \in [a_1, b_1] \times \mathbb{R} \times \mathbb{R}$.

Then, the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1) has at least one solution on $[a_1, b_1]$, provided that

$$(\mu_1 + \mu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + (\nu_1 + \nu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} < 1. \tag{25}$$

Proof. In order to verify the hypothesis of Krasnosel’skii’s fixed-point theorem [27], we decompose the operator \mathbb{K} as follows:

$$\mathbb{K}_{1,1}(\sigma, \tau)(z) = \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z, \sigma(z), \tau(z)), \quad z \in [a_1, b_1], \tag{26}$$

$$\begin{aligned} \mathbb{K}_{1,2}(\sigma, \tau)(z) &= \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\ &\times \left\{ \mathbb{A}_2 \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right. \right. \\ &+ \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \\ &+ (\mathbb{B}_1 + \mathbb{C}_1) \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right. \\ &\left. \left. + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z_j, \sigma(z_j), \tau(z_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right) \right\}, \quad z \in [a_1, b_1], \tag{27} \end{aligned}$$

$$\mathbb{K}_{2,1}(\sigma, \tau)(z) = \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(z, \sigma(z), \tau(z)), \quad z \in [a_1, b_1], \tag{28}$$

$$\begin{aligned} \mathbb{K}_{2,2}(\sigma, \tau)(z) &= \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\bar{\psi}_*(z)-\bar{\psi}_*(a_1))} (\bar{\psi}_*(z) - \bar{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L} \vartheta_*^{\gamma_1-1} \Gamma(\gamma_1)} \\ &\times \left\{ \mathbb{A}_1 \left(\sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right. \right. \\ &+ \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(z_j, \sigma(z_j), \tau(z_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \\ &+ (\mathbb{B}_2 + \mathbb{C}_2) \left(\sum_{i=1}^n \kappa_i \int_{\zeta_i}^{\eta_i} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(s, \sigma(s), \tau(s)) ds \right. \\ &\left. \left. + \sum_{j=1}^m \theta_j \mathbb{I}_{a_1+}^{\rho_2, \vartheta_*, \bar{\psi}_*} \Psi_2(\zeta_j, \sigma(\zeta_j), \tau(\zeta_j)) - \int_{a_1}^{b_1} \bar{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho_1, \vartheta_*, \bar{\psi}_*} \Psi_1(s, \sigma(s), \tau(s)) ds \right) \right\}, \quad z \in [a_1, b_1]. \tag{29} \end{aligned}$$

Let us set $\mathbb{K}_1(\sigma, \tau)(z) = \mathbb{K}_{1,1}(\sigma, \tau)(z) + \mathbb{K}_{1,2}(\sigma, \tau)(z)$ and $\mathbb{K}_2(\sigma, \tau)(z) = \mathbb{K}_{2,1}(\sigma, \tau)(z) + \mathbb{K}_{2,2}(\sigma, \tau)(z)$ and introduce the set $\mathbb{B}_\theta = \{(\sigma, \tau) \in \mathbb{X} \times \mathbb{X}; \|(\sigma, \tau)\| \leq \theta\}$, with

$$\theta \geq (\mathbb{Q}_1 + \mathbb{Q}_3) \|\Phi_1\| + (\mathbb{Q}_2 + \mathbb{Q}_4) \|\Phi_2\|.$$

As in the proof of Theorem 2, we can obtain that

$$\begin{aligned} |\mathbb{K}_{1,1}(\sigma, \tau)(z) + \mathbb{K}_{1,2}(\sigma, \tau)(z)| &\leq \mathbb{Q}_1 \|\Phi_1\| + \mathbb{Q}_2 \|\Phi_2\|, \\ |\mathbb{K}_{2,1}(\sigma, \tau)(z) + \mathbb{K}_{2,2}(\sigma, \tau)(z)| &\leq \mathbb{Q}_3 \|\Phi_1\| + \mathbb{Q}_4 \|\Phi_2\|. \end{aligned}$$

As a consequence, it follows that

$$\|\mathbb{K}_1(\sigma, \tau) + \mathbb{K}_2(\sigma, \tau)\| \leq (Q_1 + Q_3)\|\Phi_1\| + (Q_2 + Q_4)\|\Phi_2\| \leq \theta.$$

Hence, $\mathbb{K}_1(\sigma, \tau) + \mathbb{K}_2(\sigma, \tau) \in \mathbb{B}_\theta$.

