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From Algebro Geometric Solutions of the Toda Equation to Sato Formulas

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Abstract: We know that the degeneracy of solutions to PDEs, given in terms of theta functions on Riemann surfaces, provides important results about particular solutions, as in the case of the NLS equation. Here, we degenerate the so called finite gap solutions of the Toda lattice equation from the general formulation in terms of abelian functions when the gaps tend to points. This degeneracy allows us to recover the Sato formulas without using inverse scattering theory or geometric or representation theoretic methods.

Keywords: Riemann surfaces; theta functions; abelian integrals; Baker Akhiezer functions; Sato formulas

MSC: 35C99, 35Q35, 35Q53

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1. Introduction

Different formulations can be chosen to represent the Toda equation. In the original work of 1967 [1], the vibration of a uniform chain of particles with nonlinear interaction was studied, and Toda considered the following equation for the n-th particle in the chain:

$$m\partial_x^2(u_n) = -\partial_r(\phi)(u_n - u_{n-1}) + \partial_r(\phi)(u_{n+1} - u_n), \tag{1}$$

where m stands for the mass of the particles, and $\phi(r)$ is the interaction energy between adjacent particles. Some solutions were constructed for particular interactions ϕ.

The same equation was considered with $m = 1$ by Date and Tanaka in 1976 [2], and they constructed some general solutions but in terms of integrals that are difficult to use.

In the appendix of the paper of Dubrovin [3], published in 1981, Krichever considered the non-abelian version of this equation and provided solutions in terms of the theta function on Riemann surfaces.

In 1995, Matveev and Stahlhoffen [4] considered the following version of the Toda equation:

$$\partial_x^2(u_n) = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}). \tag{2}$$

They used the Darboux transformation to contruct solutions to this equation in terms of Casoratis.

In 2014, Zhang and Zhou [5] used the following representation of the Toda equation

$$\partial_x\partial_t(u_n) = (\partial_t(u_n) + \alpha(t))(u_{n-1} - 2u_n + u_{n+1}). \tag{3}$$

They used the generalization of the exp-function to contruct multiwave solutions to this equation.

More recently, Sun, Ma, and Yu [5] examined the following representation of the Toda equation:

$$\Delta(u_n) = 4(\exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1})). \tag{4}$$

They used a logarithmic transformation to obtain some particular solutions in terms of logarithms.

In 2020, Duarte [6] considered the following representation of the Toda equation:

$$\alpha\partial_x^2(u(x,y,n)) + \beta\partial_y^2(u(x,y,n))$$
$$= \exp(u(x,y,n-1) - u(x,y,n)) - \exp(u(x,y,n) - u(x,y,n+1)).$$

He used a particular ansatz combined with the properties of Laplace's equation to contruct some solutions in terms of trigonometric functions.

Also in 2020, Schiebold and Nilson [7] studied another version of the Toda equation in the form

$$\partial^2_{xy}(\log(1+u_n)) = u_{n+1} - 2u_n + u_{n+1}. \tag{5}$$

They constructed solutions in the frame of linear algebra by means of determinants.

Here, we consider the Toda equation in the following representation [8–11]:

$$\begin{cases} \dfrac{r_n}{r_{n+1}}\sqrt{c_n}\psi_{n+1} + v_n\psi_n + \dfrac{r_n}{r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1} = \lambda\psi_n \\ \partial_t\psi_n = \dfrac{r_n}{2r_{n+1}}\sqrt{c_n}\psi_{n+1} + \partial_t(\ln r_n)\psi_n - \dfrac{r_n}{2r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1}, \end{cases} \tag{6}$$

with

$$c_n = \frac{e^{2I_0}\theta^2(tV'' + nD + Z)}{\theta(tV'' + (n+1)D + Z)\theta(tV'' + (n-1)D + Z)} \tag{7}$$

$$v_n = -R_0 + \partial_t \ln \frac{\theta(tV'' + (n-1)D + Z)}{\theta(tV'' + nD + Z)}, \tag{8}$$

where θ is the classical Riemann theta function.

We know that the degeneracy of the solutions to PDEs given in terms of Riemann theta functions provides some important particular solutions. In the case of the NLS equation, we have managed to construct a quasi-rational solution involving the determinant of order N for each positive integer N depending on $2N-2$ real parameters [12]. In the case of the KdV equation, we constructed solutions in terms of Fredholm determinants and Wronskians [13], from Baker–Akhiezer functions.

In this study, we recovered Sato formulas for the Toda equation using this method.

From finite gap solutions given in terms of Riemann theta functions, we constructed some quasi-rational solutions and recovered the Sato formulas, using degeneracy, as given in the frame of the NLS equation [14].

