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Quantum Mechanics as Hamilton–Killing Flows on a Statistical Manifold †

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Abstract: The mathematical formalism of quantum mechanics is derived or "reconstructed" from more basic considerations of the probability theory and information geometry. The starting point is the recognition that probabilities are central to QM; the formalism of QM is derived as a particular kind of flow on a finite dimensional statistical manifold—a simplex. The cotangent bundle associated to the simplex has a natural symplectic structure and it inherits its own natural metric structure from the information geometry of the underlying simplex. We seek flows that preserve (in the sense of vanishing Lie derivatives) both the symplectic structure (a Hamilton flow) and the metric structure (a Killing flow). The result is a formalism in which the Fubini–Study metric, the linearity of the Schrödinger equation, the emergence of complex numbers, Hilbert spaces and the Born rule are derived rather than postulated.

Keywords: information geometry; symplectic geometry; Hamilton–Killing flows; entropic dynamics

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1. Introduction

In the traditional approach to quantum mechanics (QM), the Hilbert space plays a central, dominant role and probabilities are introduced, almost as an afterthought, in order to provide the phenomenological link for handling measurements. The uneasy coexistence of the Hilbert and the probabilistic structures is reflected in the two separate modes of wave-function evolution; one is the linear and deterministic Schrödinger evolution and the other is the discontinuous and stochastic wave function collapse. It has given rise to longstanding problems in the interpretation of the quantum state itself [1–5].

These difficulties have motivated alternative approaches in which, rather than postulating Hilbert spaces as the starting point, one recognizes that probabilities play the dominant role; probabilities are not just an accidental feature peculiar to quantum measurements. The goal there is to derive or "reconstruct" the mathematical formalism of QM from more basic considerations of probability theory and geometry. (See, e.g., [6–11] and references therein.)

In the entropic dynamics (ED) approach the central object is the epistemic configuration space, which is a statistical manifold—a space in which each point represents a probability distribution [11]. In this paper, our goal is to discuss those special curves that could potentially play the role of trajectories. What makes those curves special is that they are adapted to the natural geometric structures on the statistical manifold.

Two such structures are of central importance. The first is familiar from statistics, i.e., all statistical manifolds have an intrinsic metric structure given by the information metric [12,13]. The second is familiar from classical mechanics [14–16]. Since we are interested in trajectories, we are naturally led to consider the vectors that are tangent to such curves, as well as the dual vectors, or covectors—it is these objects that are used to represent the analogues of the velocities of probabilities and their momenta. Vectors and covectors live in the so-called tangent and cotangent spaces, respectively. It turns out that the

statistical manifold plus all its cotangent spaces is itself a manifold—the cotangent bundle—that can be endowed with a second natural structure called *symplectic*. In mechanics, the cotangent bundle is known as phase space and the symplectic transformations are known as canonical transformations.

There is extensive literature on the symplectic and metric structures inherent to QM. They have been discovered, independently rediscovered, and extensively studied by many authors [17–26]. Their crucial insight is that those structures, being of purely geometrical nature, are not just central to classical mechanics, they are also central to quantum mechanics. Furthermore, the potential connection and relevance of information geometry to various aspects of QM, including its metric structure, has also been studied [9–11,27–30].

To characterize congruences of curves in the epistemic phase space—or, equivalently, the *flows* on the cotangent bundle—we must address two problems. First, we must characterize the particular cotangent space and the symplectic structure that is relevant to QM. This amounts to establishing the correct conjugate momenta to be paired to the coordinates, which, in our case, are probabilities. In classical mechanics, this pairing is accomplished with the help of a Lagrangian $L(q, \dot{q})$ and the prescription $p = \partial L/\partial \dot{q}$. In the present problem, we have no access to a Lagrangian and a different criterion is adopted [11]. The second problem is to provide the cotangent bundle with a metric structure that is compatible with the information metric of the underlying statistical manifold. The issue is that cotangent bundles are not statistical manifolds and the challenge is to identify the natural set of assumptions that leads to the right metric structure.