Now, it will be proven that the operator $(\mathbb{K}_{1,1}, \mathbb{K}_{2,1})$ is a contraction mapping. For $(\sigma_2, \tau_2), (\sigma_1, \tau_1) \in \mathbb{B}_\theta$, and for any $z \in [a_1, b_1]$, we have

$$\begin{aligned} & |\mathbb{K}_{1,1}(\sigma_2, \tau_2)(z) - \mathbb{K}_{1,1}(\sigma_1, \tau_1)(z)| \\ & \leq {}^k I^{\bar{\alpha}; \varphi} |\Psi_1(z, \sigma_2(z), \tau_2(z)) - \Psi_1(z, \sigma_1(z), \tau_1(z))| \\ & \leq \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} (\mu_1 \|\sigma_2 - \sigma_1\| + \mu_2 \|\tau_2 - \tau_1\|), \end{aligned}$$

and hence

$$\|\mathbb{K}_{1,1}(\sigma_2, \tau_2) - \mathbb{K}_{1,1}(\sigma_1, \tau_1)\| \leq \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} (\mu_1 \|\sigma_2 - \sigma_1\| + \mu_2 \|\tau_2 - \tau_1\|).$$

Similarly, we can obtain

$$\|\mathbb{K}_{2,1}(\sigma_2, \tau_2) - \mathbb{K}_{2,1}(\sigma_1, \tau_1)\| \leq \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} (\nu_1 \|\sigma\| + \nu_2 \|\tau\|).$$

Consequently, we obtain

$$\begin{aligned} & \|(\mathbb{K}_{1,1}, \mathbb{K}_{2,1})(\sigma_2, \tau_2) - (\mathbb{K}_{1,1}, \mathbb{K}_{2,1})(\sigma_1, \tau_1)\| \\ & \leq \left[(\mu_1 + \mu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + (\nu_1 + \nu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \right] (\|\sigma_2 - \sigma_1\| + \|\tau_2 - \tau_1\|), \end{aligned}$$

which, according to Equation (25), implies that $(\mathbb{K}_{1,1}, \mathbb{K}_{2,1})$ is a contraction.

It remains to be verified that the operator $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})$ is completely continuous. Under the continuity of functions Ψ_1 and Ψ_2 , we deduce that the operator $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})$ is continuous. For all $(\sigma, \tau) \in \mathbb{B}_\theta$, following the arguments employed in the proof of Theorem 2, we find

$$\|\mathbb{K}_{1,2}(\sigma, \tau)\| \leq \left(Q_1 - \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} \right) \|\Phi_1\| + Q_2 \|\Phi_2\|.$$

Similarly, we have that

$$\|\mathbb{K}_{2,2}(\sigma, \tau)\| \leq Q_3 \|\Phi_1\| + \left(Q_4 - \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \right) \|\Phi_2\|.$$

Consequently, we have

$$\begin{aligned} \|(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})(\sigma, \tau)\| & \leq \left(Q_1 - \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + Q_3 \right) \|\Phi_1\| \\ & \quad + \left(Q_2 + Q_4 - \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \right) \|\Phi_2\|, \end{aligned}$$

Thus, set $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})\mathbb{B}_\theta$ is uniformly bounded.