Precisely, we derived multisoliton solutions from finite gap solutions by a limit transition, i.e., by making gaps tend toward points in a certain Riemann surface. This was accomplished in the spirit of [14] or more recently [13]. One strength of this approach is that it does not rely on inverse scattering theory or geometric and representation theoretic methods, which offers a fresh perspective on the problem.

We consider the Riemann surface Γ represented by $\cup_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}$ of the algebraic curve defined by [8,9]

$$\omega^2 = \prod_{j=1}^{2g+2}(z - E_j),$$

with $E_j \neq E_k$, $j \neq k$.

Let us consider Ω_1'' and Ω_0' [11] abelian integrals, verifying

$$\int_{a_k} d\Omega_1''(P) = \int_{a_k} d\Omega_0'(P) = 0 \qquad k = 1,\ldots,g, \tag{9}$$

with the following asymptotic behavior for $P = (\omega, z) \in \Gamma$ [11],

$$\Omega_1''(P) = \pm\frac{1}{2}(z + R_0 + O(z^{-1})), \qquad P \to P_\infty^\pm \tag{10}$$
$$\Omega_0'(P) = \pm(\ln z - I_0 + O(z^{-1})), \qquad P \to P_\infty^\pm, \tag{11}$$

$$\omega = \pm(z^{g+1} + O(z^g)), \qquad P \to P_\infty^\pm. \tag{12}$$

The vectors V'', D, X, and Z are defined by

$$V_j'' = \frac{1}{2\pi i}\int_{b_j} d\Omega_1'', \tag{13}$$

$$D = U(P_\infty^+) - U(P_\infty^-), \tag{14}$$

$$X = K + \sum_{j=1}^{g} U(P_j), \qquad Z = U(P_\infty^+) - X, \tag{15}$$

where K is the vector of Riemann constants.

$U(P)$ is the classical abelian integral $\int_{P_0}^{P} dU$.

The solutions to the system (6) can be written [10,11] as

$$\psi_n(P,t) = \tilde{r}_n(t)e^{n\Omega_0'(P)+t\Omega_1''(P)}\left(\frac{\theta(U(P)+tV''+nD-X)}{\theta(U(P)-X)}\right), \tag{16}$$

with

$$\tilde{r}_n(t) = r_n(t)\left(\frac{\theta(Z)\theta(Z-D)}{\theta(tV''+nD+Z)\theta(tV''+(n-1)D+Z)}\right)^{\frac{1}{2}}. \tag{17}$$

2. Degeneracy of Solutions

Let us suppose that E_j are real, $E_m < E_j$ if $m < j$, and try to evaluate the limits of all objects in Formula (16) when E_{2m}, E_{2m-1} tends to $-\alpha_m$, $\alpha_m = \kappa_m^2$, $\kappa_m > 0$, for $1 \le m \le g$.

As in the previous section, θ is constructed from the matrix of the B-periods of the surface Γ, the coefficients c_{jk} are related with abelian differentials dU_j by

$$dU_j = \frac{\sum_{k=1}^{g} c_{jk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z-E_k)}} dz, \tag{18}$$

and the coefficients c_{jk} can be obtained by solving the system of linear equations

$$\int_{a_k} dU_j = \delta_{jk}, \quad 1 \le j \le g, \quad 1 \le k \le g.$$

In the remainder of this article, we use the following notations:

$$\begin{array}{l}
\kappa_{mj} = \sqrt{(\alpha_m + E_{2g+1})(\alpha_j + E_{2g+2})}, \quad \kappa_{jm} = \sqrt{(\alpha_j + E_{2g+1})(\alpha_m + E_{2g+2})}, \\
C_k(z) = \sqrt{(z - E_{2g+1})(\alpha_k - E_{2g+2})}, \quad D_k(z) = \sqrt{(z - E_{2g+2})(\alpha_k - E_{2g+1})}, \\
L_k = \sqrt{\alpha_k + E_{2g+1}}, \quad M_k = \sqrt{\alpha_k + E_{2g+2}}, \\
F(z) = \sqrt{z - E_{2g+1}}, \quad G(z) = \sqrt{z - E_{2g+2}}.
\end{array} \tag{19}$$

2.1. *Limit of* $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$

Now, we study the limit of $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$.

The limit of $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$ is obviously equal to $\tilde{P}(z) = \prod_{j=1}^{g}(z+\alpha_j)^2(z - E_{2g+1})(z - E_{2g+2})$ or, with (19),

$$\tilde{P}(z) = \prod_{j=1}^{g}((z+\alpha_j)F(z)G(z))^2. \tag{20}$$

2.2. Limit of $dU_m = \frac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z-E_k)}} dz$

Now, we study the limit of $dU_m = \dfrac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z - E_k)}} dz$.

