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On the Convergence of a Kurchatov-Type Method for Solving Nonlinear Equations and Its Applications

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Abstract: A local and a semi-local convergence analysis are presented for the Kurchatov-type method to solve numerically nonlinear equations in a Banach space. The method depends on a real parameter. By specializing the parameter, we obtain methods already studied in the literature under different types of conditions, such us Newton's, and Steffensen's, and Kurchatov's methods, the Secant method, and other methods. This study is carried out under generalized conditions for first-order divided differences, as well as first-order derivatives. Both in the local case and in the semi-local case, the error estimates, the radii of the region of convergence, and the regions of the solution's uniqueness are determined. A numerical majorizing sequence is constructed for studying semi-local convergence. The approach of restricted convergence regions is used to develop a convergence analysis of the considered method. The new approach allows a comparison of the convergence of different methods under a uniform set of conditions. In particular, the assumption of generalized continuity used to control the divided difference provides more precise knowledge on the location of the solution as well as tighter error estimates. Moreover, the generality of the approach makes it useful for studying other methods in an analogous way. Numerical examples demonstrate the applicability of our theoretical results.

Keywords: Kurchatov-type method; derivative-free method; local and semi-local convergence; convergence ball; Banach space

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1. Introduction

Many mathematical models that describe physical or technological processes require solving nonlinear problems. These can include systems of nonlinear algebraic or transcendental equations, nonlinear integral equations, nonlinear boundary value problems for ordinary differential equations, and more complex problems described by nonlinear partial differential equations. Generally, these problems are represented by an equation of the form [1–3]

$$F(z) = 0. \tag{1}$$

Here, $F : D \subset B_1 \to B_2$ is a nonlinear operator, B_1 and B_2 denote Banach spaces, and D is an open and convex set. Recall that Banach is a complete linear normed space, that is, a linear space equipped with some norm such that every Cauchy sequence converges [4]. Moreover, the operator $F : D \to B_2$ is said to be Fréchet-differentiable at $x \in D$ if there exists a bounded linear operator A from D into B_2 such that

$$\lim_{h\to 0} \frac{1}{\|h\|} \|F(x+h) - F(x) - A(h)\| = 0.$$

The linear operator A is denoted by $F'(x)$, and is called the Fréchet derivative of F at x [4]. Furthermore, let $\{x_n\}$ be a sequence in B_1. Then, a sequence $\{m_n\} \subset [0, \infty)$ for which

$$\|x_{n+1} - x_n\| \leq m_{n+1} - m_n \text{ for each } n = 0, 1, 2, \dots$$

holds is a majorizing sequence for $\{x_n\}$ [4].

It is very rare to find an exact solution to such problems. Therefore, an important task is the development and study of numerical methods for solving (1). Nonlinear problems are usually solved by iterative methods, in particular, by methods with derivatives and methods with divided differences.

The most widely used method for solving the nonlinear Equation (1) is Newton's with the quadratic convergence order [1,2]

$$\begin{aligned} z_0 &\in D, \\ z_{n+1} &= z_n - [F'(z_n)]^{-1} F(z_n), \; n \geq 0. \end{aligned} \tag{2}$$

But it can be applied only for the differentiable operator F. If there are difficulties with the calculation of the derivative, then we can apply the approximation of the derivative by the first-order divided differences [3,5,6].

Definition 1. *Let F be a nonlinear operator defined on a subset D of a Banach space B_1 with values in a Banach space B_2, and let x, y be two points of D. A linear operator from B_1 to B_2 which is denoted by $[x, y; F]$ and satisfies the following conditions is called a first-order divided difference of F at the points x and y:*

(1) For all points, $x, y \in D$ and $x \neq y$

$$[x, y; F](x - y) = F(x) - F(y),$$

(2) If there exists a Fréchet derivative $F'(x)$, then

$$[x, x; F] = F'(x),$$

For Fréchet-differentiable operators, the following equality holds:

$$[x, y; F] = \int_0^1 F'(x + t(y - x)) dt.$$

One of the methods with divided differences is the Secant method [3,7]:

$$\begin{aligned} z_{-1}, z_0 &\in D, \\ z_{n+1} &= z_n - [z_n, z_{n-1}; F]^{-1} F(z_n), \; n \geq 0 \end{aligned} \tag{3}$$

with a convergence order that is equal to $\frac{1+\sqrt{5}}{2}$. The method of linear interpolation (Kurchatov method), such as Newton's, has a quadratic convergence order and is described by the formula [1,6]

$$\begin{aligned} z_{-1}, z_0 &\in D, \\ z_{n+1} &= z_n - [2z_n - z_{n-1}, z_{n-1}; F]^{-1} F(z_n), \; n \geq 0. \end{aligned} \tag{4}$$

The order of convergence of method (4) is theoretically obtained under the assumption that the first- and second-order divided differences of the nonlinear operator satisfy the classical Lipschitz conditions. Derivative-free methods are often employed to solve nonlinear problems involving a non-differentiable operator.

In this article, we study the uniparametric family of Kurchatov-type methods

$$\begin{aligned} \lambda &\in \mathbb{R},\, z_{-1}, z_0 \in D, \\ x_n &= (1-\lambda)z_n + \lambda z_{n-1}, \\ y_n &= (1+\lambda)z_n - \lambda z_{n-1},\, A_n = [x_n, y_n; F], \\ \textit{and } z_{n+1} &= z_n - A_n^{-1}F(z_n),\, n \geq 0. \end{aligned} \tag{5}$$

It should be noted that by setting $\lambda = 1$ in method (5), the Kurchatov method (4) is obtained. If $\lambda = 0$ and the operator F is differentiable, then we obtain the Newton method (2). Other choices of λ are possible, leading to other methods [1,3,6].

Motivation for the paper.

There are certain restrictions limiting the applicability of (5). This method was proposed in [6]. The local convergence was studied under the condition that $F \in C^4(D)$, while the semi-local convergence was analyzed for a non-differentiable integral operator F. Let us look at a toy example. Choose $D = (-1.5, 1.5)$ and define the function $f : D \to \mathbb{R}$ as

$$f(t) = \begin{cases} \alpha_1 t^3 \log t^4 + \alpha_2 t^5 + \alpha_3 t^4, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and satisfy $\alpha_1 \neq 0, \alpha_2 + \alpha_3 = 0$. It follows based on the definition of f that $f^{(3)}(t)$ is not continuous at $t = 0 \in D$. Consequently the results in [6] cannot assure that $\lim_{n\to\infty} z_n = z^*$, which denotes a solution to the equation $f(t) = 0$. However, method (5) converges to the solution $z^* = 1 \in D$, if, e.g., $z_{-1} = 0.95$, $z_0 = 1.05$, $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = -1$. These observations indicate that the conditions in [6] can be replaced by new ones that are weaker.