Lastly, we show that set $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})\mathbb{B}_\theta$ is equicontinuous. Let $z_1, z_2 \in [a_1, b_1]$ such that $z_1 < z_2$. For all $(\sigma, \tau) \in \mathbb{B}_\theta$, due to the equicontinuous property of operators \mathbb{K}_1 and \mathbb{K}_2 , we can show that $|\mathbb{K}_1(\sigma, \tau)(z_2) - \mathbb{K}_1(\sigma, \tau)(z_1)| \rightarrow 0$, $|\mathbb{K}_2(\sigma, \tau)(z_2) - \mathbb{K}_2(\sigma, \tau)(z_1)| \rightarrow 0$ as $z_1 \rightarrow z_2$ independent of $(\sigma, \tau) \in \mathbb{B}_\theta$. Consequently, set $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})\mathbb{B}_\theta$ is equicontinuous. Now, under the Arzelá–Ascoli theorem, the compactness property of operator $(\mathbb{K}_{1,2}, \mathbb{K}_{2,2})$

on \mathbb{B}_θ is established. Hence, under the conclusion of Krasnosel’skiĭ’s fixed-point theorem, the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1) has at least one solution on $[a_1, b_1]$. \square

4. Illustrative Examples

Example 1. Let us consider a coupled system of nonlinear proportional fractional differential equations of the Hilfer type:

$$\begin{cases} \mathbb{D}_{\frac{1}{7}}^{\frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{z+2}{z+3}} \sigma(z) = \Psi_1(z, \sigma(z), \tau(z)), & z \in \left[\frac{1}{7}, \frac{20}{7} \right], \\ \mathbb{D}_{\frac{1}{7}}^{\frac{6}{5}, \frac{5}{6}, \frac{2}{5}, \frac{z+2}{z+3}} \tau(z) = \Psi_2(z, \sigma(z), \tau(z)), & z \in \left[\frac{1}{7}, \frac{20}{7} \right], \end{cases} \tag{30}$$

supplemented with multi-point and integro-multi-strip boundary conditions of the form

$$\begin{cases} \sigma\left(\frac{1}{7}\right) = 0, & \int_{\frac{1}{7}}^{\frac{20}{7}} \frac{\sigma(s)}{(s+3)^2} ds = \frac{1}{11} \int_{\frac{2}{7}}^{\frac{3}{7}} \frac{\tau(s)}{(s+3)^2} ds + \frac{2}{13} \int_{\frac{8}{7}}^{\frac{9}{7}} \frac{\tau(s)}{(s+3)^2} ds \\ & + \frac{3}{17} \int_{\frac{15}{7}}^{\frac{16}{7}} \frac{\tau(s)}{(s+3)^2} ds + \frac{4}{19} \tau\left(\frac{4}{7}\right) + \frac{5}{21} \tau\left(\frac{10}{7}\right) + \frac{6}{23} \tau\left(\frac{16}{7}\right), \\ \tau\left(\frac{1}{7}\right) = 0, & \int_{\frac{1}{7}}^{\frac{20}{7}} \frac{\tau(s)}{(s+3)^2} ds = \frac{7}{24} \int_{\frac{6}{7}}^{\frac{6}{7}} \frac{\sigma(s)}{(s+3)^2} ds + \frac{8}{27} \int_{\frac{12}{7}}^{\frac{12}{7}} \frac{\sigma(s)}{(s+3)^2} ds \\ & + \frac{9}{29} \int_{\frac{18}{7}}^{\frac{18}{7}} \frac{\sigma(s)}{(s+3)^2} ds + \frac{10}{31} \sigma(1) + \frac{11}{34} \sigma\left(\frac{13}{7}\right) + \frac{12}{37} \sigma\left(\frac{19}{7}\right). \end{cases} \tag{31}$$