The limit of dU_m is equal to $d\tilde{U}_m = \dfrac{\varphi_m(z)}{\prod_{j=1}^{g}(z+\alpha_j)\sqrt{(z-E_{2g+1})(z-E_{2g+2})}} dz$, where $\varphi_m(z) = \sum_{k=1}^{g} \tilde{c}_{mk} z^{g-k}$. The normalization condition takes the form in the limit

$$\int_{a_k} dU_j \to \frac{2\pi i \varphi_j(-\alpha_k)}{\prod_{m\neq k}(\alpha_m - \alpha_k)\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})}} = \delta_{kj}, \tag{21}$$

which proves that the numbers $-\alpha_m$, $m \neq k$ are the zeros of the polynomials $\varphi_k(z)$; hence, $\varphi_k(z)$ can be written as $\varphi_k(z) = \tilde{c}_{k1}\prod_{m\neq k}(z+\alpha_m)$.

By (21), we obtain in the limit

$$\tilde{c_{k1}} = \frac{\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})}}{2\pi i}.$$

So,

$$\varphi_k(z) = \frac{\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})}}{2\pi i}\prod_{m\neq k}(z+\alpha_m).$$

Moreover,

$$d\tilde{U}_k(z) = \frac{\varphi_k(z)dz}{\sqrt{(z-E_{2g+1})(z-E_{2g+2})}\prod_{m=1}^{g}(z+\alpha_m)};$$

in other words,

$$d\tilde{U}_k(z) = \frac{\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})}}{2\pi i\sqrt{(z-E_{2g+1})(z-E_{2g+2})}(z+\alpha_k)} dz,$$

and with (19),

$$d\tilde{U}_k(z) = \frac{L_k M_k}{2\pi i F(z)G(z)(z+\alpha_k)} dz. \tag{22}$$

2.3. Limit of B_{mk}

The subject of this subsection is the study of the limit of B_{mk}.

We have

$$I = \int_{\alpha_m}^{E_{2g+2}} dU_k \to \frac{1}{2}\tilde{B}_{mk}.$$

The integral I can be easily evaluated along the real axis on the upper sheet of surface Γ, and we obtain

$$\tilde{B}_{mk} = \frac{i}{\pi}\ln\left|\frac{\sqrt{(\alpha_m + E_{2g+1})(\alpha_k + E_{2g+2})} + \sqrt{(\alpha_k + E_{2g+1})(\alpha_m + E_{2g+2})}}{\sqrt{(\alpha_m + E_{2g+1})(\alpha_k + E_{2g+2})} - \sqrt{(\alpha_k + E_{2g+1})(\alpha_m + E_{2g+2})}}\right|,$$

or with the previous notations (19),

$$\tilde{B}_{mk} = \frac{i}{\pi}\ln\left|\frac{\kappa_{mk} + \kappa_{km}}{\kappa_{mk} - \kappa_{km}}\right|. \tag{23}$$

So, iB_{kk} tends to $-\infty$.

Moreover, we have

$$\int_{E_{2g+2}}^{P} d\tilde{U}_k = \frac{-i}{2\pi}\ln\left|\frac{\sqrt{(z - E_{2g+1})(\alpha_k + E_{2g+2})} + \sqrt{(z - E_{2g+2})(\alpha_k + E_{2g+1})}}{\sqrt{(z - E_{2g+1})(\alpha_k + E_{2g+2})} - \sqrt{(z - E_{2g+2})(\alpha_k + E_{2g+1})}}\right|,$$

or with the previous notations (19),

$$\int_{E_{2g+2}}^{P} d\tilde{U}_k = \frac{-i}{2\pi}\ln\left|\frac{C_k(z) + D_k(z)}{C_k(z) - D_k(z)}\right|. \tag{24}$$

2.4. Limit of Ω_1''

In this subsection, we study the limit of Ω_1''.

Ω_1'' is an abelian integral of the second kind satisfying the conditions

$$\int_{a_k} d\Omega_1''(P) = 0, \quad 1 \le k \le g,$$

such that

$$\Omega_1''(P) = \pm\frac{1}{2}(z + R_0 + O(z^{-1})), \qquad P \to P_\infty^\pm$$

The limit of $d\Omega_1''(P)$ is equal to $d\tilde{\Omega}_1''(P) = \dfrac{\phi(z)}{\prod_{j=1}^{g}(z+\alpha_j)\sqrt{(z - E_{2g+1})(z - E_{2g+2})}}dz,$
where $\phi(z) = \sum_{k=0}^{g+1}\tilde{c}_k z^{g+1-k}$.