The convergence analysis in [6] uses conditions on $F \in C^{(4)}(D)$. But such derivatives do not appear in the method.

Novelty of the paper.

The new local and the semi-local convergence analyses are shown using conditions only on the operators which are present in method (5), that is to say F and its divided difference of order one. The analysis is valid in the Banach space for operators more general than an integral equation. The generalized continuity used to control the divided difference allows for tighter estimates of $||z_n - z^*||$ as well as better knowledge on the location of the solution z^*.

As can be seen in Sections 2 and 3, the developed approach is very general. Thus, it can be used to extend the applicability of other methods along the same lines [4,8–15]. Another advantage of this approach is that a comparison between different methods studied under different conditions becomes possible.

This paper is structured as follows: We conduct a local and a semi-local convergence analysis of method (5) under generalized conditions for first-order divided differences, as well as first-order derivatives, using the approach of restricted convergence regions. These results are presented in Sections 2 and 3, respectively. In both cases, the uniqueness regions for the solution of the nonlinear problem are obtained. Furthermore, in Section 4, we present numerical examples to demonstrate the reliability of the theoretical results. The concept of our investigation is succinctly represented in Figure 1.

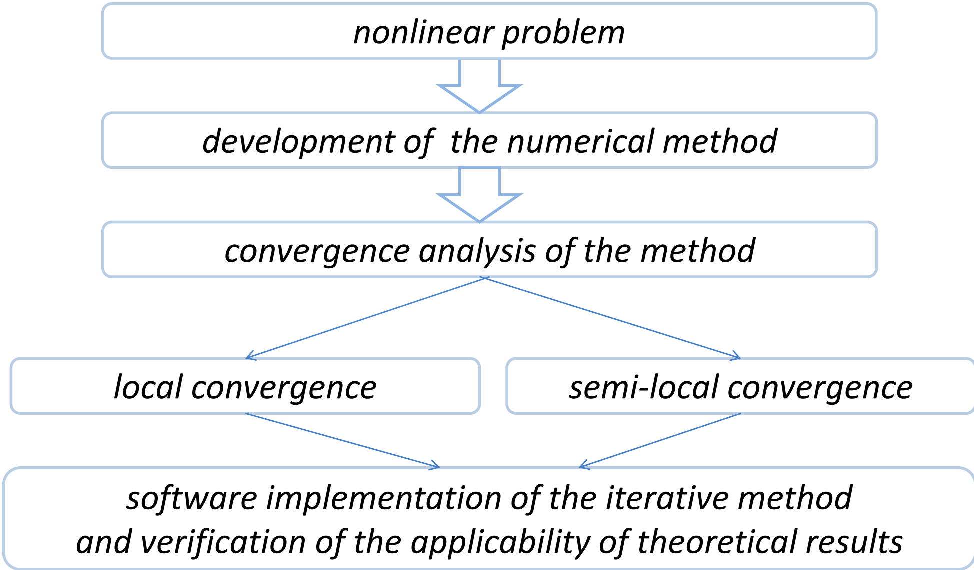

Figure 1. The concept of the investigation.

2. Local Convergence

The results of the local analysis are important, since they illustrate how difficult it is to select initial points.

Certain real functions play a role in the local convergence analysis of method (5). The notations $U(x,r)$ and $U[x,r]$ denote open and closed balls, respectively, centered at the point x with the radius r. Let us set $S = [0, \infty)$.

Define the functions $g_1 : S \to S$ and $g_2 : S \to S$ as

$$g_1(t) = (|1-\lambda| + |\lambda|)t \quad and \quad g_2(t) = (|1+\lambda| + |\lambda|)t. \tag{6}$$

Suppose the following:

(C_1) There exists the function $\omega_0 : S \times S \to S$, which is continuous on $S \times S$ and strictly increasing in both variables such that equation $\omega_0(g_1(t), g_2(t)) - 1 = 0$ has at least one positive root. We denote using ρ_0 the smallest such root and set $S_0 = [0, \rho_0)$.

(C_2) There exists the function $\omega : S_0 \times S_0 \to S$, which is continuous on $S_0 \times S_0$ and strictly increasing in both variables such that for function $h : S_0 \to S$, given by

$$h(t) = \frac{\omega(t + g_1(t), g_2(t))}{1 - \omega_0(g_1(t), g_2(t))} \tag{7}$$

equation $h(t) - 1 = 0$ has at least one root in the interval $(0, \rho_0)$. We denote using r^* the smallest such root and set $S_1 = [0, r^*)$.

It follows according to these definitions that for each $t \in S_1$,

$$0 \le \omega_0(g_1(t), g_2(t)) < 1 \tag{8}$$

and

$$0 \le h(t) < 1. \tag{9}$$

The parameter r^* is shown to be a radius of convergence for method (5) in Theorem 1. Define the parameter

$$\rho^* = \max\{|1-\lambda| + |\lambda|, |1+\lambda| + |\lambda|\}r^*. \tag{10}$$

There is a connection between the real functions ω_0 and ω and the operators in method (5).

(C_3) There exists a solution $z^* \in D$ to the equation $F(z) = 0$, $L \in \mathcal{L}(B_1, B_2)$ such that $L^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|L^{-1}([x, y; F] - L)\| \leq \omega_0(\|x - z^*\|, \|y - z^*\|)$$

for each $x, y \in D$. Set $D_0 = D \cap U(z^*, \rho_0)$.

(C_4) $\|L^{-1}([x, y; F] - [z, z^*; F])\| \leq \omega(\|x - z\|, \|y - z^*\|)$ for each $z, y, x \in D_0$.

(C_5) $U(z^*, \rho^*) \subset D_0$.

Remark 1.

(1) *Some popular selections, but not necessarily the most flexible for the operator, are $L = I$ or $L = F'(\bar{x})$, or in particular, $L = F'(z^*)$, where $\bar{x}$ is an auxiliary point. In the case of $L = F'(z^*)$, the solution z^* is simple. However, this assumption is not made or implied by the conditions (C_1)–(C_5). Consequently, our results can be used to find solutions to multiplicity greater than one using method (5).*

(2) *The proof of Theorem 1 that follows shows that the condition (C_4) can be replaced by (C_4') $\|L^{-1}([x, y; F] - [z, z^*; F])\| \leq \overline{\omega}(\|x - z\|, \|y - z^*\|)$ for each $y, x \in D_1$ and $z = z_{n+1} = z_n - A_n^{-1}F(z_n)$, $z \in U(z^*, \rho_0)$, where $\overline{\omega}$ is as ω. In this case, $\overline{\omega} \leq \omega$ and the results are more precise. However, the condition (C_4') is verified only in special cases.*

Next, the local convergence of method (5) relies on the conditions (C_1)–(C_5) and the preceding notation.

