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Existence of Heteroclinic Solutions in Nonlinear Differential Equations of the Second-Order Incorporating Generalized Impulse Effects with the Possibility of Application to Bird Population Growth

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Abstract: This work considers the existence of solutions of the heteroclinic type in nonlinear second-order differential equations with ϕ-Laplacians, incorporating generalized impulsive conditions on the real line. For the construction of the results, it was only imposed that ϕ be a homeomorphism, using Schauder's fixed-point theorem, coupled with concepts of L^1-Carathéodory sequences and functions along with impulsive points equiconvergence and equiconvergence at infinity. Finally, a practical part illustrates the main theorem and a possible application to bird population growth.

Keywords: heteroclinic solutions; impulsive points equiconvergence and equiconvergence at infinity; L^1-Carathéodory sequences and functions

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1. Introduction

In the theory of differential equations, there are several aspects and interesting features to study or explore further. Among these is the study of impulsive differential equations or systems and the qualitative analysis of solutions in the real line. In this context, this article aims to study the existence of solutions of the heteroclinic type in nonlinear second-order differential equations with generalized, infinite impulse effects, which, to the best of our knowledge is rarely addressed in the existing literature. More specifically, we extend the results in [1], adding infinite impulses in the system with ϕ-Laplacian. This can be very interesting for modeling phenomena with minor changes and with different intensities, which occur very quickly and for long periods of time, opening new fields for investigation on the subject.

Impulses incorporated in differential equations are intended to describe and represent the effects of small and sudden changes in a given system over certain periods of time. In the literature, there are several areas of study associated with impulses such as biotechnology, medicine, population dynamics, logging, etc. (see [2,3]) and references therein. Various theoretical approaches, as well as numerous applications of second-order nonlinear differential equations featuring impulses, can be found in ([1,4–10]).

On the other hand, in the analysis of the qualitative aspects of differential equations, the investigation into the existence of heteroclinic or homoclinic solutions is useful and necessary. When a system of ordinary differential equations has equilibria (that is, constant solutions), studying the connections between them through the trajectories of the system's solutions, known as homoclinic or heteroclinic solutions, becomes an essential task. It is common for homoclinic and heteroclinic solutions to emerge in mathematical models dealing with dynamical systems, bifurcations, mechanics, chemistry and biology [11–13].

The existence of heteroclinic orbits is also crucial for analyzing spatiotemporal chaotic patterns of nonlinear evolution equations [14].

Over the years, some studies have been carried out on the topic of heteroclinic solutions. In [15], the author investigates heteroclinic solutions pertaining to a second-order equation that is asymptotically autonomous $\ddot{x} = a(t)V'(x(t))$. In [16], Coti Zelati along with Rabinowitz investigated heteroclinic orbits for a non-autonomous differential equation that connects stationary points with distinct energy levels. For a fourth-degree ordinary differential equation, heteroclinic solutions linking nonconsecutive equilibria of a triple-well potential were found [17]. Cabada et al., in [18], examine the existence of heteroclinic-type solutions in semi-linear second-order difference equations pertaining to the Fisher–Kolmogorov's equation. Monotonicity and continuity arguments form the basis of the proof for these results. Hale and Rybakowski also demonstrated the existence of heteroclinic solutions for retarded functional differential equations [19]. Furthermore, works involving heteroclinics and impulses can be found in [20–26] and references therein.

Using findings concerning the existence of non-principal solutions, in [27], the authors study Leighton and Wong theorems of oscillation regarding a class of second-order impulsive equations having the form

$$\begin{cases} (p(t)x')' + q(t)x = 0, \quad t \neq \theta_i \\ \Delta x + a_i x = 0, \;\; \Delta p(t)x' + b_i x + c_i x' = 0, \quad t = \theta_i, \end{cases}$$

and

$$\begin{cases} (p(t)x\prime)' + q(t)x = f(t), \quad t \neq \theta_i \\ \Delta x + a_i x = f_i, \;\; \Delta p(t)x' + b_i x + c_i x' = g_i, \quad t = \theta_i, \end{cases}$$

in which $p > 0, q, f$ are left continuous piece-wise functions in $[0, \infty)$, and $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ are real number sequences with $i \geq 1$. The set $\{\theta_i\}$, of impulse points, constitutes a strictly increasing, unbounded sequence of positive real numbers.

In [28], Cupini, Marcelli and Papalini present the strongly nonlinear boundary-value problem

$$\begin{cases} (a(x(t))\Phi(x'(t)))' = f(t, x(t), x'(t)) \;\text{ a.e. } t \in \mathbb{R} \\ x(-\infty) = v^-, \;\; x(+\infty) = v^+ \end{cases}$$

In this work, the authors consider nonlinear mixed differential operators depending both on x and x'. Here, $v^- < v^+$, and $\Phi : \mathbb{R} \to \mathbb{R}$ is a general increasing homeomorphism, $\Phi(0) = 0$, a is a positive continuous function, and f is a nonlinear Caratheódory function.