For $\Omega_1''(P)$, satisfying the condition $\Omega_1''(P) = \pm\frac{1}{2}z + O(1)$ when $z \to \pm\infty$, we have $\tilde{c_0} = \frac{1}{2}$. Moreover, the conditions

$$\int_{a_k} d\Omega_1''(P) = 0, \quad 1 \le k \le g$$

prove that $-\alpha_1, \ldots, -\alpha_g$ are the zeros of ϕ; thus, $\phi(z) = \frac{1}{2}\prod_{m=1}^{g}(z + \alpha_m)(z - \beta)$.

We have that $P(z) = \sqrt{\prod_{k=1}^{2g+2}(z - E_k)}$ tends to $\tilde{P}(z) = \sqrt{(z - E_{2g+1})(z - E_{2g+2})}\prod_{m=1}^{g}(z + \alpha_m)$, and we obtain

$$d\tilde{\Omega}_1''(z) = \frac{z}{2\sqrt{(z - E_{2g+1})(z - E_{2g+2})}}dz;$$

$\Omega_1''(P)$ tends to

$$\tilde{\Omega}_1''(P) = \int_{E_{2g+2}}^{z}\frac{u}{2\sqrt{(u - E_{2g+1})(u - E_{2g+2})}}du.$$

This can be evaluated, and it gives

$$\tilde{\Omega}_1''(P) = \frac{E_{2g+1} + E_{2g+2}}{4} \ln\left|\frac{\sqrt{z - E_{2g+1}} + \sqrt{z - E_{2g+2}}}{\sqrt{z - E_{2g+1}} - \sqrt{z - E_{2g+2}}}\right| + \frac{1}{2}\sqrt{(z - E_{2g+1})(z - E_{2g+2})}.$$

With the notations defined in (19), it can be rewritten as

$$\tilde{\Omega}_1''(P) = \frac{E_{2g+1} + E_{2g+2}}{4} \ln\left|\frac{F(z) + G(z)}{F(z) - G(z)}\right| + \frac{1}{2}F(z)G(z). \tag{25}$$

2.5. Limit of $\Omega_0^{'}$

We consider $\Omega_0^{'}$, and we study its limit.

$\Omega_0^{'}$ is an abelian integral of the third kind satisfying the conditions

$$\int_{a_k} d\Omega_0^{'}(P) = 0, \quad 1 \le k \le g,$$

such that

$$\Omega_0'(P) = \pm(\ln z - I_0 + O(z^{-1})), \qquad P \to P_\infty^{\pm}.$$

The limit of $d\Omega_0'(P)$ is equal to

$$d\tilde{\Omega}'_0(P) = \frac{\sum_{k=0}^{g} \tilde{c}_k z^{g-k}}{\prod_{j=1}^{g}(z + \alpha_j)\sqrt{(z - E_{2g+1})(z - E_{2g+2})}} dz.$$

$\Omega_0'(P)$ satisfies the condition $\Omega_0'(P) = \pm\ln(z) + O(1)$, and when $z \to \pm\infty$, we have $\tilde{c_0} = 1$.

Moreover, the conditions

$$\int_{a_k} d\Omega_0'(P) = 0, \quad 1 \le k \le g$$

prove that $-\alpha_1, \ldots, -\alpha_g$ are the zeros of ϕ; thus, $\phi(z)$, defined by $\phi(z) = \sum_{k=0}^{g} \tilde{c}_k z^{g-k}$, can be written as $\phi(z) = \prod_{m=1}^{g}(z + \alpha_m)$.

As $P(z) = \sqrt{\prod_{k=1}^{2g+2}(z - E_k)}$ tends to $\tilde{P}(z) = \sqrt{(z - E_{2g+1})(z - E_{2g+2})}\prod_{m=1}^{g}(z + \alpha_m)$, we obtain

$$d\tilde{\Omega}'_0(z) = \frac{1}{\sqrt{(z - E_{2g+1})(z - E_{2g+2})}} dz,$$

and $\Omega_0'(P)$ tends to $\tilde{\Omega}'_1(P) = \int_{E_{2g+2}}^{z} \dfrac{1}{\sqrt{(u - E_{2g+1})(u - E_{2g+2})}} du$.