Theorem 1. *Suppose that conditions (C_1)–(C_5) hold and choose $z_0, z_{-1} \in U(z^*, r^*)$ such that $z_0 \neq z_{-1}$. Then, for $x_0 = (1 - \lambda)z_0 + \lambda z_{-1}$ and $y_0 = (1 + \lambda)z_0 - \lambda z_{-1}$, the sequence $\{z_n\}$ generated by method (5) is well defined in $U(z^*, r^*)$ for each $n = 0, 1, \ldots$ and converges to the solution $z^* \in U(z^*, r^*)$ of the equation. Moreover, the following error estimates hold for each $n = 0, 1, \ldots$:*

$$\begin{aligned} \|z_{n+1} - z^*\| &\leq \frac{\omega(\|z_n - z^*\| + \|x_n - z^*\|, \|y_n - z^*\|)}{1 - \omega_0(\|x_n - z^*\|, \|y_n - z^*\|)} \|z_n - z^*\| \\ &< h(r^*)\|z_n - z^*\| = \|z_n - z^*\| < r^*. \end{aligned} \tag{11}$$

Proof. The estimate (11) is shown through mathematical induction. According to hypothesis $z_0, z_{-1} \in U(z^*, r^*)$. We can write, in turn, that

$$x_0 - z^* = (1 - \lambda)z_0 + \lambda z_{-1} - z^* = (1 - \lambda)(z_0 - z^*) + \lambda(z_{-1} - z^*),$$

so

$$\|x_0 - z^*\| \leq |1 - \lambda|\|z_0 - z^*\| + |\lambda|\|z_{-1} - z^*\| = (|1 - \lambda| + |\lambda|)r^* \leq \rho^*. \tag{12}$$

Similarly, we obtain

$$\|y_0 - z^*\| \leq (|1 + \lambda| + |\lambda|)r^* \leq \rho^*. \tag{13}$$

Thus, according to condition (C_5), we have $y_0, x_0 \in U[z^*, \rho^*]$. Notice that $y_0 \neq x_0$, since $z_0 \neq z_{-1}$. Thus, the divided difference A_0 is well defined. Next, we show that $A_0 = [x_0, y_0; F]$ is invertible.

Using (6), (8), (12) and (13) and condition (C_3), we determine, in turn, that

$$\|L^{-1}(A_0 - L)\| \leq \omega_0(\|x_0 - z^*\|, \|y_0 - z^*\|) \leq \omega_0(\rho^*, \rho^*) < 1. \tag{14}$$

The Banach Lemma on invertible linear operators [4] and (14) implies that $A_0^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|A_0^{-1}L\| \leq \frac{1}{1 - \omega_0(\|x_0 - z^*\|, \|y_0 - z^*\|)}. \tag{15}$$

Moreover, the iterate z_1 is well defined by the third substep of method (5) for $n = 0$. We need to show that $z_1 \in U(z^*, r^*)$ and (9) holds if $n = 0$. The third substep of method (5) gives

$$z_1 - z^* = z_0 - z^* - A_0^{-1}F(z_0) = A_0^{-1}(A_0 - [z_0, z^*; F])(z_0 - z^*). \tag{16}$$

According to (8)–(11), (15) and (16) and conditions (C_3) and (C_4), we determine, in turn, that

$$\begin{aligned}
\|z_1 - z^*\| &\leq \|A_0^{-1}L\|\|L^{-1}(A_0 - [z_0, z^*; F])\|\|z_0 - z^*\| \\
&\leq \frac{\omega(\|x_0 - z_0\|, \|y_0 - z^*\|)}{1 - \omega_0(\|x_0 - z^*\|, \|y_0 - z^*\|)}\|z_0 - z^*\| \\
&\leq \frac{\omega(\|z_0 - z^*\| + \|x_0 - z^*\|, \|y_0 - z^*\|)}{1 - \omega_0(\|x_0 - z^*\|, \|y_0 - z^*\|)}\|z_0 - z^*\| \\
&< h(r^*)\|z_0 - z^*\| \leq \|z_0 - z^*\| < r^*
\end{aligned} \tag{17}$$

showing that (11) if $n = 0$ and the iterate $z_1 \in U(z^*, r^*)$, where we used

$$\|x_0 - z_0\| \leq \|x_0 - z^*\| + \|z_0 - z^*\|.$$

The preceding calculations can be repeated simply by exchanging z_{-1}, z_0, A_0 with z_{m-1}, z_m, A_m, respectively, where m is a natural number. So, we obtain

$$\begin{aligned}
\|z_{m+1} - z^*\| &\leq \frac{\omega(\|x_m - z_m\|, \|y_m - z^*\|)}{1 - \omega_0(\|x_m - z^*\|, \|y_m - z^*\|)}\|z_m - z^*\| \\
&\leq \frac{\omega(\|z_m - z^*\| + \|x_m - z^*\|, \|y_m - z^*\|)}{1 - \omega_0(\|x_m - z^*\|, \|y_m - z^*\|)}\|z_m - z^*\| \\
&< h(r^*)\|z_m - z^*\| = \|z_m - z^*\| < r^*
\end{aligned} \tag{18}$$

which completes the induction for (11) and also shows that the iterate $z_{m+1} \in U(z^*, r^*)$.

Finally, according to (18), there exists $\alpha \in [0, 1)$ such that

$$\|z_{m+1} - z^*\| \leq \alpha\|z_m - z^*\| \leq \alpha^{m+1}\|z_0 - z^*\| < r^*. \tag{19}$$

Consequently, according to (19), we conclude that the iterate $z_{m+1} \in U(z^*, r^*)$ and $\lim_{m\to\infty} z_m = z^*$. □

A region is determined in the next result which contains only z^* as a solution to the equation $F(z) = 0$.

Proposition 1. *Suppose the following:*

(a) The condition (C_3) holds in $U(z^*, r_1)$ for some $r_1 > 0$.

(b) There exists $r_2 \geq r_1$ such that

$$\omega_0(r_2, 0) < 1. \tag{20}$$

Set $D_1 = D \cap U[z^*, r_2]$.

Then, the only solution to the equation $F(z) = 0$ in the region D_1 is z^*.