In a recent paper [1], Sousa and Minhós consider the following coupled system

$$\begin{cases} (a(t)\phi(u'(t)))' = f(t, u(t), v(t), u'(t), v'(t)), \\ (b(t)\psi(v'(t)))' = h(t, u(t), v(t), u'(t), v'(t)), t \in \mathbb{R}. \end{cases}$$

where ϕ and ψ are increasing homeomorphisms satisfying adequate relations on their inverses, with $a, b : \mathbb{R} \to (0, +\infty[$ being continuous functions, and $f, h : \mathbb{R}^5 \to \mathbb{R}$, L^1-Carathéodory functions, along with the following asymptotic conditions

$$u(-\infty) = A, \;\; u'(+\infty) = 0, v(-\infty) = B, v'(+\infty) = 0$$

for $A, B \in \mathbb{R}$.

Motivated by these works, in our paper, we consider a similar problem but with the inclusion of infinite impulsive conditions; more precisely, we study the following real nonlinear second-order differential equation

$$\big(a(t)\phi(u'(t))\big)' = f(t, u(t), u'(t)), \; t \in \mathbb{R} \backslash \{t_k\}, \;\; k \in \mathbb{Z} \tag{1}$$

with ϕ being an increasing homeomorphism satisfying adequate relations on its inverse, $a : \mathbb{R} \to (0, +\infty[$ being a continuous function, and $f : \mathbb{R}^3 \to \mathbb{R}$ being an L^1-Carathéodory function, considering the following asymptotic conditions

$$u(-\infty) = C, \ u(+\infty) = L \tag{2}$$

for $C, L \in \mathbb{R}$, together with the generalized and infinite impulse conditions

$$\begin{cases} \Delta u(t_k) = I_k(t_k, u(t_k), u'(t_k)), \\ \Delta\phi(u'(t_k)) = J_k(t_k, u(t_k), u'(t_k)), \end{cases} \tag{3}$$

where, for $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers), $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta\phi(u'(t_k)) = \phi(u'(t_k^+)) - \phi(u'(t_k^-))$, and $u(t_k^+)$, $u(t_k^-)$ are the right and left limits for $u(t_k)$, respectively. $\Delta\phi(u'(t_k^+))$ and $\Delta\phi(u'(t_k^-))$ have a similar meaning for $\Delta\phi(u'(t_k))$. $I_k, J_k \in C(\mathbb{R}^3, \mathbb{R})$, are Carathéodory sequences and t_k are moments such that $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$, and

$$\lim_{k\to-\infty} t_k = -\infty, \quad \lim_{k\to+\infty} t_k = +\infty.$$

It is worth noting that problems involving ψ and ϕ Laplacians, which are more general forms of the one-dimensional p-Laplacian equation ($\phi(x) = |x|^{p-2}x$), appear frequently in the study of periodic solutions for differential equations (see [29] and references therein for more information).

Another concept to highlight is the generalization of measurable functions (sequences); that is, the concept of Carathéodory sequences was used in the work precisely to control the behavior of the infinite moments of impulse (see [30,31] for more information).

The outline of the present paper is given as follows: Section 2 comprises the functional backgrounds and the main theorem of existence, while Section 3 presents an example application which illustrates the main result.

2. Auxiliary Results, Definitions and the Main Theorem

Let us define

$$u(t_k^\pm) := \lim_{t\to t_k^\pm} u(t),$$

and consider the set

$$PC(\mathbb{R}) = \left\{ \begin{array}{c} u : u \in C^n(\mathbb{R}) \text{ is continuous for } t \neq t_k, u^{(n)}(t_k) = u^{(n)}(t_k^-), \\ u^{(n)}(t_k^+) \text{ exists for } k \in \mathbb{Z} \text{ and } n = 0,1 \end{array} \right\}.$$

Considering the space

$$X := \left\{ x : x \in PC(\mathbb{R}), \lim_{t\to\pm\infty} x^{(i)}(t) \in \mathbb{R}, \ i = 0,1 \right\}, \tag{4}$$

with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\},$$

being

$$\|u\|_\infty := \sup_{t\in\mathbb{R}} |u(t)|.$$

Lemma 1. *$(X, \|\cdot\|_X)$ given in (4) is a real Banach space.*

Proof. Let $x, y \in X$, and $\lambda \in \mathbb{R}$. In order to show that the space $(X, \|\cdot\|_X)$ is Banach, the following points need to be proven:

1. *X is a vector space.*

(a) *Vector addition*

$$\lim_{t\to\pm\infty}(x+y)^{(i)}(t) = \lim_{t\to\pm\infty}\left(x^{(i)}(t)+y^{(i)}(t)\right)$$
$$= \left(\lim_{t\to\pm\infty}x^{(i)}(t)+\lim_{t\to\pm\infty}y^{(i)}(t)\right)\in\mathbb{R}$$

Since from (4), both $\lim_{t\to\pm\infty}x^{(i)}(t)\in\mathbb{R}$ and $\lim_{t\to\pm\infty}y^{(i)}(t)\in\mathbb{R}$. Hence, X is closed under addition, and one can see that this addition is commutative, associative and there exists a *zero vector* 0, and $-x$ for each x such that

$$\lim_{t\to\pm\infty}(x+(-x)+0)^{(i)}(t) = \lim_{t\to\pm\infty}\left(x^{(i)}(t)+(-x)^{(i)}(t)\right)$$
$$= \left(\lim_{t\to\pm\infty}x^{(i)}(t)-\lim_{t\to\pm\infty}x^{(i)}(t)\right)=0\in\mathbb{R}$$

(b) *Multiplication by scalars*
From derivative and limit rules we obtain that

$$\lim_{t\to\pm\infty}(\lambda x)^{(i)}(t) = \lim_{t\to\pm\infty}\lambda(x)^{(i)}(t)$$
$$= \lambda\left(\lim_{t\to\pm\infty}(x)^{(i)}(t)\right)\in\mathbb{R}$$

Thus, X is also closed under multiplication by scalars, satisfying the commutative and distributive laws as well. And therefore, X is a vector space.