This can be calculated, and it gives

$$\tilde{\Omega}_0'(P) = \ln\left|\frac{\sqrt{z - E_{2g+1}} + \sqrt{z - E_{2g+2}}}{\sqrt{z - E_{2g+1}} - \sqrt{z - E_{2g+2}}}\right|.$$

It can be written with (19) as

$$\tilde{\Omega}_0'(P) = \ln\left|\frac{F(z) + G(z)}{F(z) - G(z)}\right|. \tag{26}$$

2.6. Limit of X

We deduce the limit of X.

From the previous sections,

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln\left|\frac{\kappa_{mk} + \kappa_{km}}{\kappa_{mk} - \kappa_{km}}\right|,$$

$$\tilde{U}_k(P) = \frac{i}{2\pi} \ln\left| \frac{C_k(z) + D_k(z))}{C_k(z) - D_k(z)} \right|.$$

As K is defined by $K_m = \sum_{j=1}^{g} B_{jm} - \frac{m}{2}$, and $X = K + \sum_{j=1}^{g} U(P_j)$, we can write

$$X_m \to \tilde{X_m} = \frac{i}{\pi} \sum_{j=1, j\neq m}^{g} \ln\left| \frac{\kappa_{mj} + \kappa_{jm}}{\kappa_{mj} - \kappa_{jm}} \right| - \frac{m}{2} + \frac{i}{2\pi} \sum_{j=1}^{g} \ln\left| \frac{C_m(z_j) + D_m(z_j)}{C_m(z_j) - D_m(z_j)} \right| + i\infty. \tag{27}$$

2.7. Limit of $U(P_\infty^+)$, $U(P_\infty^-)$, D and Z

In this subsection, we study the limit of $U(P_\infty^+)$, $U(P_\infty^-)$, D, and Z.
From the previous sections, it is easy to obtain the limits of $\tilde{U}_k$ as

$$\lim_{P \to P_\infty^+} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln\left| \frac{\sqrt{\alpha_k + E_{2g+2}} + \sqrt{\alpha_k + E_{2g+1}}}{\sqrt{\alpha_k + E_{2g+2}} - \sqrt{\alpha_k + E_{2g+1}}} \right|,$$

or

$$\lim_{P \to P_\infty^+} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln\left| \frac{M_k + L_k}{M_k + L_k} \right|. \tag{28}$$

$$D_k = U_k(P_\infty^+) - U_k(P_\infty^-) \to \tilde{D}_k = -\frac{i}{2\pi} \ln \frac{M_k + iL_k}{M_k - iL_k}. \tag{29}$$

Thus,

$$\lim_{P \to P_\infty^-} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln\left| \frac{\sqrt{\alpha_k + E_{2g+2}} + \sqrt{\alpha_k + E_{2g+1}}}{\sqrt{\alpha_k + E_{2g+2}} - \sqrt{\alpha_k + E_{2g+1}}} \right| - D_k,$$

or

$$\lim_{P \to P_\infty^-} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln\left| \frac{M_k + L_k}{M_k + L_k} \right| + \frac{i}{2\pi} \ln\left| \frac{M_k + iL_k}{M_k + iL_k} \right|. \tag{30}$$

Therefore, the limit of $Z = U(P_\infty^+) - X$ is given by

$$Z_k \to \tilde{Z}_k = -\frac{i}{\pi} \sum_{j=1, j\neq k}^{g} \ln\left| \frac{\kappa_{kj} + \kappa_{jk}}{\kappa_{kj} - \kappa_{jk}} \right| + \frac{k}{2} - \frac{i}{2\pi} \sum_{j=1,}^{g} \ln\left| \frac{C_k(z_j) + D_k(z_j)}{C_k(z_j) - D_k(z_j)} \right| - i\infty - \frac{i}{2\pi} \ln\left| \frac{M_k + L_k}{M_k - L_k} \right| \tag{31}$$

2.8. Limit of V''

In this paragraph, the limit of V'' is studied.
$V''_k = \frac{1}{2\pi i} \int_{b_k} d\Omega''_1$ is defined in (13).
We have $I = \int_{-\alpha_k}^{E_{2g+2}} d\Omega''_1 \to \frac{1}{2} \int_{b_k} d\tilde{\Omega}''_1$.
As in the determination of the limit of Ω''_1, we have

$$\int_{-\alpha_k}^{E_{2g+2}} d\tilde{\Omega}''_1(P) = -\frac{E_{2g+1} + E_{2g+2}}{4} \ln\left| \frac{\sqrt{\alpha_k + E_{2g+1}} + \sqrt{\alpha_k + E_{2g+2}}}{\sqrt{\alpha_k + E_{2g+1}} - \sqrt{\alpha_k + E_{2g+2}}} \right|$$
$$-\frac{1}{2}\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})}.$$