We show that the flows that are relevant to quantum mechanics are those that preserve (in the sense of vanishing Lie derivatives) both the symplectic structure (a Hamilton flow) and the metric structure (a Killing flow). The characterization of these Hamilton–Killing (HK) flows results in a formalism that includes states described by rays, a geometry given by the Fubini–Study metric, flows that obey a linear Schrödinger equation, the emergence of a complex structure, the Born rule, and Hilbert spaces. All these elements are derived rather than postulated.

The present discussion includes two new developments. First, our focus is on isolating the essential geometrical aspects of the problem (a discussion of the physical aspects is given in [11]) and the main ideas are presented in the simpler context of a finite-dimensional manifold—a simplex. Thus, what we derive here is the geometrical framework that applies to a toy model—an n-sided quantum die. Second, the metric structure of the cotangent bundle is found by a new argument involving the minimal assumption that the metric of phase space is determined by the only metric structure at our disposal, namely, the information metric of the simplex.

Is this all there is to quantum mechanics? We conclude with a word of caution. The framework developed here takes us a long way towards justifying the mathematical formalism that underlies quantum mechanics, but it is only a kinematical prelude to the true dynamics. The point is that not every HK curve is a trajectory and not every parameter that labels points along a curve is time. All changes of probabilities, including the changes we call dynamics, must be compatible with the entropic and Bayesian rules that have been found to be of universal applicability in inference. It is this additional requirement that further restricts the HK flows to an entropic dynamics that describes an evolution in a suitably constructed entropic concept of time [7,11].

This paper focuses on deriving the mathematical formalism of quantum mechanics, but the ED approach has been applied to a variety of other topics in quantum theory. These include the quantum measurement problem [31,32]; momentum and uncertainty relations [33,34]; the Bohmian limit [34,35] and the classical limit [36]; extensions to curved spaces [37]; to relativistic fields [38–40]; and the ED of spin [41].

2. Some Background

We deal with several distinct spaces. One is the *ontic configuration space* of microstates labeled by $i = 1 \ldots n$, which are the unknown variables we are trying to predict. Another

is the space of probability distributions $\rho = (\rho^1 \dots \rho^n)$, which is the *epistemic configuration space* or, to use a shorter name, the *e-configuration space*. This $(n-1)$-dimensional statistical manifold is a simplex $\mathcal{S}$,

$$\mathcal{S} = \left\{ \rho |\, \rho^i \geq 0 \,;\; \textstyle\sum_{i=1}^n \rho^i = 1 \right\} . \tag{1}$$

As coordinates for a generic point ρ on $\mathcal{S}$, we shall use the probabilities ρ^i themselves.

Given the manifold $\mathcal{S}$, we can construct two other special manifolds that will turn out to be useful, the *tangent bundle* $T\mathcal{S}$ and the *cotangent bundle* $T^*\mathcal{S}$. These are fiber bundles; the *base manifold* is $\mathcal{S}$ and the fibers at each point ρ are respectively the tangent $T\mathcal{S}_\rho$ and cotangent $T^*\mathcal{S}_\rho$ spaces at ρ. The tangent space at ρ, $T\mathcal{S}_\rho$, is the vector space composed of all vectors that are tangent to curves through the point ρ. While this space is obviously important (it is the space of "velocities" of probabilities), in what follows, we will not have much to say about it. Much more central to our discussion is the cotangent space at ρ, $T^*\mathcal{S}_\rho$ which is the vector space of all covectors at ρ.