2. $(X,\|\cdot\|_X)$ is a normed space.
Let us show that the norm is well defined by verifying its properties. We can see from the definition that $\|x\|_X\geq 0$, $\forall t\in\mathbb{R}$. When $\|x\|_X=0$, by definition, we must have $\|x\|_\infty=\|x'\|_\infty=0$, which in turn means that $x(t)=0$ for all $t\in\mathbb{R}$. Conversely, when $x(t)=0$ for all $t\in\mathbb{R}$ we get that $\|x\|_\infty=\|x'\|_\infty=0$, and therefore $\|x\|_X=0$.

Given some $\lambda\in\mathbb{R}$, we have that

$$\|\lambda x\|_X = \max\{\sup\|\lambda x(t)\|,\sup\|(\lambda x)'(t)\|\}$$
$$= \max\{|\lambda|\sup\|x(t)\|,|\lambda|\sup\|x'(t)\|\}$$
$$= |\lambda|\|x\|_X.$$

Now, we show the triangle inequality

$$\|x+y\|_X\leq\|x\|_X+\|y\|_X.$$

We know that
$$|x+y|\leq|x|+|y|\leq\sup|x|+\sup|y|,$$
and
$$|x'+y'|\leq|x'|+|y'|\leq\sup|x'|+\sup|y'|.$$
But since the supremum is the least upper bound, we obtain from the above inequalities that
$$|x+y|\leq\sup|x+y|\leq\sup|x|+\sup|y|,$$
$$|x'+y'|\leq\sup|x'+y'|\leq\sup|x'|+\sup|y'|.$$

So

$$\sup|x+y|\leq\sup|x|+\sup|y|$$
$$\leq\max\{\sup|x|,\sup|x'|\}+\max\{\sup|y|,\sup|y'|\}$$

and

$$\sup |(x+y)'| \leq \sup |x'| + \sup |y'| \\ \leq \max\{\sup |x|, \sup |x'|\} + \max\{\sup |y|, \sup |y'|\}$$

So, the maximum between $\sup |x+y|$ and $\sup |(x+y)'|$ is bounded as

$$\max\{\sup |x+y|, \sup |(x+y)'|\} \leq \max\{\sup |x|, \sup |x'|\} + \max\{\sup |y|, \sup |y'|\}.$$

Rewriting by the definition, we obtain

$$\|x+y\|_X \leq \|x\|_X + \|y\|_X,$$

thus showing that the norm is well defined.

3. $(X, \|\cdot\|_X)$ is complete under the metric induced by the following norm:

$$d(x,y) = \|x-y\|_X = \max\{\sup |x-y|, \sup |(x-y)'|\}.$$

Let (x_m) be an arbitrary Cauchy sequence in X. Then, for any $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ such that for all $m, n > N$,

$$d(x_m, x_n) = \max\{\sup |x_m - x_n|, \sup |(x_m - x_n)'|\} < \varepsilon.$$

So, for every fixed $t_0 \in \mathbb{R}$, we have

$$\max\{|x_m(t_0) - x_n(t_0)|, |(x_m(t_0) - x_n(t_0))'|\} < \varepsilon$$

that is,

$$|x_m(t_0) - x_n(t_0)| < \varepsilon$$

and

$$|(x_m(t_0) - x_n(t_0))'| < \varepsilon$$

And so, the sequence of numbers $(x_1(t_0), x_2(t_0), x_3(t_0), \ldots)$ and $(x_1'(t_0), x_2'(t_0), x_3'(t_0), \ldots)$ are Cauchy, and each of them converge, (see [32], Theorem 1.4-4), say

$$x_m(t_0) \to x(t_0) \in \mathbb{R}$$

and

$$x_m'(t_0) \to x'(t_0) \in \mathbb{R},$$

as $m \to \infty$. Therefore,

$$\lim_{t \to \pm\infty} x(t) \in \mathbb{R}$$

and

$$\lim_{t \to \pm\infty} x'(t) \in \mathbb{R};$$

And so $x(t) \in X$, the space X is complete, and because it satisfies all the conditions, it is also a Banach space.

□

For the reader's convenience, we consider the definition of L^1-Carathéodory functions:

Definition 1. *A function $f : \mathbb{R}^3 \to \mathbb{R}$ is L^1- Carathéodory if*

(*i*) *For each $(x,y) \in \mathbb{R}^2$, $t \mapsto f(t,x,y)$ is measurable on $\mathbb{R}$;*

(*ii*) *For a.e. $t \in \mathbb{R}$, $(x,y) \mapsto f(t,x,y)$ is continuous on $\mathbb{R}^2$;*

(*iii*) *For each* $\rho > 0$, *there exists a positive function* $\omega_\rho \in L^1(\mathbb{R})$ *such that, whenever* $x, y \in [-\rho, \rho]$, *then*

$$|f(t,x,y)| \leq \omega_\rho(t), \text{ a.e. } t \in \mathbb{R}. \tag{5}$$

Definition 2. *A sequence* $I_k : \mathbb{R}^3 \to \mathbb{R}$, $k \in \mathbb{Z}$ *is Carathéodory if it verifies*

(*i*) *For each* $(a,b) \in \mathbb{R}^2$, $(a,b) \longmapsto I_k(t_k, a, b)$ *is continuous for all* $k \in \mathbb{Z}$;
(*ii*) *For each* $\rho > 0$, *there are non-negative constants* $\chi_{k,\rho} \geq 0$ *with* $\sum_{-\infty<k<+\infty} \chi_{k,\rho} < +\infty$ *such that for* $|a| < \rho$ *and* $|b| < \rho$ *we have* $|I_k(t_k,a,b)| \leq \chi_{k,\rho}$, *for every* $k \in \mathbb{Z}$.