Proof. Suppose that there exists $\tilde{x} \in D_1$, solving the equation $F(z) = 0$, and $\tilde{z} \neq z^*$. It follows that the divided difference $M = [\tilde{z}, z^*; F]$ is well defined. Then, according to (a)–(b), we obtain

$$\|L^{-1}(M - L)\| \leq \omega_0(\|\tilde{z} - z^*\|, 0) \leq \omega_0(r_2, 0) < 1. \tag{21}$$

Then, according to (21) and the Banach Lemma on invertible linear operators, we determine that $M^{-1} \in \mathcal{L}(B_2, B_1)$. Thus, from the identity

$$\tilde{z} - z^* = M^{-1}(F(\tilde{z}) - F(z^*)) = M^{-1}(0 - 0) = M^{-1}(0) = 0,$$

we determine that $\tilde{z} = z^*$. $\square$

Remark 2. *Clearly, if all the conditions* (C_1)–(C_5) *hold in Proposition 1, then we can certainly choose* $r_1 = r^*$.

3. Semi-Local Convergence

This analysis uses majorizing sequences [3,4] developed to control the iterate $\{z_n\}$.

The conditions and computations are similar to the local analysis of method (5). But the roles of z^*, ω_0, and ω are exchanged with z_0, v_0, and v, respectively, where v_0 and v are real functions.

Suppose the following:

(H_1) There exists the function $v_0 : S \times S \to S$, which is continuous on $S \times S$ and nondecreasing in both variables such that equation $v_0(g_1(t), g_2(t)) - 1 = 0$ has at least one positive root. We denote using R_0 the smallest such root. Set $S_2 = [0, R_0)$.

(H_2) There exists the function $v : S_2 \times S_2 \to S$, which is continuous on $S_2 \times S_2$ and nondecreasing in both variables. Define the sequence $\{a_n\}$ for $a_{-1} = 0$, $a_0 \geq 0$, $a_1 \geq a_0$, and each $n = 0, 1, \ldots$ as

$$a_{n+2} = a_{n+1} + \frac{v(a_{n+1} - a_n + |\lambda|(a_n - a_{n-1}), |\lambda|(a_n - a_{n-1}))}{1 - v_0(|1 - \lambda|a_n + |\lambda|a_{n-1}|, |1 + \lambda|a_n + |\lambda|a_{n-1})}(a_{n+1} - a_n). \quad (22)$$

The sequence $\{a_n\}$ shall be shown to be majorizing for $\{z_n\}$ in Theorem 2. But first, a general convergence condition is given for the sequence $\{a_n\}$.

(H_3) There exists $R \in [0, R_0)$ such that for each $n = 0, 1, \ldots$,

$$v_0(|1 - \lambda|a_n + |\lambda|a_{n-1}|, 1 + \lambda|a_n + |\lambda|a_{n-1}) < 1 \quad \textit{and} \quad a_n \leq R_0.$$

It follows based on the initial conditions that $a_{-1} \leq a_0 \leq a_1$. Then, according to (22) for $n = 0$, the condition (H_3), and the hypothesis that the functions v_0 and v are nondecreasing in each variable, it follows that $a_1 \leq a_2$. Suppose that $a_m \leq a_{m+1}$ for all integers $m = 0, 1, 2, \ldots, n$. Then, according to the same hypothesis about the functions v_0 and v and (H_3), it follows that $a_{m+1} \leq a_{m+2}$, which completes the induction for

$$0 \leq a_n \leq a_{n+1} \leq R_0$$

and there exists $a^* \in [0, R_0]$ such that

$$\lim_{n \to +\infty} a_n = a^*.$$

There is a connection between the real functions v_0 and v and the operators in method (5).

(H_4) There exists a point $z_0 \in D$, $L \in \mathcal{L}(B_1, B_2)$ such that $L^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|L^{-1}([x, y; F] - L)\| \leq v_0(\|x - z_0\|, \|y - z_0\|).$$

Let z_{-1}, $z_0 \in D$. Then take $\|z_0 - z_{-1}\| \leq a_0$. We can write, based on the first two substeps of method (5),

$$x_n - z_0 = (1 - \lambda)z_n + \lambda z_{n-1} - z_0 = (1 - \lambda)(z_n - z_0) + \lambda(z_{n-1} - z_0),$$

$$\|x_n - z_0\| \leq |1 - \lambda|\|z_n - z_0\| + |\lambda|\|z_{n-1} - z_0\| \leq (|1 - \lambda| + |\lambda|)a^*$$

and similarly,

$$\|y_n - z_0\| \le |1+\lambda|\|z_n - z_0\| + |\lambda|\|z_{n-1} - z_0\| \le (|1+\lambda| + |\lambda|)a^*$$

provided that these iterates exists and belong in $U(z_0, \gamma)$.
In particular, for $n = 0$, the condition (H_4) gives

$$\|L^{-1}(A_0 - L)\| \le v_0((|1-\lambda| + |\lambda|)a^*, (|1+\lambda| + |\lambda|)a^*) < 1.$$

Hence, $A_0^{-1} \in \mathcal{L}(B_2, B_1)$ and the iterate z_1 is well defined by the third substep of method (5). Let us choose $a_1 \ge a_0 + \|A_0^{-1}F(z_0)\|$.
Set $D_3 = D \cap U(z_0, R_0)$.

(H_5) $\|L^{-1}([x, y; F] - [z, u; F])\| \le v(\|x - z\|, \|y - u\|)$ for each $y, x, z, u \in D_3$.
and

(H_6) $U(z_0, \gamma) \subset D_3$, where $\gamma = \max\{|1-\lambda| + |\lambda|, |1+\lambda| + |\lambda|\}a^*$.

Remark 3. *As in the local convergence analysis, possible choices for* $L = I$ *or* $L = F'(z_0)$ *or* $L = [u_1, u_2; F]$, *where* u_1, u_2 *are auxiliary points with* $u_1 \ne u_2$, *the last choice can be taken in the case when the operator* F *is not necessarily differentiable.*

The main semi-local convergence analysis of method (5) follows.

Theorem 2. *Suppose that the conditions* (H_1)–(H_6) *hold. Then, the sequence* $\{z_n\}$ *generated by method (5) is well defined in* $U(z_0, a^*)$ *and remains in* $U(z_0, a^*)$ *for each* $n = 0, 1, \ldots$, *and there exists a solution* $z^* \in U[z_0, a^*]$ *such that the sequence* $\{z_n\}$ *converges to* z^* *and*

$$\|z_n - z^*\| \le a^* - a_n. \tag{23}$$

Proof. Mathematical induction is used to establish the estimate

$$\|z_{m+1} - z_m\| \le a_{m+1} - a_m \tag{24}$$

for each $m = -1, 0, \ldots$. Estimate (24) holds for $m = -1, 0$ based on the initial conditions $\|z_0 - z_{-1}\| \le a_0 < a^*$ and $\|z_1 - z_0\| = \|A_0^{-1}F(z_0)\| \le a_1 - a_0 < a^*$. Moreover, we determine that the iterates z_1, $z_{-1} \in U(z_0, a^*)$. According to the arguments below the condition (H_3), the iterates $y_{m+1}, x_{m+1} \in U(z_0, \gamma)$.
We also have the estimate