As already mentioned, the reason we care about vectors and covectors is that these are the objects that are used to represent velocities and momenta. The cotangent bundle $T^*\mathcal{S}$, plays the central role of the *epistemic phase space*, or *e-phase space.*

A point $X \in T^*\mathcal{S}$ is represented as $X = (\rho, \pi)$, where $\rho = (\rho^1 \dots \rho^n)$ are coordinates on the base manifold $\mathcal{S}$ and $\pi = (\pi_1 \dots \pi_n)$ are some generic coordinates on the cotangent space $T^*\mathcal{S}_\rho$ at ρ. Curves on $T^*\mathcal{S}$ allow us to define vectors on the tangent spaces $T(T^*\mathcal{S})_X$. Let $X = X(\lambda)$ be a curve parameterized by λ; then, the vector $\bar{V}$ tangent to the curve at $X = (\rho, \pi)$ has components $d\rho^i/d\lambda$ and $d\pi_i/d\lambda$ and is written as

$$\bar{V} = \frac{d}{d\lambda} = \frac{d\rho^i}{d\lambda}\bar{\rho}_i + \frac{d\pi_i}{d\lambda}\bar{\pi}^i = \frac{d\rho^i}{d\lambda}\frac{\partial}{\partial\rho^i} + \frac{d\pi_i}{d\lambda}\frac{\partial}{\partial\pi_i} , \tag{2}$$

where $\bar{\rho}_i$ and $\bar{\pi}^i$ are the basis vectors, the index $i = 1 \dots n$ is summed over and we adopt the standard notation in differential geometry, $\bar{\rho}_i = \partial/\partial\rho^i$ and $\bar{\pi}^i = \partial/\partial\pi_i$. The directional derivative of a function $F(X)$ along the curve $X(\lambda)$ is

$$\frac{dF}{d\lambda} = \frac{\partial F}{\partial\rho^i}\frac{d\rho^i}{d\lambda} + \frac{\partial F}{\partial\pi_i}\frac{d\pi_i}{d\lambda} \stackrel{\text{def}}{=} \tilde{\nabla} F[\bar{V}] , \tag{3}$$

where $\tilde{\nabla}$ is the gradient in $T^*\mathcal{S}$, that is, the gradient of a generic function $F(X) = F(\rho, \pi)$ is

$$\tilde{\nabla} F = \frac{\partial F}{\partial\rho^i}\tilde{\nabla}\rho^i + \frac{\partial F}{\partial\pi_i}\tilde{\nabla}\pi_i , \tag{4}$$

where $\tilde{\nabla}\rho^i$ and $\tilde{\nabla}\pi_i$ are the basis covectors and the tilde '~' serves to distinguish the gradient $\tilde{\nabla}$ on the bundle $T^*\mathcal{S}$ from the gradient ∇ on the simplex $\mathcal{S}$.

Here, unfortunately, we encounter a technical difficulty due to the fact that the space $\mathcal{S}$ is constrained to normalized probabilities so that the coordinates ρ^i cannot be varied independently. This problem is handled, without loss of generality, by embedding the $(n-1)$-dimensional manifold $\mathcal{S}$ into a manifold of one dimension higher, the so-called positive-cone, denoted $\mathcal{S}^+$, where the coordinates ρ^i are unconstrained.

To simplify the notation, a point $X = (\rho, \pi)$ in the $2n$-dimensional $T^*\mathcal{S}^+$ is labeled by its coordinates $X^{\alpha i} = (X^{1i}, X^{2i}) = (\rho^i, \pi_i)$, where αi is a composite index. The first index α (chosen from the beginning of the Greek alphabet) takes two values, $\alpha = 1, 2$. Since α keeps track of whether i is an upper ρ^i index ($\alpha = 1$) or a lower π_i index ($\alpha = 2$), from now on we can set $\rho_i = \rho^i$. Then, Equations (2) and (4) are written as

$$\bar{V} = \frac{d}{d\lambda} = V^{\alpha i}\frac{\partial}{\partial X^{\alpha i}} \quad \text{and} \quad \tilde{\nabla} F = \frac{\partial F}{\partial X^{\alpha i}}\tilde{\nabla} X^{\alpha i} . \tag{5}$$

The repeated indices indicate a double summation over α and i. The action of the basis covectors $\tilde{\nabla} X^{\alpha i}$ on the basis vectors, $\partial/\partial X^{\beta j} = \partial_{\beta j}$, is given by

$$\tilde{\nabla} X^{\alpha i}[\partial_{\beta j}] = \frac{\partial X^{\alpha i}}{\partial X^{\beta j}} = \delta^{\alpha i}_{\beta j} \quad \text{so that} \quad \tilde{\nabla} F[\bar{V}] = \frac{\partial F}{\partial X^{\alpha i}} V^{\alpha i} = \frac{dF}{d\lambda} \tag{6}$$

is the directional derivative of F along the vector $\bar{V}$.