So,

$$V''_k = \frac{1}{2\pi i} \int_{b_k} d\Omega''_1 \to \tilde{V}'' = -\frac{E_{2g+1} + E_{2g+2}}{4\pi i} \ln\left| \frac{\sqrt{\alpha_k + E_{2g+1}} + \sqrt{\alpha_k + E_{2g+2}}}{\sqrt{\alpha_k + E_{2g+1}} - \sqrt{\alpha_k + E_{2g+2}}} \right|$$
$$-\frac{1}{2\pi i}\sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})},$$

$$\tilde{V}''_k = \frac{E_{2g+1} + E_{2g+2}}{4\pi i} \ln\left| \frac{L_k - M_k}{L_k + M_k} \right| - \frac{1}{2\pi i} L_k M_k. \tag{32}$$

2.9. Limit of $\theta(p)$

We determine the limit of $\theta(p)$.

Let us denote A as the argument of $\theta(p) = \sum_{k\in\mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|p)\}$.

A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_mk_j + \sum_{j=1}^{g} \pi i(2p_j + B_{jj})k_j.$$

Using the inequality $k_j(k_j-1) \geq 0$ for all $k \in \mathbf{Z}^g$ and the fact that iB_{kk} tends to $-\infty$, we can reduce the limit $\tilde{\theta}$ of $\theta(p)$ to a finite sum taken over vectors $k \in \mathbf{Z}^g$, such that each k_j must be equal to 0 or 1.

In this section, we compute the limit of all the terms in the expression of the solution $\psi(P,t)$ in θ. We denote p_i as the arguments of these different expressions.

We first study the terms of $\tilde{r}_n$ corresponding to the arguments $p_1 = Z = U(P_\infty^+) - X$, $p_2 = Z - D$, $p_3 = tV'' + nD - Z$, $p_4 = tV'' + (n-1)D - Z$.

Then, we study the remaining terms of ψ_n corresponding to the arguments $p_5 = U(P) - X$, $p_6 = U(P) + tV'' + nD - X$.

With these notations, the solution $\psi(n,,t,z)$ can be written as

$$\psi_n(P,z) = r_n(t)\left(\frac{\theta(p_1)\theta(p_2)}{\theta(p_3)\theta(p_4)}\right)^{\frac{1}{2}} \frac{\theta(p_6)}{\theta(p_5)} e^{n\Omega_0'(P)+t\Omega_1''(P)}.$$

2.10. Limit of $\theta(p_1)$

We denote p_1 as the term $Z = U(P_\infty^+) - X$.

We study its limit.

Then, $\theta(p_1) \to \tilde{\theta}_1$:

$$\tilde{\theta}_1 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\ 1} \exp\left\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_jk_l + \sum_{j=1}^{g} k_j\left(\ln\left|\frac{M_j+L_j}{M_j-L_j}\right|\right.\right.$$
$$\left.\left. + \sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g}\ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right|\right)\right\}.$$

2.11. Limit of $\theta(p_2)$

We consider $p_2 = Z - D$. We study its limit.

Then, $\theta(p_2) \to \tilde{\theta}_2$. As $D_j \to \tilde{D}_j$, it is easy to see that

$$\tilde{\theta}_2 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\ 1} \exp\left\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_jk_l + \sum_{j=1}^{g} k_j\left(\ln\left|\frac{M_j+L_j}{M_j-L_j}\right|\right.\right.$$
$$\left.\left. + \sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g}\ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right| + \sum_{j=1}^{g}\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|\right)\right\}.$$

2.12. Limit of $\theta(p_3)$

Let p_3 be $tV'' + nD + Z$. We study its limit.

Then, $\theta(p_3) \to \tilde{\theta}_3$:

$$\tilde{\theta}_3 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\ 1} \exp\left\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_jk_l + \sum_{j=1}^{g} k_j\left(\ln\left|\frac{L_j+M_j}{L_j-M_j}\right|\right.\right.$$
$$+t\left(\left(\frac{E_{g+1}+E_{2g+2}}{2}\right)\ln\left|\frac{L_j-M_j}{L_j+M_j}\right| - L_jM_j\right) + n\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$$

$$+\sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g} \ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right|\Bigg)\Bigg\}.$$

2.13. Limit of $\theta(p_4)$

We consider $p_4 = tV'' + (n-1)D + Z$; $\theta(p_4) \to \tilde{\theta}_4$. We determine its limit.
As $D_j \to \tilde{D}_j$, it is easy to verify that

$$\tilde{\theta}_4 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\,or\,\,1} \exp\Bigg\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_j k_l + \sum_{j=1}^{g} k_j \Bigg(\ln\left|\frac{L_j+M_j}{L_j-M_j}\right|$$

$$+t\left(\left(\frac{E_{g+1}+E_{2g+2}}{2}\right)\ln\left|\frac{L_j-M_j}{L_j+M_j}\right| + L_j M_j\right) + (n-1)\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$$

$$+\sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g} \ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right|\Bigg)\Bigg\}.$$