$$\|L^{-1}(A_{m+1} - L)\| \le v_0(\|x_{m+1} - z_0\|, \|y_{m+1} - z_0\|) < 1.$$

Thus, $A_{m+1}^{-1} \in \mathcal{L}(B_2, B_1)$,

$$\|A_{m+1}^{-1}L\| \le \frac{1}{1 - v_0(\|x_{m+1} - z_0\|, \|y_{m+1} - z_0\|)}, \tag{25}$$

and the iterate z_{m+2} is well defined by the third substep of method (5). Furthermore, based on the condition (H_6), the iterate $z_{m+1} \in U(z_0, a^*)$. Then, we can write, based on the third substep of method (5),

$$F(z_{m+1}) = F(z_{m+1}) - F(z_m) - A_m(z_{m+1} - z_m) = ([z_{m+1}, z_m; F] - A_m)(z_{m+1} - z_m)$$

leading to

$$\begin{aligned}
\|z_{m+1}-z_m\| &\leq \|A_{m+1}^{-1}L\|\|L^{-1}F(z_{m+1})\| \\
&\leq \frac{v(\|z_{m+1}-x_m\|,\|z_m-y_m\|)\|z_{m+1}-z_m\|}{1-v_0(\|x_{m+1}-x_0\|,\|y_{m+1}-x_0\|)} \\
&\leq \frac{v(a_{m+1}-a_m+|\lambda|(a_m-a_{m-1}),|\lambda|(a_m-a_{m-1}))(a_{m+1}-a_m)}{1-v_0(|1-\lambda|a_m+|\lambda|a_{m-1},|1+\lambda|a_m+|\lambda|a_{m-1})} \\
&\leq a_{m+2}-a_{m+1}.
\end{aligned} \tag{26}$$

The induction for (24) is completed, and

$$\begin{aligned}
\|z_{m+2}-z_0\| &\leq \|z_{m+2}-z_{m+1}\|+\|z_{m+1}-z_0\| \\
&\leq a_{m+2}-a_{m+1}+a_{m+1}-a_0=a_{m+2}-a_0\leq a^*-a_0.
\end{aligned}$$

Hence, the iterate $z_{m+2}\in U(z_0,a^*)$. Therefore, the sequence $\{a_m\}$ is majorizing for $\{z_m\}$. So, there exists $z^*\in U[z_0,a^*]$ such that $\lim_{m\to+\infty} z_m=z^*$. According to (26), we obtained the estimate

$$\|L^{-1}F(z_{m+1})\|\leq v(a_{m+1}-a_m+|\lambda|(a_m-a_{m-1}),|\lambda|(a_m-a_{m-1}))(a_{m+1}-a_m). \tag{27}$$

If $m\to+\infty$ in (27), we conclude that $F(z^*)=0$. Finally, from (24) and the triangle inequality,

$$\|z_{m+i}-z_m\|\leq a_{m+i}-a_m,\quad i=0,1,2,\ldots. \tag{28}$$

Thus, by letting $i\to+\infty$ in (28), we deduce (23). □

Next, a region is specified that contains only one solution.

Proposition 2. *Suppose the following:*

(i) *The equation $F(z)=0$ has a solution $y^*\in U(z_0,R_2)$ for some $R_2>0$.*
(ii) *The condition (H_3) holds in the ball $U(z_0,R_2)$.*
(iii) *There exists $R_3\geq R_2$ such that*

$$v_0(R_3,R_2)<1. \tag{29}$$

Define the region $D_4=D\cap U[z_0,R_3]$.

Then, the only solution to the equation $F(z)=0$ in the region D_4 is y^*.

Proof. Suppose that the equation $F(z)=0$ has a solution $q\in D_4$ such that $q\neq y^*$. Then, the divided difference $T=[q,y^*;F]$ is well defined. In view of the conditions (ii) and (29), we determine, in turn, that

$$\|L^{-1}(T-L)\|\leq v_0(\|q-z_0\|,\|y^*-z_0\|)\leq v_0(R_3,R_2)<1.$$

Hence, T is invertible.
Finally, from the identity

$$q-y^*=T^{-1}(F(q)-F(y^*))=T^{-1}(0)=0,$$

we deduce that $q=y^*$. □

Remark 4.

(i) *Under the conditions (H_1)–(H_6), we can let $y_*=z^*$ and $R_2=a^*$.*
(ii) *It follows from the proof of Theorem 2 that the iterates $\{z_n\}\subset U(z_0,a^*-a_0)$.*

4. Numerical Examples

This section presents the results of verifying the convergence conditions of theorems 1 and 2 for method (5) and shows the applicability of the considered method for solving different nonlinear problems. The study was conducted for a nonlinear equation, a system of nonlinear equations, a Hammerstein integral equation, and a boundary value problem. These problems and similar ones are often used to test the applicability of iterative methods (see [1,3,4]). The nonlinear Hammerstein integral equations are a special case of Fredholm integral equations of the second kind and have a physical foundation, as they originate from electromagnetic fluid dynamics.The experiments were conducted in GNU Octave 7.3.0 software. The condition $\|z_{n+1} - z_n\| \leq \varepsilon$ was used for stopping the iterative process. The calculations were performed with $\varepsilon = 10^{-8}$ (for problems 1 and 2) and $\varepsilon = 10^{-5}$ (for problem 3), and the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{C[a,b]}$ were used.

Example 1. *Consider the system of m nonlinear equations*

$$F_i(z) = \sum_{j=1}^{m} z_j + e^{z_i} - 1 = 0, \quad i = 1, \ldots, m.$$

Here, $B_1 = B_2 = \mathbb{R}^m$, $D = (-1,1)^m \subset \mathbb{R}^m$ *and the exact solution is* $z^* = (0, \ldots, 0)^T$.