Here, $\rho_1 = 4/3, \rho_2 = 6/5, \varphi_1 = 3/4, \varphi_2 = 5/6, \vartheta_* = 2/5, \bar{\psi}_* = (z + 2)/(z + 3), a_1 = 1/7, b_1 = 20/7, n = 3, m = 3, \kappa_1 = 1/11, \kappa_2 = 2/13, \kappa_3 = 3/17, \eta_1 = 3/7, \eta_2 = 9/7, \eta_3 = 15/7, \zeta_1 = 2/7, \zeta_2 = 8/7, \zeta_3 = 2, \theta_1 = 4/19, \theta_2 = 5/21, \theta_3 = 6/23, \zeta_1 = 4/7, \zeta_2 = 10/7, \zeta_3 = 16/7, \phi_1 = 7/24, \phi_2 = 8/27, \phi_3 = 9/29, \epsilon_1 = 6/7, \epsilon_2 = 12/7, \epsilon_3 = 18/7, \delta_1 = 5/7, \delta_2 = 11/7, \delta_3 = 17/7, \vartheta_1 = 10/31, \vartheta_2 = 11/34, \vartheta_3 = 12/37, z_1 = 1, z_2 = 13/7 and z_3 = 19/7. Using these values, we find that $\gamma_1 = 11/6, \gamma_2 = 28/15, \mathbb{A}_1 \approx 0.03230568122, \mathbb{B}_1 \approx 0.0006815253467, \mathbb{C}_1 \approx 0.1733066601, \mathbb{A}_2 \approx 0.03033569188, \mathbb{B}_2 \approx 0.001631277305, \mathbb{C}_2 \approx 0.2898589804, |\mathbb{L}| \approx 0.04973584581, \mathbb{Q}_1 \approx 0.4615448602, \mathbb{Q}_2 \approx 0.06076238415, \mathbb{Q}_3 \approx 0.6458576234 and \mathbb{Q}_4 \approx 0.5776922344.$$

(i) For illustrating Theorem 1, let us take the Lipschitzian functions Ψ_1 and Ψ_2 on $[1/7, 20/7]$ defined by

$$\begin{cases} \Psi_1(z, \sigma, \tau) = \frac{1}{7z+5} \left(\frac{\sigma^2 + |\sigma|}{1 + |\sigma|} \right) + \frac{1}{14z+2} \sin \tau + \sqrt{z} + 3, \\ \Psi_2(z, \sigma, \tau) = \frac{1}{(7z+1)^2} \left(\frac{|\sigma|}{1 + |\sigma|} \right) + \frac{1}{28z+1} \tan^{-1} \tau + z^3 + \frac{1}{2}. \end{cases} \tag{32}$$

Notice that

$$|\Psi_1(z, \sigma_1, \tau_1) - \Psi_1(z, \sigma_2, \tau_2)| \leq \frac{1}{3} |\sigma_1 - \sigma_2| + \frac{1}{4} |\tau_1 - \tau_2|$$

and

$$|\Psi_2(z, \sigma_1, \tau_1) - \Psi_2(z, \sigma_2, \tau_2)| \leq \frac{1}{4} |\sigma_1 - \sigma_2| + \frac{1}{5} |\tau_1 - \tau_2|,$$

for all $\sigma_i, \tau_i \in \mathbb{R}, i = 1, 2$ and $z \in [1/7, 20/7]$. By setting the Lipschitz constants to $\mu_1 = 1/3, \mu_2 = 1/4, \nu_1 = 1/4$ and $\nu_2 = 1/5$, we obtain

$$(\mathbb{Q}_1 + \mathbb{Q}_3)(\mu_1 + \mu_2) + (\mathbb{Q}_2 + \mathbb{Q}_4)(\nu_1 + \nu_2) \approx 0.9332893607 < 1.$$

Clearly, all the assumptions of Theorem 1 are fulfilled, and hence its conclusion implies that the system (30) with multi-point and integro-multi-strip boundary conditions (31) and the functions Ψ_1 and Ψ_2 given in Equation (32) has a unique solution on $[1/7, 20/7]$.