3. Hamiltonian Flows

Just as a manifold can be supplied with a symmetric bilinear form, the metric tensor, which gives it the fairly rigid structure described as its *metric geometry*, cotangent bundles can be supplied with an *antisymmetric* bilinear form, the *symplectic form*, which gives them the somewhat floppier structure called *symplectic geometry* (Arnold 1997 [15,16]).

A vector field $\bar{V}(X)$ defines a space-filling congruence of curves $X^i = X^i(\lambda)$ that are tangent to the field $\bar{V}(X)$ at every point X. We seek those special congruences or *flows* that reflect the symplectic geometry.

3.1. hE Symplectic Form

Once the local coordinates (ρ^i, π_i) on $T^*\mathcal{S}^+$ are established there is a natural choice of symplectic form

$$\Omega = \tilde{\nabla}\rho^i \otimes \tilde{\nabla}\pi_i - \tilde{\nabla}\pi_i \otimes \tilde{\nabla}\rho^i \,. \tag{7}$$

The question of how to choose those local coordinates, which are Darboux coordinates for the cotangent bundle, remains open. The answer is not to be found in mathematics but in physics. In classical mechanics, the criterion for choosing a canonical momentum is provided by a Lagrangian; however, here, we do not have a Lagrangian. An alternative criterion more closely tailored to the framework presented here is provided by entropic dynamics [11]. From now on, we assume that the correct π_i coordinates have been identified.

The action of $\Omega[\cdot,\cdot]$ on two vectors $\bar{V} = d/d\lambda$ and $\bar{U} = d/d\mu$ is obtained using (6).

$$\tilde{\nabla}\rho^i(\bar{V}) = V^{1i} \quad \text{and} \quad \tilde{\nabla}\pi_i(\bar{V}) = V^{2i} \,. \tag{8}$$

The result is

$$\Omega[\bar{V},\bar{U}] = V^{1i}U^{2i} - V^{2i}U^{1i} = \Omega_{\alpha i,\beta j} V^{\alpha i} U^{\beta j} \quad \text{where} \quad \Omega_{\alpha i,\beta j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{ij} \,. \tag{9}$$

3.2. Hamilton's Equations and Poisson Brackets

Next, we derive the $2n$-dimensional $T^*\mathcal{S}^+$ analogues of the results that are standard in classical mechanics [14–16]. We seek those vector fields $\bar{V}(X)$ that generate flows (the congruence of integral curves) that preserve the symplectic structure in the sense that

$$\pounds_V \Omega = 0 \,, \tag{10}$$

where the Lie derivative [16] is

$$(\pounds_V \Omega)_{\alpha i,\beta j} = V^{\gamma k}\partial_{\gamma k}\Omega_{\alpha i,\beta j} + \Omega_{\gamma k,\beta j}\partial_{\alpha i}V^{\gamma k} + \Omega_{\alpha i,\gamma k}\partial_{\beta j}V^{\gamma k} \,. \tag{11}$$