It is easy to see that the elements of the Jacobian matrix and the divided difference matrix have the following forms:

$$F'(z)_{i,j} = \begin{cases} e^{z_i} + 1, & i = j, \\ 1, & i \neq j, \end{cases} \quad \text{and} \quad [x, y; F]_{i,j} = \begin{cases} \frac{e^{x_i} - e^{y_i}}{x_i - y_i} + 1, & i = j, \\ 1, & i \neq j. \end{cases}$$

Let us consider a local case and choose $L = F'(z^*)$. Then, we have

$$F'(z^*)_{i,j} = \begin{cases} 2, & i = j, \\ 1, & i \neq j, \end{cases} \quad [F'(z^*)]^{-1}_{i,j} = \begin{cases} \alpha, & i = j, \\ \beta, & i \neq j, \end{cases}$$

and $L^{-1}([x, y; F] - L) = L^{-1} diag\left\{ \frac{e^{x_1} - e^{y_1}}{x_1 - y_1} - 1, \ldots, \frac{e^{x_m} - e^{y_m}}{x_m - y_m} - 1 \right\}$. Therefore, we can write that function ω_0 and ω have the following forms:

$$\omega_0(\|x - z^*\|, \|y - z^*\|) = \frac{(e-1)\|L^{-1}\|}{2}(\|x - z^*\| + \|y - z^*\|)$$

and

$$\omega(\|x - z\|, \|y - z^*\|) = \frac{e^{\min\{1, \rho_0\}}\|L^{-1}\|}{2}(\|x - z\| + \|y - z^*\|).$$

Let $m = 25$ and $\lambda = 0.4$. Then, $\rho_0 \approx 0.2206$, $D_0 \approx U(z^*, 0.2206)$, $r^* \approx 0.1111$, $U(z^*, r^*) \approx (-0.1111, 0.1111)$, $\rho^* \approx \max\{0.1111, 0.2000\} = 0.2000$, $U(z^*, \rho^*) \subset D_0$.

Table 1 shows results that are obtained for the initial approximations $z_0 = (0.1, \ldots, 0.1)^T$ and $z_{-1} = (0.11, \ldots, 0.11)$. Method (5) converges at three iterations. Thus, error estimate (11) holds for all $n \geq 0$, and the sequence $\{z_n\}_{n \geq -1}$ remains in $U(z^*, r^*)$ and converges to an exact solution.

Table 1. Error estimates (11) for Example 1.

n	$\|z_{n+1} - z^*\|$	**Right Side of Estimate (11)**
-1	1.0000×10^{-1}	-
0	2.0480×10^{-4}	3.4786×10^{-2}
1	3.3397×10^{-9}	2.4639×10^{-5}
2	4.1561×10^{-14}	4.0265×10^{-10}

Let us consider a semi-local case. Choosing L as $L = [x_0, y_0; F]$, we obtain the following functions:

$$v_0(\|x - z_0\|, \|y - z_0\|) = \frac{e\|L^{-1}\|}{2}(\|x - z_0\| + \|y - z_0\| + \|x_0 - z_0\| + \|y_0 - z_0\|)$$

and

$$v(\|x - z\|, \|y - u\|) = \frac{e^{\kappa_0}\|L^{-1}\|}{2}(\|x - z\| + \|y - u\|).$$

Let $m = 25$, $\lambda = 0.2$ and the initial approximations $z_0 = (0.1, \ldots, 0.1)^T$, $z_{-1} = (0.11, \ldots, 0.11)^T$. Then, we determine that $R_0 \approx 0.1784$, $D_3 \approx (-0.0785,\, 0.2784)^m$, and $\kappa_0 = 0.2784$, and the majorizing sequence

$$\{a_n\} = \{0,\, 0.0100,\, 0.1098,\, 0.1221,\, \ldots 0.1237\},$$

converges to $a^* \approx 0.1237$. The convergence ball is $U(z_0, a^*) \approx (-0.0237,\, 0.2237)^m$, and $\gamma \approx \max\{0.1237, 0.1732\} = 0.1732$ and $U(z_0, \gamma) \approx (-0.0732,\, 0.2732)^m \subset D_3$.

Table 2 shows that the error estimates (23) hold for all $n \geq 0$ and (24) holds for all $n \geq -1$. The sequence $\{z_n\}_{n \geq -1}$ remains in $U(z_0, a^*)$ and converges to an exact solution.

Table 2. Error estimates (23) and (24) for Example 1.

n	$\|z_{n+1} - z^*\|$	$a^* - a_{n+1}$	$\|z_{n+1} - z_n\|$	$a_{n+1} - a_n$
-2	1.1000×10^{-1}	1.2369×10^{-1}	-	-
-1	1.0000×10^{-1}	1.1369×10^{-1}	1.0000×10^{-2}	1.0000×10^{-2}
0	2.0480×10^{-4}	1.3894×10^{-2}	9.9795×10^{-2}	9.9795×10^{-2}
1	1.4398×10^{-9}	1.5640×10^{-3}	2.0480×10^{-4}	1.2330×10^{-2}
2	4.5643×10^{-15}	3.4364×10^{-5}	1.4398×10^{-9}	1.5296×10^{-3}

Example 2. *Consider the nonlinear integral equation*

$$F(z(t)) = z(t) - \alpha \int_0^1 tsz^3(s)ds = 0.$$

Here, $B_1 = B_2 = C_{[0,1]}$, $\alpha > 0$ is some constant and the exact solution is $z^*(t) = 0$.

Then, we can write

$$F'(z(t))h(t) = h(t) - 3\alpha \int_0^1 tsz^2(s)h(s)ds$$

and

$$[x(t), y(t); F]h(t) = h(t) - \alpha \int_0^1 ts\Big(x^2(s) + x(s)y(s) + y^2(s)\Big)h(s)ds.$$

Since $z^*(t) = 0$, then $F'(z^*(t))h(t) = h(t) - 3\alpha \int_0^1 ts(z^*(s))^2 h(s)ds = h(t) = Ih(t)$, where I is the identity operator. In the local case, we obtain for $L = F'(z^*(t))$ the following functions:

$$\omega_0(\|x - z^*\|, \|y - z^*\|) = 2\alpha(\|x - z^*\| + \|y - z^*\|)$$

and

$$\omega(\|x - z\|, \|y - z^*\|) = 2\min\{1, \rho_0\}\alpha(\|x - z\| + \|y - z^*\|).$$

Let us choose $\lambda = 0.1$ and $\alpha = 1$. Then, $\rho_0 \approx 0.2273$, $D_0 \approx U(z^*,\, 0.2273)$, $r^* \approx 0.1708$, $\rho^* \approx \max\{0.1708, 0.2050\} = 0.2050$, and $U(z^*, \rho^*) \subset D_0$.

Let us choose $\lambda = 0.01$ and $\alpha = 1$. Then $\rho_0 \approx 0.2475$, $D_0 \approx U(z^*, 0.2475)$, $r^* \approx 0.1807$, $\rho^* \approx \max\{0.1807, 0.1843\} = 0.1843$, and $U(z^*, \rho^*) \subset D_0$.

To solve the integral equation, the quadrature method using Simpson's rule with $h = \frac{1}{m}$ was applied. The calculation was carried out for $m = 50$, $\alpha = 1$, and $\lambda = 0.1$. The initial approximations were $z_0(t) = 0.1t$ and $z_{-1}(t) = 0.1t + 0.01$ for $t \in [0, 1]$. Table 3 shows that the error estimates (11) hold for all $n \geq 0$, and the sequence $\{z_n\}_{n \geq -1}$ remains in $U(z^*, r^*)$ and converges to an exact solution.