(ii) We demonstrate the application of Theorem 2 by considering the following nonlinear non-Lipschitzian functions:

$$\begin{cases} \Psi_1(z, \sigma, \tau) = \frac{3\sigma\tau^{2022}}{(7z+4)(1+\tau^{2022})} + \frac{4e^{-\sigma^2}|\tau|^{2023}}{(7z+6)(1+\tau^{2022})} + \frac{2}{3}, \\ \Psi_2(z, \sigma, \tau) = \frac{2\sigma^2 \sin^2 \tau^2}{(14z+3)(1+|\sigma|)} + \frac{3\tau \cos^2 \sigma^4}{14z+5} + \frac{1}{4}. \end{cases} \tag{33}$$

Note that Ψ_1 and Ψ_2 are bounded as

$$|\Psi_1(z, \sigma, \tau)| \leq \frac{2}{3} + \frac{3}{5}|\sigma| + \frac{4}{7}|\tau|$$

and

$$|\Psi_2(z, \sigma, \tau)| \leq \frac{1}{4} + \frac{2}{5}|\sigma| + \frac{3}{7}|\tau|,$$

for all $z \in [1/7, 20/7]$ and $\sigma, \tau \in \mathbb{R}$. By fixing $\pi_0 = 2/3, \pi_1 = 3/5, \pi_2 = 4/7, \omega_0 = 1/4, \omega_1 = 2/5$ and $\omega_2 = 3/7$, we obtain $(Q_1 + Q_3)\pi_1 + (Q_2 + Q_4)\omega_1 \approx 0.9198233378 < 1$ and $(Q_1 + Q_3)\pi_2 + (Q_2 + Q_4)\omega_2 \approx 0.9064248274 < 1$. Therefore, it follows with the conclusion of Theorem 2 that there exists at least one solution (σ, τ) on the interval $[1/7, 20/7]$ of the system (30) with multi-point and integro-multi-strip boundary conditions (31) and two nonlinear functions Ψ_1 and Ψ_2 given in Equation (33).

(iii) Let us use the following functions for explaining the application of Theorem 3:

$$\begin{cases} \Psi_1(z, \sigma, \tau) = \frac{6}{14z+5} \sin \sigma + \frac{8|\tau|}{(7z+2)^2(1+|\tau|)} + z^2 + \frac{1}{4}, \\ \Psi_2(z, \sigma, \tau) = \frac{7|\sigma|}{(7z+1)^3(1+|\sigma|)} + \frac{9}{5(7z+1)} \tan^{-1} \tau + z^4 + \frac{1}{5}, \end{cases} \tag{34}$$

which are obviously bounded as

$$|\Psi_1(z, \sigma, \tau)| \leq \frac{6}{14z+5} + \frac{8}{(7z+2)^2} + z^2 + \frac{1}{4} := \Phi_1(z),$$

and

$$|\Psi_2(z, \sigma, \tau)| \leq \frac{7}{(7z+1)^3} + \frac{9\pi}{10(7z+1)} + z^4 + \frac{1}{5} := \Phi_2(z)$$

for all $z \in [1/7, 20/7]$ and $\sigma, \tau \in \mathbb{R}$. Moreover, these functions are Lipschitz functions since

$$|\Psi_1(z, \sigma_1, \tau_1) - \Psi_1(z, \sigma_2, \tau_2)| \leq \frac{6}{7}|\sigma_1 - \sigma_2| + \frac{8}{9}|\tau_1 - \tau_2|$$

and

$$|\Psi_2(z, \sigma_1, \tau_1) - \Psi_2(z, \sigma_2, \tau_2)| \leq \frac{7}{8}|\sigma_1 - \sigma_2| + \frac{9}{10}|\tau_1 - \tau_2|.$$