2.14. Limit of $\theta(p_5)$

The term $p_5 = U(P) - X$ tends to $\tilde{p}_5 = \tilde{U}(P) - \tilde{X}$.
We determine its limit.
Then, $\theta(p_5) \to \tilde{\theta}_5$:

$$\tilde{\theta}_5 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\,or\,\,1} \exp\Bigg\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_j k_l + \sum_{j=1}^{g} k_j \Bigg(\ln\left|\frac{C_j(z)-D_j(z)}{C_j(z)+D_j(z)}\right|$$

$$+\sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g} \ln\left|\frac{C_j(z_l+D_j(z_l}{C_j(z_l-D_j(z_l}\right|\Bigg)\Bigg\}.$$

2.15. Limit of $\theta(p_6)$

The term $p_6 = U(P) + tV'' + nD - X$ tends to $\tilde{p}_6 = \tilde{U}(P) + t\tilde{V}'' - \tilde{X} + n\tilde{D}$.
Then, we determine its limit. $\theta(p_6) \to \tilde{\theta}_6$:

$$\tilde{\theta}_6 = \sum_{k\in\mathbf{Z}^g,\, k_j=0\,or\,\,1} \exp\Bigg\{\sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_j k_l + \sum_{j=1}^{g} k_j \Bigg(\ln\left|\frac{L_j+M_j}{L_j-M_j}\right|$$

$$+t\left(\left(\frac{E_{g+1}+E_{2g+2}}{2}\right)\ln\left|\frac{L_j-M_j}{L_j+M_j}\right| - L_j M_j\right) + n\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$$

$$+\sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i - \sum_{l=1}^{g} \ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right| + \ln\left|\frac{C_j(z)-D_j(z)}{C_j(z)+D_j(z)}\right|\Bigg)\Bigg\}.$$

2.16. Limit of Solutions Ψ

From the previous section, we can give the limit $\tilde{\psi}$ of ψ. It takes the following form:

$$\tilde{\psi}(z,n,t)) = \frac{r_n(t)\left(\tilde{\theta_1}\tilde{\theta_2}\right)^{\frac{1}{2}}}{\tilde{\theta_5}} \times \frac{e^{\left\{n\ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right| + t\left(\frac{E_{2g+1}+E_{2g+2}}{4}\ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right| + \frac{1}{2}F(z)G(z)\right)\right\}}\tilde{\theta_6}}{\left(\tilde{\theta_3}\tilde{\theta_4}\right)^{\frac{1}{2}}}.$$

In the previous expression, $\tilde{\theta}_1$, $\tilde{\theta}_2$, and $\tilde{\theta}_5$ are independent of n and t.
Only $\tilde{\theta}_3$, $\tilde{\theta}_4$, and $\tilde{\theta}_6$ depend on n and t.
We choose the particular case in which $t = 0$.
We replace n by x.
We denote c as the coefficient defined by

$$c = \frac{r_n(0)\left(\tilde{\theta}_1\tilde{\theta}_2\right)^{\frac{1}{2}}}{\tilde{\theta}_5}.$$

We denote H as the function defined by

$$H(z) = \left|\frac{F(z) - G(z)}{F(z) + G(z)}\right|.$$

In the different sums involving θ, only terms with $k_j = 1$ remain. So, the sums can be reduced only on the subsets J of $[1;g]$ of integer $\mathbf{N}$. We denote a_j as the term $a_j = -\frac{1}{2}\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$. Then, the term $\tilde{\theta}_6$ can be written as

$$\tilde{\theta}_6 = \sum_{J\subset\{1,\ldots,g\}} \prod_{j,l\in J, j<l} \ln\left|\frac{\kappa_{lj} - \kappa_{jl}}{\kappa_{lj} + \kappa_{jl}}\right|^2 \prod_{j\in J}\left|\frac{L_j + M_j}{L_j - M_j}\right|$$

$$\prod_{l\in J l\neq j} \ln\left|\frac{\kappa_{jl} + \kappa_{lj}}{\kappa_{jl} - \kappa_{lj}}\right|^2 \prod_{j\in J}\left|\frac{C_j(z_l) + D_j(z_l)}{C_j(z_l) + D_j(z_l)}\right|\left|\frac{C_j(z) - D_j(z)}{C_j(z) + D_j(z)}\right| e^{\sum_{j\in J} j\pi i} e^{-2x\sum_{j\in J} a_j}.$$