Table 3. Error estimate (11) for Example 2.

n	$\|z_{n+1} - z^*\|$	Right Side of Estimate (11)
−1	1.0000×10^{-1}	—
0	4.0245×10^{-4}	1.5167×10^{-2}
1	1.0295×10^{-8}	4.2258×10^{-6}
2	2.6081×10^{-13}	1.0769×10^{-10}
3	6.6072×10^{-18}	2.7281×10^{-15}

Figure 2 shows the error value $|z^*(t) - z_n(t)|$ at each iteration. These graphs illustrate the decrease in the error at each iteration and its distribution over the specified interval. The maximum error values at each iteration are presented in Table 3.

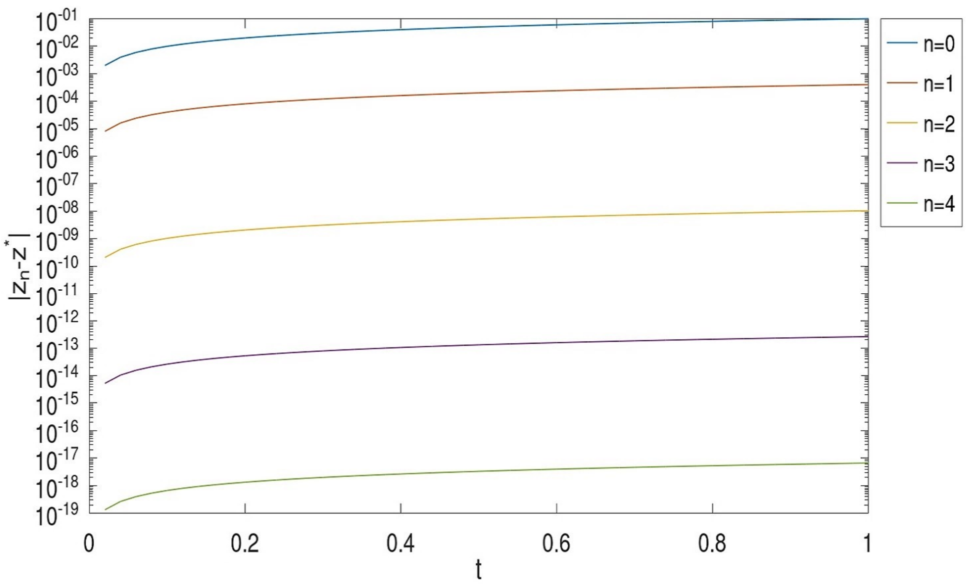

Figure 2. Error for problem 2.

Let us consider a semi-local case and choose $z_0(t) = 0.1t$, $z_{-1}(t) = 0.1t + 0.01$, $t \in [0, 1]$, $L = [x_0, y_0; F]$. Then, $x_0(t) = 0.1t + 0.01\lambda$, $y_0(t) = 0.1t - 0.01\lambda$, and

$$\|I - L\| \leq \alpha \Big[(0.1 + 0.01\lambda)^2 + |(0.1 + 0.01\lambda)(0.1 - 0.01\lambda)| + (0.1 - 0.01\lambda)^2\Big] = p.$$

Moreover, the values α and λ are chosen so that $p < 1$. As a result, we obtain the estimate

$$\|L^{-1}\| \leq \frac{1}{1 - p},$$

and the functions

$$v_0(\|x - z_0\|, \|y - z_0\|) = \frac{\alpha(2 + |0.1 + 0.01\lambda|)}{1 - p}(\|x - z_0\| + \|y - z_0\|) + \frac{\alpha}{1 - p}\|3z_0^2 - x_0^2 - y_0^2 - x_0 y_0\|$$

and

$$v(\|x - z\|, \|y - u\|) = \frac{3\alpha\kappa_0}{1 - p}(\|x - z\| + \|y - u\|).$$

Let us choose $\lambda = 0.1$ and $\alpha = 1$. Then, $p = 0.030001$, $R_0 \approx 0.2099$, $D_3 \approx U(z_0, 0.2099)$, $\kappa_0 = 0.3099$, $a^* \approx 0.1213$, $\gamma \approx \max\{0.1213, 0.1455\} = 0.1455$, and $U(z_0, \gamma) \subset D_3$.

Let us choose $\lambda = 0.01$ and $\alpha = 1$. Then, $p = 0.0300$, $R_0 \approx 0.2287$, $D_3 \approx U(z_0, 0.2287)$, $\kappa_0 = 0.3287$, $a^* \approx 0.1214$, $\gamma \approx \max\{0.1214, 0.1238\} = 0.1238$, and $U(z_0, \gamma) \subset D_3$.

Table 4 shows that the error estimates (23) hold for all $n \geq 0$ and (24) holds for all $n \geq -1$. The sequence $\{z_n\}_{n \geq -1}$ remains in $U(z_0, a^*)$ and converges to an exact solution. These results are obtained for $\alpha = 1$, $m = 50$, and $\lambda = 0.1$.

Table 4. Error estimates (23) and (24) for Example 2.

n	$\|z_{n+1} - z^*\|$	$a^* - a_{n+1}$	$\|z_{n+1} - z_n\|$	$a_{n+1} - a_n$
-2	1.1000×10^{-1}	1.2128×10^{-1}	–	–
-1	1.0000×10^{-1}	1.1128×10^{-1}	1.0000×10^{-2}	1.0000×10^{-2}
0	4.0245×10^{-4}	1.0882×10^{-2}	1.0040×10^{-1}	1.0040×10^{-1}
1	1.0295×10^{-8}	5.8292×10^{-4}	4.0246×10^{-4}	1.0299×10^{-2}
2	2.6081×10^{-13}	3.4152×10^{-6}	1.0295×10^{-8}	5.7951×10^{-4}
3	6.6072×10^{-18}	9.2440×10^{-10}	2.6081×10^{-13}	3.4143×10^{-6}

Example 3. *Consider the nonlinear boundary-value problem [4]*

$$\begin{cases} u''(t) = 2(u(t) - 0.5t + 1)^3, \ 0 < t < 1, \\ u(0) = 0, \ u(1) = 0. \end{cases}$$

Here, $B_1 = B_2 = C_{[0,1]}$ and the exact solution is $u^*(t) = \frac{1}{1+t} + \frac{1}{2}t - 2$.

Let $t_i = ih$, $i = 0, \ldots, m$, $h = \frac{1}{m}$, and m be a natural number and denote $\theta_i \approx u(t_i)$, $i = 1, \ldots, m-1$. To solve problem 3, we use the finite difference method. As a result, we obtain the system of nonlinear equations $F(z) = 0$, where

$$F_i(z) = \theta_{i+1} - 2\theta_i + \theta_{i-1} - 2h^2(\theta_i - 0.5t_i + 1)^3 = 0, \ i = 1, \ldots, m-1, \ \theta_0 = \theta_m = 0$$

and $z = (\theta_1, \ldots, \theta_{m-1})^T$.