Lemma 2. *Assume that* $f : \mathbb{R}^3 \to \mathbb{R}$ *is an* L^1*-Carathéodory function and* $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ *are Carathéodory sequences for* $k \in \mathbb{Z}$. *Then, Equation (1) with conditions (2), (3) has a solution* $u \in X$ *expressed by*

$$u(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)),$$

with $C, M \in \mathbb{R}$ *satisfying condition (2). Namely,* M *is such that the following expression is verified:*

$$u(+\infty) = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<+\infty} I_k(t_k,u(t_k),u'(t_k)) = L. \tag{6}$$

Proof. Assuming the appropriate convergence conditions are met, the first boundary condition is satisfied as

$$u(-\infty) = \int_{-\infty}^{-\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<-\infty} I_k(t_k,u(t_k),u'(t_k)) = C.$$

Also, $M \in \mathbb{R}$ is such that (6) is satisfied.
Working the expression with $u(t)$, we obtain

$$u'(t) = \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)$$
$$\Leftrightarrow a(t)\phi(u'(t)) = \int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))$$
$$\Leftrightarrow (a(t)\phi(u'(t)))' = f(t,u(t),u'(t)).$$

□

The following theorem presents a useful criterion for the operator's compactness.

Theorem 1 ([1], Theorem 3)**.** *A set* $M \subset X$ *is relatively compact if the following conditions hold:*

(*i*) *Both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are uniformly bounded;*
(*ii*) *Both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are equicontinuous on any compact interval of* $\mathbb{R}$;

(*iii*) *Both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are equiconvergent at* $\pm\infty$*, that is, for any given* $\epsilon > 0$*, there exists* $t_\epsilon > 0$ *such that*

$$|f(t) - f(\pm\infty)| < \epsilon, \ \left|f'(t) - f'(\pm\infty)\right| < \epsilon, \forall |t| > t_\epsilon, f \in M.$$

Schauder's fixed point theorem will provide the means to establish existence.

Theorem 2 ([33])**.** *Let* Y *be a nonempty, closed, bounded and convex subset of a Banach space* X*, and suppose that* $P : Y \to Y$ *is a compact operator. Then,* P *has at least one fixed point in* Y*.*

Along this paper, we assume that

(H1) $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0.$

(H2) $a : \mathbb{R} \to (0, +\infty[$ is a positive continuous function such that

$$\lim_{t \to \pm\infty} \frac{1}{a(t)} \in \mathbb{R}.$$

Main Theorem

Here, we present the main result of this work, that is, the theorem that guarantees the existence of a solution to the problem (1)–(3), for $C, L \in \mathbb{R}$.

Theorem 3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an increasing homeomorphism and* $a : \mathbb{R} \to (0, +\infty[$ *a continuous function satisfying (H1) and (H2). Assume that* $f : \mathbb{R}^3 \to \mathbb{R}$ *is an* L^1*-Carathéodory function,* $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ *are Carathéodory sequences and there are* $\rho > 0$*,* $\omega_\rho \in L^1(\mathbb{R})$ *and non-negative constants* $\chi_{k,\rho}, \Psi_{k,\rho} \geq 0$ *such that*

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(r)dr + M + \chi_{k,\rho}}{a(s)}\right) ds + C + \Psi_{k,\rho} < +\infty, \tag{7}$$

with

$$\sup_{t \in \mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(r)dr + M + \chi_{k,\rho}}{a(t)}\right) < +\infty, \tag{8}$$

$$\begin{gathered} |f(t,x,y)| \leq \omega_\rho(t), \\ |J_k(t_k, x(t_k), y(t_k))| \leq \chi_{k,\rho}, \\ |I_k(t_k, x(t_k), y(t_k))| \leq \Psi_{k,\rho}. \end{gathered}$$

when $x, y \in [-\rho, \rho]$.

Then, for $C, L \in \mathbb{R}$*, satisfying condition (2) and* $M \in \mathbb{R}$ *such that (6) is satisfied, the problem (1)–(3) has, at least heteroclinic solutions* $u \in X$*.*

Proof. Let us define the operator

$$\begin{aligned} T : \quad X &\to \quad X \\ u &\to \quad T(u) \end{aligned}$$

with

$$(T(u))(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)),$$

with $C \in \mathbb{R}$, satisfying condition (2) and $M \in \mathbb{R}$ such that (6) is satisfied.

To apply Theorem 2, we will prove that T is compact and that it has a fixed point; that is, the proof follows five steps.