Since, by Equation (9), the components $\Omega_{\alpha i,\beta j}$ are constant, $\partial_{\gamma k}\Omega_{\alpha i,\beta j} = 0$, we can rewrite $\pounds_V \Omega$ as

$$(\pounds_V \Omega)_{\alpha i,\beta j} = \partial_{\alpha i}(\Omega_{\gamma k,\beta j}V^{\gamma k}) - \partial_{\beta j}(\Omega_{\gamma k,\alpha i}V^{\gamma k}) \,, \tag{12}$$

which is the exterior derivative (roughly, the curl) of the covector $\Omega_{\gamma k,\alpha i} V^{\gamma k}$. By Poincare's lemma, requiring $\pounds_V \Omega = 0$ (a vanishing curl) implies that $\Omega_{\gamma k,\alpha i} V^{\gamma k}$ is the gradient of a scalar function, which we denote by $\tilde{V}(X)$,

$$\Omega_{\gamma k,\alpha i} V^{\gamma k} = \partial_{\alpha i} \tilde{V} \quad \text{or} \quad \Omega(\bar{V}, \cdot) = \tilde{\nabla} \tilde{V}(\cdot) \,. \tag{13}$$

In the opposite direction, we can easily check that (13) implies $\pounds_V \Omega = 0$. Using (9), Equation (13) is more explicitly written as

$$\frac{d\rho^i}{d\lambda} \tilde{\nabla} \pi_i - \frac{d\pi_i}{d\lambda} \tilde{\nabla} \rho^i = \frac{\partial \tilde{V}}{\partial \rho^i} \tilde{\nabla} \rho^i + \frac{\partial \tilde{V}}{\partial \pi_i} \tilde{\nabla} \pi_i \,, \tag{14}$$

or

$$\frac{d\rho^i}{d\lambda} = \frac{\partial \tilde{V}}{\partial \pi_i} \quad \text{and} \quad \frac{d\pi_i}{d\lambda} = -\frac{\partial \tilde{V}}{\partial \rho^i} \,, \tag{15}$$

which we recognize as Hamilton's equations for a Hamiltonian function $\tilde{V}$. This justifies calling $\bar{V}$ the *Hamiltonian vector field* associated to the *Hamiltonian function* $\tilde{V}$. In other words, the flows that preserve the symplectic structure, $\pounds_V \Omega = 0$, are generated by Hamiltonian vector fields $\bar{V}$ associated to Hamiltonian functions $\tilde{V}$.

From (9) and (15) the action of the symplectic form Ω on two Hamiltonian vector fields $\bar{V} = d/d\lambda$ and $\bar{U} = d/d\mu$ generated, respectively, by $\tilde{V}$ and $\tilde{U}$, is

$$\Omega[\bar{V}, \bar{U}] = \frac{d\rho^i}{d\lambda} \frac{d\pi_i}{d\mu} - \frac{d\pi_i}{d\lambda} \frac{d\rho^i}{d\mu} = \frac{\partial \tilde{V}}{\partial \rho^i} \frac{\delta \tilde{U}}{\delta \pi_i} - \frac{\partial \tilde{V}}{\partial \pi_i} \frac{\partial \tilde{U}}{\partial \rho^i} \stackrel{\text{def}}{=} \{\tilde{V}, \tilde{U}\} \,, \tag{16}$$

where, on the right hand side, we have introduced the Poisson bracket notation. In other words, the action of Ω on two Hamiltonian vector fields is the Poisson bracket of the associated Hamiltonian functions. We can also check that the derivative of an arbitrary function $F(X)$ along the vector field $\bar{V} = d/d\lambda$ is

$$\frac{dF}{d\lambda} = \{F, \tilde{V}\} \,. \tag{17}$$

Thus, *the Hamiltonian formalism that is so familiar in physics emerges from purely geometrical considerations*. It might be desirable to adopt a more suggestive notation; instead of $(\tilde{V}, \lambda)$ let us write $(\tilde{H}, \tau)$. Then, the flow generated by a Hamiltonian function $\tilde{H}$ and parameterized by "time" τ is given by Hamilton's equations in the standard form,

$$\frac{d\rho^i}{d\tau} = \frac{\partial \tilde{H}}{\partial \pi_i} \quad \text{and} \quad \frac{d\pi_i}{d\tau} = -\frac{\partial \tilde{H}}{\partial \rho^i} \,, \tag{18}$$

and the τ evolution of any well-behaved function $f(X)$ is given by

$$\frac{df}{d\tau} = \bar{H}(f) = \{f, \tilde{H}\} \quad \text{with} \quad \bar{H} = \frac{\partial \tilde{H}}{\partial \pi_i} \frac{\partial}{\partial \rho^i} - \frac{\partial \tilde{H}}{\partial \rho^i} \frac{\partial}{\partial \pi_i} \,. \tag{19}$$

The difference with classical mechanics is that, here, the degrees of freedom are probabilities and not ontic variables such as, for example, the positions of particles.