We denote τ as the function defined by

$$\tau(x) = \tilde{\theta}_3(x) = \sum_{J\subset\{1,\ldots,g\}} \prod_{j,l\in J, j<l} \ln\left|\frac{\kappa_{lj} - \kappa_{jl}}{\kappa_{lj} + \kappa_{jl}}\right|^2 \prod_{j\in J}\left|\frac{L_j + M_j}{L_j - M_j}\right|$$

$$\prod_{l\in J l\neq j} \ln\left|\frac{\kappa_{jl} + \kappa_{lj}}{\kappa_{jl} - \kappa_{lj}}\right|^2 \prod_{j\in J}\left|\frac{C_j(z_l) + D_j(z_l)}{C_j(z_l) - D_j(z_l)}\right| e^{\sum_{j\in J} j\pi i} e^{-2x\sum_{j\in J} a_j}.$$

Thus, the solution $\tilde{\Psi}$ can be written as

$$\tilde{\Psi}(x,z) = c\frac{\exp(xH(z))\chi(x,z)}{\sqrt{\tau(x)\tau(x-1)}}. \quad (33)$$

Here, we return to the Sato formulation of the solution, as given in [15].

This expression is very similar to the solution expressed on page 5830, and the expression of the solution is given by

$$\tilde{\Psi}(x,z) = \frac{\exp(xz)\chi(x,z)}{\sqrt{\tau(x)\tau(x-1)}}. \quad (34)$$

The difference between these two statements comes from the fact that these two equations are treated differently:

Van Diejen considers the equation

$$a(x)f(x+1) + b(x)f'x) + a(x-1)f(x-1) = \lambda f(x);$$

here, we consider (1)

$$\frac{r_n}{r_{n+1}}\sqrt{c_n}\psi_{n+1} + v_n\psi_n + \frac{r_n}{r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1} = \lambda\psi_n.$$

2.17. Limit of the Associated Potentials

In this subsection, we determine the limit of the associated potentials.

The potentials $u_n = \sqrt{c_n}$ and v_n are described in (6) and (7) as

$$c_n = \frac{e^{2I_0}\theta^2(tV'' + nD + Z)}{\theta(tV'' + (n+1)D + Z)\theta(tV'' + (n-1)D + Z)}$$

$$v_n = -R_0 + \partial_t \ln \frac{\theta(tV'' + (n-1)D + Z)}{\theta(tV'' + nD + Z)}.$$

So, the limit of u_n is equal to

$$\tilde{u_n} = k_1 \frac{\tau(x)}{\sqrt{\tau(x+1)\tau(x-1)}},$$

which is the same as in [15] (p. 5830), up to the constant k_1.

Denoting $\sigma(x) = \partial_t \theta(t\tilde{V}'' + x\tilde{D} + \tilde{Z})_{t=0}$, the limit of v_n is equal to

$$\tilde{v_n} = k_2 + \frac{\sigma(x-1)}{\tau(x-1)} - \frac{\sigma(x)}{\tau(x)},$$

which is similar to the one written in [15] (p. 5830), up to the constant k_2.

3. Conclusions

We have used the degeneracy of solutions of some PDEs given in terms of Riemann theta functions to obtain some important solutions. In particular, in the case of the NLS equation, we have managed to construct a quasi-rational solution involving the determinant of order N for each positive integer N depending on $2N-2$ real parameters [12]. In the case of the KdV equation, from abelian functions, we constructed solutions in terms of the Fredholm determinants and Wronskians [13].

In this study, we have managed to recover Sato formulas using this method for the Toda equation. From solutions given in terms of the Baker–Akiezer functions, we succeeded to construct by degeneracy, as given in the frame of the NLS equation [14], some quasi-rational solutions, and we recovered the Sato formulas for the Toda equation.

Precisely, we derived multisoliton solutions from finite gap solutions by a limit transition, i.e., by making gaps tend toward points in a certain Riemann surface. One strength of this approach is that it does not rely on inverse scattering theory or geometric and representation theoretic methods, which offers another perspective on the problem.

The degeneracy of the solutions to the NLS equation has allowed building new quasi-rational solutions of order N depending on $2N-2$ real parameters and their construction up to order 23, which were previously unknown. In the case of the KdV equation, the degeneracy of the solutions made it possible to find the solutions given by the Darboux method, which constituted a bridge between the geometric algebra approach and the Darboux transformations framework. In this article, this is somewhat the same situation as that of the KdV equation, where we linked the algebro–geometric approach to the framework of the inverse scattering method.

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