Step 1. *T is well defined and it is continuous in X.*

Allow $u \in X$ and let us take $\rho > 0$ such that $\|u\|_X < \rho$. Being f an L^1-Carathéodory function and I_k, J_k Carathéodory sequences, there exists a positive function $\omega_\rho \in L^1(\mathbb{R})$, and non-negative constants $\chi_{k,\rho}, \Psi_{k,\rho} \geq 0$ such that

$$|J_k(t_k,u(t_k),u'(t_k))| \leq \chi_{k,\rho}, \quad |I_k(t_k,u(t_k),u'(t_k))| \leq \Psi_{k,\rho}.$$

Thus, $T \in C^1(\mathbb{R})$, as

$$\begin{aligned}
&\int_{-\infty}^{t} \left|f(r,u(r),u'(r))\right|dr + |M| + \sum_{-\infty<t_k<t<+\infty} \left|J_k(t_k,u(t_k),u'(t_k))\right| \\
&\leq \int_{-\infty}^{+\infty} \left|f(r,u(r),u'(r))\right|dr + |M| + \sum_{-\infty<t_k<t<+\infty} \left|J_k(t_k,u(t_k),u'(t_k))\right| \\
&\leq \int_{-\infty}^{+\infty} \omega_\rho(t)\,dt + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho} < +\infty,
\end{aligned}$$

$$\sum_{-\infty<t_k<t<+\infty} \left|I_k(t_k,u(t_k),u'(t_k))\right| \leq \sum_{-\infty<t_k<+\infty} \Psi_{k,\rho} < +\infty$$

and

$$\begin{aligned}
(T(u))'(t) \quad &= \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right) \\
&\leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(t)\,dt + M + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(t)}\right) < +\infty.
\end{aligned}$$

Furthermore, by (2), (7), (8) and (H2),

$$\lim_{t\to-\infty} T(u)(t)$$
$$= \lim_{t\to-\infty}\left(\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds\right.$$
$$\left.+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k))\right) = C \in \mathbb{R},$$

$$\lim_{t\to+\infty} T(u)(t)$$
$$= \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) = L \in \mathbb{R},$$

and

$$\lim_{t\to\pm\infty} (T(u))'(t)$$
$$= \lim_{t\to\pm\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)$$
$$\leq \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(r)\,dr + M + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(+\infty)}\right) < +\infty.$$

Therefore, $T(u) \in X$.

Step 2. *TK is uniformly bounded on $K \subseteq X$, for some bounded K.*

Take K to be a bounded set of X, with the definition

$$K := \{u \in X : \max\{\|u\|_\infty, \|u'\|_\infty\} \leq \rho_1\} \tag{9}$$

for some $\rho_1 > 0$.

By (7), (8), (H1) and (H2), we have

$$\|T(u)(t)\|_\infty$$
$$= \sup_{t\in\mathbb{R}}\left(\left|\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds\right.\right.$$
$$\left.\left. +C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k))\right|\right)$$
$$\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t}\left|\phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)\right|ds$$
$$+|C| + \sup_{t\in\mathbb{R}}\left(\sum_{-\infty<t_k<t<+\infty} |I_k(t_k,u(t_k),u'(t_k))|\right)$$
$$\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} |f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<s_k<s<+\infty} |J_k(s_k,u(s_k),u'(s_k))|}{a(s)}\right)ds$$
$$+|C| + \sup_{t\in\mathbb{R}}\left(\sum_{-\infty<t_k<t<+\infty} |I_k(t_k,u(t_k),u'(t_k))|\right)$$
$$\leq \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} \omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<s_k<s<+\infty} \chi_{k,\rho_1}}{a(s)}\right)ds$$
$$+|C| + \sum_{-\infty<t_k<+\infty} \Psi_{k,\rho_1} < +\infty$$

and

$$
\begin{aligned}
&\|T(u)'(t)\|_{\infty} \\
&= \sup_{t\in\mathbb{R}} \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right| \\
&\leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{t} |f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<t_k<t<+\infty} |J_k(t_k,u(t_k),u'(t_k))|}{a(t)} \right) \\
&\leq \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho_1}}{a(+\infty)} \right) < +\infty
\end{aligned}
$$

So, $\|T(u)(t)\|_X < +\infty$, that is, TK is uniformly bounded on X.

Step 3. *TK is equicontinuous, on each* $]t_k, t_{k+1}]$ interval, *for* $k \in \mathbb{Z}$.

Consider $t_1, t_2 \in I \subseteq]t_k, t_{k+1}]$ and let us suppose, without losing generality, that $t_1 \leq t_2$. So, for $u \in K$ and by (7), (8), and (H1), follow

$$
\begin{aligned}
&|T(u)(t_1) - T(u)(t_2)| \\
&= \left| \int_{-\infty}^{t_1} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&+C + \sum_{-\infty<t_k<t_1<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \\
&\left(\int_{-\infty}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&\left. \left. +C + \sum_{-\infty<t_k<t_2<+\infty} I_k(t_k,u(t_k),u'(t_k)) \right) \right| \\
&\leq \int_{t_1}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \\
&+ \sum_{t_1\leq t_k<t_2<+\infty} I_k(t_k,u(t_k),u'(t_k)) \\
&\leq \int_{t_1}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(t)\,dt + |M| + \sum_{-\infty<s_k<s<+\infty} \chi_{k,\rho_1}}{a(s)} \right) ds + \sum_{t_1\leq t_k<t_2<+\infty} \Psi_{k,\rho_1} \to 0
\end{aligned}
$$

uniformly for $u \in K$, as $t_1 \to t_2$,

$$
\begin{aligned}
&|T(u)'(t_1) - T(u)'(t_2)| \\
&= \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t_1} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_1<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_1)} \right) \right. \\
&\left. -\phi^{-1}\left(\frac{\int_{-\infty}^{t_2} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_2<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_2)} \right) \right| \to 0
\end{aligned}
$$

uniformly for $u \in K$, as $t_1 \to t_2$. Then, TK *is equicontinuous on each interval* $]t_k, t_{k+1}]$, *for* $k \in \mathbb{Z}$.