By setting $\mu_1 = 6/7, \mu_2 = 8/9, \nu_1 = 7/8$ and $\nu_2 = 9/10$, we obtain

$$(\mu_1 + \mu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_1}}{\vartheta_*^{\rho_1} \Gamma(\rho_1 + 1)} + (\nu_1 + \nu_2) \frac{(\bar{\psi}_*(b_1) - \bar{\psi}_*(a_1))^{\rho_2}}{\vartheta_*^{\rho_2} \Gamma(\rho_2 + 1)} \approx 0.8740050020 < 1.$$

Therefore, the hypothesis of Theorem 3 holds true, and consequently, the coupled system of nonlinear proportional fractional differential equations of the Hilfer type (30) with multi-point and integro-multi-strip boundary conditions (31) and Ψ_1 and Ψ_2 given in Equation (34) has least one solution (σ, τ) on the interval $[1/7, 20/7]$.

Remark 2. We cannot use Theorem 3 in case (i) as the function Ψ_1 is unbounded. On the other hand, in (iii), we have $(Q_1 + Q_3)((6/7) + (8/9)) + (Q_2 + Q_4)((7/8) + (9/10)) \approx 3.066816841 > 1$, which contradicts the condition in Equation (20) in the statement of Theorem 1.

5. Conclusions

In this paper, we presented the criteria for ensuring the existence and uniqueness of solutions for a coupled system of $\bar{\psi}_*$ Hilfer fractional proportional differential equations complemented with nonlocal integro-multi-strip and multi-point boundary conditions. We relied on the standard fixed-point theorems to establish the desired results, which were illustrated well by constructing numerical examples. Our results are novel and contribute to the existing literature on nonlocal boundary value problems for systems of nonlinear $\bar{\psi}_*$ Hilfer fractional proportional differential equations. It is worthwhile to point out that the results presented in this paper are wider in scope and produced a variety of new results as special cases. For instance, fixing the parameters in the nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer generalized proportional fractional system (1), we obtained some new results as special cases associated with the following:

- Integral multi-strip nonlocal $\bar{\psi}_*$ Hilfer fractional proportional systems of an order within $(1, 2]$ if $\theta_j = 0, \vartheta_j = 0, j = 1, 2, \dots, m$;
- Integral multi-point nonlocal $\bar{\psi}_*$ Hilfer fractional proportional systems of an order within $(1, 2]$ if $\kappa_i = 0, \varphi_i = 0, i = 1, 2, \dots, n$;
- Integral multi-strip nonlocal Hilfer fractional proportional systems of an order within $(1, 2]$ if $\bar{\psi}_*(z) = z$;
- Nonlocal integro-multi-strip and multi-point $\bar{\psi}_*$ Hilfer fractional systems of an order within $(1, 2]$ if $\vartheta_* = 1$.

Furthermore, some more new results can be recorded as special cases for different combinations of the parameters $\theta_j, \vartheta_j, j = 1, 2, \dots, m$ and $\kappa_i, \varphi_i, i = 1, 2, \dots, n$ involved in the system (1). For example, by taking all values where $\kappa_i = 0, i = 1, 2, \dots, n$, we obtain the results for a coupled system of nonlinear $\bar{\psi}_*$ Hilfer fractional proportional differential equations supplemented by the following nonlocal boundary conditions:

$$\begin{aligned} \sigma(a_1) &= 0, \quad \int_{a_1}^{b_1} \bar{\psi}'_*(s)\sigma(s)ds = \sum_{j=1}^m \theta_j \tau(\zeta_j), \\ \tau(a_1) &= 0, \quad \int_{a_1}^{b_1} \bar{\psi}'_*(s)\tau(s)ds = \sum_{i=1}^n \phi_i \int_{\delta_i}^{\epsilon_i} \bar{\psi}'_*(s)\sigma(s)ds + \sum_{j=1}^m \vartheta_j \sigma(z_j). \end{aligned}$$

In a nutshell, the work established in this paper was of a more general nature and yielded several new results as special cases.

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