The considered method (5) converges at five iterations for $m = 100$, $\lambda = 0.5$, and $\varepsilon = 10^{-5}$. The initial approximation $z_0 = u^*(t) - 0.5$ and $z_{-1} = u^*(t) - 0.51$, $t = t_i$, $i = 1, \ldots, m-1$. Figure 3 shows the error value $|u^*(t_i) - \theta_i|$, $i = 0, \ldots, m$ at the last iteration, $\max_{i=0,\ldots,m} |u^*(t_i) - \theta_i| \approx 1.5021 \times 10^{-5}$.

Example 4. *Let $B_1 = B_2 = \mathbb{R}$ and $D = (0, 2)$ and let $F : \Omega \to \mathbb{R}$ be defined by*

$$F(z) = z^3 - 1.$$

The exact solution for $F(z) = 0$ is $z^* = 1$.

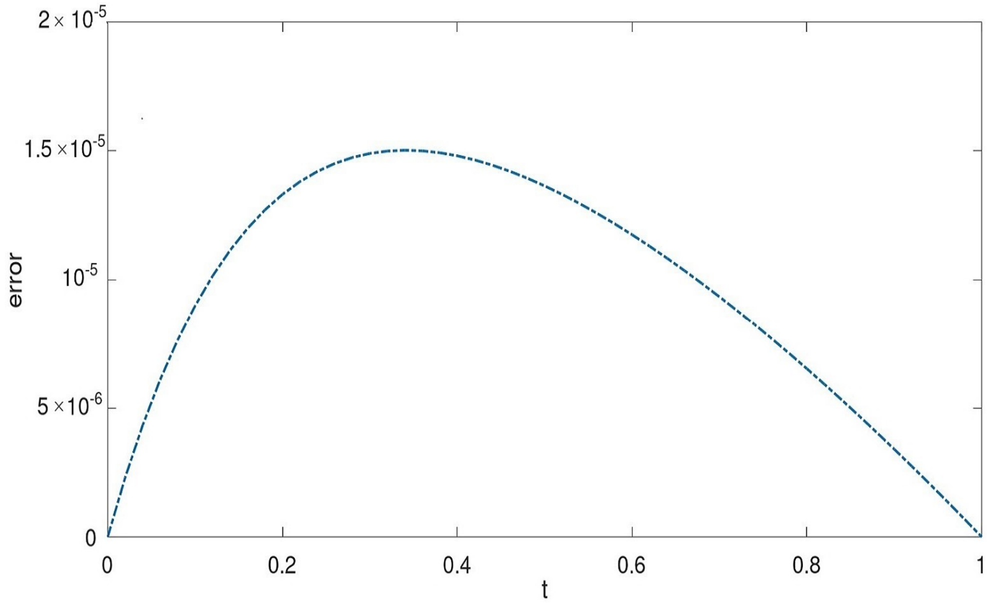

Figure 3. Error for problem 3.

Let us show that the assumptions C_1–C_5 hold. We can write that $F'(z) = 3z^2$ and $[x, y; F] = x^2 + xy + y^2$. Let us choose $L = F'(z^*)$. Next, we obtain

$$\begin{aligned}
[x, y; F] - F'(z^*) &= x^2 + xy + y^2 - 3(z^*)^2 = x^2 - (z^*)^2 + y^2 - (z^*)^2 + xy - xz^* \\
&\quad + xz^* - (z^*)^2 = (x - z^*)(x + 2z^*) + (y - z^*)(x + y + z^*), \\
[x, y; F] - [z, z^*; F] &= x^2 + xy + y^2 - z^2 - zz^* - (z^*)^2 = x^2 - z^2 + xy - zy \\
&\quad + zy - zz^* + y^2 - (z^*)^2 \\
&= (x - z)(x + z + y) + (y - z^*)(y + z + z^*),
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega_0(|x - z^*|, |y - z^*|) &= A_0|x - z^*| + B_0|y - z^*|, \\
&\quad A_0 = \max_{x \in D} \frac{|x + 2z^*|}{3(z^*)^2},\ B_0 = \max_{x,y \in D} \frac{|x + y + z^*|}{3(z^*)^2}, \\
\omega(|x - z|, |y - z^*|) &= A|x - z| + B|y - z^*|, \\
&\quad A \max_{x,y,z \in D_0} \frac{|x + y + z|}{3(z^*)^2},\ B = \max_{x,y \in D_0} \frac{|x + y + z^*|}{3(z^*)^2}.
\end{aligned}$$

Let $\lambda = 0.1$. Then, $g_1(t) = t$, $g_2(t) = \frac{6}{5}t$, $A_0 = \frac{4}{3}, B_0 = \frac{5}{3}$, and

$$\omega_0(g_1(t), g_2(t)) - 1 = \frac{4}{3}t + 2t - 1 = \frac{10}{3}t - 1 = 0.$$

The last equation has root $\rho_0 = \frac{3}{10}$ and $D_0 = \left(\frac{7}{10}, \frac{13}{10}\right)$. Then, $A = \frac{13}{10}$, $B = \frac{6}{5}$, and the equation $h(t) - 1 = 0$ has the form

$$\frac{(2A + \frac{6}{5}B)t}{1 - \frac{10}{3}t} - 1 = \frac{\frac{101}{25}t}{1 - \frac{10}{3}t} = 0.$$

The solution of this equation is $r^* = \frac{75}{553}$, $\rho^* = \max\{\frac{75}{553}, \frac{90}{553}\} = \frac{90}{553}$, and

$$U(z^*, \rho^*) = \left(\frac{463}{553}, \frac{643}{553}\right) \subset D_0.$$

So, assumptions C_1–C_5 hold.

5. Conclusions

We have developed unified local and semi-local convergence of a family of Kurchatov-type methods depending on one parameter for solving nonlinear operator equations under generalized conditions in a Banach space. Moreover, we have studied the uniqueness of the solution of the nonlinear Equation (1). Numerical examples that demonstrate the applicability of our theoretical results are also provided. Some of the advantages of the new approach are as follows:

- A comparison between different methods becomes possible, since their convergence is studied under uniform conditions;
- The assumptions involve only the operators which are present in the method, in contrast to earlier studies using assumptions involving derivatives not in the method [6,7,16–19];
- The generalized continuity assumption imposed on the divided difference leads to better information on the location of the solution z^* and fewer iterates to obtain the error tolerance than before, since the bounds on $\|z_n - z^*\|$ are tighter;
- Finally, the generality of the new approach helps with the extension of the applicability of other methods in a similar way [4,8–15]. This is the direction of our future research.

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