Step 4. *TK is equiconvergent at each impulse point, and at $t = \pm\infty$, that is TK, is equiconvergent at $t = t_i^+$, $(i \in \mathbb{Z})$ and at infinity.*

First, let us prove that TK is equiconvergent at $t = t_i^+$, for $i \in \mathbb{Z}$. Let $u \in K$. So, by (7), (8), and (H1), it follows

$$
\begin{aligned}
&|T(u)(t) - \lim_{t\to t_i^+} T(u)(t)| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \\
&\left(\int_{-\infty}^{t_i^+} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s_i^+<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&\left. +C + \sum_{-\infty<t_k<t_i^+<+\infty} I_k(t_k,u(t_k),u'(t_k)) \right) \Bigg| \to 0
\end{aligned}
$$

uniformly in $u \in K$, as $t \to t_i^+$, for $i \in \mathbb{Z}$ and

$$
\begin{aligned}
&|T(u)'(t) - \lim_{t\to t_i^+} T(u)'(t)| \\
&= \Bigg| \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \\
&-\phi^{-1}\left(\frac{\int_{-\infty}^{t_i^+} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_i^+<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_i^+)} \right) \Bigg| \to 0
\end{aligned}
$$

uniformly in $u \in K$, as $t \to t_i^+$, for $i \in \mathbb{Z}$. Therefore, TK is equiconvergent at each point $t = t_i^+$, for $i \in \mathbb{Z}$.

Identically, we will prove that *TK is equiconvergent at $t = \pm\infty$*. In this way, we have

$$
\begin{aligned}
&|T(u)(t) - \lim_{t\to -\infty} T(u)(t)| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \lim_{t\to -\infty} \\
&\left(\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&\left. +C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) \right) \Bigg| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - C \Bigg| \to 0
\end{aligned}
$$

uniformly in $u \in K$, as $t \to -\infty$.

In turn,

$$|T(u)(t) - \lim_{t\to+\infty} T(u)(t)|$$
$$= \left| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<s<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(s)} \right) ds \right.$$
$$\left. +C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - L \right| \to 0$$

uniformly in $u \in K$, as $t \to +\infty$.
It follows for the derivative that

$$|T(u)'(t) - \lim_{t\to+\infty} T(u)'(t)|$$
$$= \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right.$$
$$\left. -\phi^{-1}\left(\lim_{t\to+\infty} \frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right| \to 0$$

uniformly in $u \in K$, as $t \to +\infty$, and

$$|T(u)'(t) - \lim_{t\to-\infty} T(u)'(t)|$$
$$= \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right.$$
$$\left. -\phi^{-1}\left(\lim_{t\to-\infty} \frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right|$$
$$\leq \phi^{-1}\left(\left| \frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right| \right)$$
$$\leq \phi^{-1}\left(\frac{\int_{-\infty}^{t} \omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<t_k<t<+\infty} \chi_{n_1,\rho_1}}{a(t)} \right) \to 0$$

uniformly in $u \in K$, as $t \to -\infty$.

Therefore, TK is equiconvergent at $\pm\infty$ and by Theorem 1, TK is relatively compact.

Step 5. *$T : X \to X$ has a fixed point.*

To be able to apply Schauder's fixed-point theorem for the operator $T(u)$, we have to prove that $TD \subset D$, for some, bounded, closed and convex $D \subset X$.

Let us consider
$$D := \{u \in X : \|u\|_X \leq \rho_2\},$$
with $\rho_2 > 0$ such that
$$\rho_2 \geq \max \begin{cases} \rho_1, \\ \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(r)dr + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(s)} \right) ds + |C| + \sum_{-\infty<t_k<+\infty} \Psi_{k,\rho}, \\ \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(r)dr + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(t)} \right) \end{cases}$$
with ρ_1 given by (9).

Following arguments similar to step 2, we have that for $u \in D$
$$\|T(u)\|_X = \max\{\|T(u)\|_\infty, \|(T(u))'\|_\infty\} \leq \rho_2,$$

and $TD \subset D$. Then, the operator $T(u)$, by Theorem 2, has a fixed point $u \in X$. Using standard arguments, we can demonstrate that this fixed point determines a pair of heteroclinic or homoclinic solutions for the problem (1)-(3). $\square$

3. Example of Application of the Main Result and a Concrete Case of Application: Model for Studying the Dynamics of Bird Population Growth in the Natural Reserve

3.1. Example of Application of the Main Result

Let us consider the following second-order nonlinear system

$$\left((1+t^6)(u'(t))^5\right)' = \frac{t^2}{(1+t^4)^2}\left[(u(t))^2 + (u'(t))^3 + u(t)u'(t)\right] \tag{10}$$

together with the boundary conditions

$$u(-\infty) = C, \;\; u(+\infty) = L. \tag{11}$$

for $C, L \in \mathbb{R}$, and the generalized, impulse conditions

$$\begin{cases} \Delta u(t_k) = \frac{1}{k^4}(\alpha_1 \sqrt[5]{u(t_k)} + \alpha_2 \sqrt[5]{u'(t_k)}), \\ \\ \Delta\phi(u')(t_k) = \frac{1}{k^2}(\alpha_3 u(t_k) + \alpha_4 u'(t_k)), \end{cases} \tag{12}$$

with $\alpha_i \in \mathbb{R}$, $i = 1,2,3,4.$ and for $k \in \mathbb{N}$, $\cdots < t_1 < \cdots < t_k < \cdots,.$

The above system (10)–(12) happens to be a particular case of problem (1)–(3). For example, for $\rho > 0$ so that

$$\rho := \max\{|x|, |y|\}, \tag{13}$$

taking $x = u(t), y = u'(t)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is an L^1-Carathéodory function being

$$\begin{aligned} f(t,x,y) &= \frac{t^2}{(1+t^4)^2}(x^2 + y^3 + xy), \\ &\leq \frac{t^2}{(1+t^4)^2}(2\rho^2 + \rho^3) := \omega_\rho(t), \end{aligned}$$

where $\omega_\rho(t) \in L^1(\mathbb{R})$,

$$\phi(y) = y^5, \;\; a(t) = 1 + t^6,$$

and I_k, J_k, are Carathéodory sequences that satisfy Definition 2, as for each k, we set

$$I_k(t_k,x,y) \leq \frac{\sqrt[5]{\rho}(|\alpha_1| + |\alpha_2|)}{k^4}, \;\; J_k(t_k,x,y) \leq \frac{\rho(|\alpha_3| + |\alpha_4|)}{k^2}.$$

with $t_k = k$, $k \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $i = 1,2,3,4.$

So, evaluating the following series, we obtain

$$\sum_{k=1}^{+\infty} I_k(t_k,x,y) \leq \sum_{k=1}^{+\infty} \frac{\sqrt[5]{\rho}(|\alpha_1| + |\alpha_2|)}{k^4} = \frac{\pi^4 \sqrt[5]{\rho}(|\alpha_1| + |\alpha_2|)}{90}$$

and

$$\sum_{k=1}^{+\infty} J_k(t_k,x,y) \leq \sum_{k=1}^{+\infty} \frac{\rho(|\alpha_3| + |\alpha_4|)}{k^2} = \frac{\pi^2 \rho(|\alpha_3| + |\alpha_4|)}{6}$$

Conditions (H1), (H2) hold, being that

- $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, $|\phi^{-1}(y)| = |\sqrt[5]{y}| = \phi^{-1}(|y|) = \sqrt[5]{|y|}$;

- $\lim_{t\to\pm\infty}\frac{1}{a(t)} = \lim_{t\to\pm\infty}\frac{1}{1+t^6} = 0.$

Finally, note that

$$\int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{\int_{-\infty}^{+\infty}\omega_\rho(r)dr+|M|+\sum_{-\infty<t_k<t<+\infty}J_k(t_k,x,y)}{a(s)}\right)ds+|C|+\sum_{-\infty<t_k<t<+\infty}I_k(t_k,x,y)$$

$$\leq\int_{-\infty}^{+\infty}\left(\sqrt[5]{\frac{\int_{-\infty}^{+\infty}\frac{r^2}{(1+r^4)^2}(2\rho^2+\rho^3)dr+|M|+\frac{\pi^2\rho(|\alpha_3|+|\alpha_4|)}{6}}{1+s^6}}\right)ds+|C|+\frac{\pi^4\sqrt[5]{\rho}(|\alpha_1|+|\alpha_2|)}{90}<\rho$$

is finite. For example, taking $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$, $M=C=0$ and using the Maple software, we find an approximated fixed point for $\rho=382.8775379$. For slightly smaller values of ρ, the above inequality is false. So, by Theorem 3, there is at least $u\in X$, that is a solution to the problem (10)–(12).

3.2. A Possible and Specific Case of Application: Model for Studying the Complex Dynamics of Bird Population Growth

The study of bird population growth is important and one of the most studied aspect of avian physiology [34] and appears in studies on the global decline of biodiversity, population ecology and others [35–38].

Differential equations involving impulses and ϕ-Laplacian are widely used to model population dynamics (see [10,39,40]) and therefore also bird population growth. Works on bird population growth involving impulses are scarce in the literature. Just to mention a few, we have [41], where an impulse control model is used to manage the bird population and [42], where the authors study impulse dispersal in single-species models. Motivated by these works, we consider a specific example adapted from existing models (see [38,43–47]) of bird population growth, which is represented by an ordinary differential equation system that describes and captures the complexity of bird population dynamics, considering both the influence of continuous factors and discrete events in a specific area.

Therefore, by adapting a nonlinear second-order differential equation based on existing models, we aim to accurately model and represent the complex dynamics of bird populations as closely as possible by

$$\Big((1+t^2)u'(t)\Big)'=\frac{t^2}{(1+t^4)^2}\left[-\beta(t)u(t)-\alpha(t)u'(t)+\gamma(t)\left(1-\frac{u(t)}{K}\right)u(t)+\eta(t)\exp(-0.02u(t))\right]$$

where

- $u(t)$ is the population size of the bird species at time t;
- $u'(t)$ represents the rate of change of the population;
- $(1+t^2)u''(t)$ represents the population acceleration, capturing rapid changes in the growth rate;
- K is the carrying capacity of the environment;
- $\beta(t)$ is a time-dependent coefficient representing factors like seasonal variations that affect the growth rate;
- $\alpha(t)$ is a time-dependent function modeling damping effects such as environmental resistance or intraspecific competition;
- $\gamma(t)$ is a time-dependent coefficient representing seasonal variations in the growth rate;

- $\eta(t)$ is an additional coefficient representing other specific environmental influences;
- $\gamma(t)\left(1-\frac{u(t)}{K}\right)u(t)$ is a modified logistic growth component.

The initial condition indicates that the population starts with 1000 individuals:

$$u(-\infty) = 1000.$$

After a long period, the population stabilizes at 50,000 individuals:

$$u(+\infty) = 50000.$$

The model also incorporates impulse terms to account for specific events that affect the bird population:

$$\begin{cases} \Delta u(t_k) = I_k(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) = J_k(t_k, u(t_k), u'(t_k)), \end{cases}$$

to model abrupt changes in population size and growth rate due to specific events with the following:

- The moments of impulse are $t_k = k$ for $k \in \mathbb{N}$;
- $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$: represents the change in population size at discrete time points t_k;
- $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$: represents the change in growth rate at t_k;
- The functions I_k and J_k are defined as

$$\begin{cases} I_k(t_k, u(t_k), u'(t_k)) = \frac{1}{k^2}\left(\alpha_1\sqrt{u(t_k)} + \alpha_2\sqrt{u'(t_k)}\right), \\ J_k(t_k, u(t_k), u'(t_k)) = \frac{1}{k}(\beta_1 u(t_k) + \beta_2 u'(t_k)), \end{cases}$$

where α_i and β_j are real constants for $i, j = 1, 2$.

The function ϕ is an increasing homeomorphism, which is often used to model nonlinear effects influencing population dynamics. For this model, we assume the following:

$$\phi(x) = x,$$

which simplifies the model by assuming linear effects.

The general form of the function f is defined as

$$f(t, u(t), u'(t)) = \frac{t^2}{(1+t^4)^2}\Bigg[-\beta(t)u(t) - \alpha(t)u'(t) + \gamma(t)\left(1-\frac{u(t)}{K}\right)u(t) + \eta(t)\exp(-0.02u(t))\Bigg]$$

representing an L^1-Carathéodory function.

4. Discussion

The model employs advanced mathematical techniques to analyze nonlinear differential equations with impulsive effects, ensuring robust solutions under specific conditions. It accurately represents bird population dynamics by integrating both continuous and discrete factors, thus supporting effective management strategies.

Impulse terms simulate sudden population changes due to environmental events or human actions. The generalized ϕ-Laplacian homeomorphism simplifies the model while capturing essential nonlinear effects. The initial and boundary conditions, generalized to accommodate various starting and stabilizing population sizes, demonstrate the model's capability to predict long-term population trends. These conditions ensure that the model aligns with observed population dynamics over extended periods.

In this study, we considered several assumptions and aspects, such as the generalized ϕ-Laplacian being a homeomorphism, specific forms of impulse conditions, asymptotic boundary conditions, and the way nonlinearities appear. Relaxing these assumptions or considering alternative forms could add depth to the analysis; however, it could also affect the problem's well-posedness. Examining the impact of these changes would enhance the understanding of the model's flexibility and its applicability to a broader range of ecological scenarios.

Changes in the assumptions or components of the boundary value problem can lead to variations in the solutions—both in their existence and behavior. Alterations in impulse conditions can affect the stability and profile of the solutions. Similarly, changes in nonlinearities can result in different solutions, such as bifurcations or chaotic behavior. Relaxing the assumptions may yield weak solutions, ill-posed problems, or necessitate both numerical and theoretical methods to find solutions.

In summary, any variations or changes in the problem may require entirely different methods and necessitate analytical and numerical tools to adequately understand and solve the problem.

5. Conclusions

This work establishes the existence of heteroclinic solutions for strongly nonlinear second-order equations using Schauder's fixed-point theorem. By incorporating a generalized nonlinear ϕ-Laplacian operator and accommodating infinite impulses of varying intensity, we extend the existing literature significantly. Our approach utilizes a generalized ϕ-Laplacian homeomorphism to manage nonlinearities, employs equiconvergence at impulsive moments to control jumps, and leverages Carathéodory sequences to ensure solution existence.

One of the problems proposed in the practical part attempts to capture bird population dynamics through a nonlinear second-order differential equation, integrating both continuous and discrete factors. This adaptability allows it to simulate a variety of ecological scenarios, including abrupt environmental changes and management interventions. Theoretical results affirm the existence of solutions, offering a robust foundation for future studies and simulations. Future research should refine the model using empirical data and incorporate additional factors like predation, disease, and migration to enhance bird population management and conservation efforts.

Our research enhances the understanding of bird population dynamics, particularly in handling discrete and sudden events that impact populations. By combining continuous and impulsive approaches, the model offers comprehensive and realistic ecological system modeling.

Overall, this model is a powerful tool for ecologists and conservationists, facilitating the better prediction and management of bird populations. Future work should refine impulse functions with empirical data, expand the model to include factors such as predation and migration, explore different forms of the ϕ-Laplacian, adjust boundary conditions, and consider higher-order differential equations. These enhancements will improve conservation strategies and habitat management.

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