3.3. The Normalization Constraint

Since our actual interest is not in flows on $T^*\mathcal{S}^+$ but on the bundle $T^*\mathcal{S}$ of normalized probabilities, we shall restrict ourselves to flows that preserve the normalization of probabilities. Let

$$|\rho| \stackrel{\text{def}}{=} \sum_{i=1}^{n} \rho^i \quad \text{and} \quad \tilde{N} \stackrel{\text{def}}{=} 1 - |\rho| \,. \tag{20}$$

We seek those special Hamiltonians $\tilde{H}$ such that the initial condition $\tilde{N} = 0$ is preserved by the flow, that is,

$$\partial_\tau \tilde{N} = \{\tilde{N}, \tilde{H}\} = 0 \quad \text{or} \quad \sum_i \frac{\partial \tilde{H}}{\partial \pi_i} = \sum_i \frac{d\rho^i}{d\tau} = 0 \,. \tag{21}$$

Indeed, the actual quantum Hamiltonians will preserve $\tilde{N} = const.$ even when the constant does not vanish [11]. Since the probabilities ρ^i must remain positive, we further require that $d\rho^i/d\tau \geq 0$ when $\rho^i = 0$.

We can also consider the Hamiltonian flow generated by $\tilde{N}$ and parameterized by ν. From Equation (15) the corresponding Hamiltonian vector field $\bar{N}$ is given by

$$\bar{N} = N^{\alpha i} \frac{\partial}{\partial X^{\alpha i}} \quad \text{with} \quad N^{\alpha i} = \frac{dX^{\alpha i}}{d\nu} = \{X^{\alpha i}, \tilde{N}\} \,, \tag{22}$$

or, more explicitly,

$$N^{1i} = \frac{d\rho^i}{d\nu} = 0 \quad \text{and} \quad N^{2i} = \frac{d\pi_i}{d\nu} = 1 \,. \tag{23}$$

The integral curves generated by $\tilde{N}$ are found by integrating (23). The result is

$$\rho^i(\nu) = \rho^i(0) \quad \text{and} \quad \pi_i(\nu) = \pi_i(0) + \nu \,, \tag{24}$$

which amounts to shifting all momenta by the i-independent parameter ν. We can also see that, if $\tilde{N}$ is conserved along $\tilde{H}$, then $\tilde{H}$ is conserved along $\bar{N}$.

$$\frac{d\tilde{H}}{d\nu} = \{\tilde{H}, \tilde{N}\} = 0 \,, \tag{25}$$

which implies that the conserved quantity $\tilde{N}$ is the generator of a symmetry transformation.

To summarize: the phase space of interest is $T^*\mathcal{S}$, but the description is simplified by using the unnormalized coordinates ρ of the larger embedding space $T^*\mathcal{S}^+$. The introduction of one superfluous ρ coordinate forces us to also introduce one superfluous π momentum. We eliminate the extra coordinate by imposing the constraint $\tilde{N} = 0$. We eliminate the extra momentum by declaring it unphysical; the shifted point $(\rho', \pi') = (\rho, \pi + \nu)$ is declared to be equivalent to (ρ, π). This equivalence is described as a global "gauge" symmetry which, as we shall see later in the paper, is the reason why quantum mechanical states are represented by rays rather than vectors in a Hilbert space.

4. The Information Geometry of E-Phase Space

Our next goal is to extend the metric of the simplex $\mathcal{S}$—given by information geometry—to the full e-phase space, $T^*\mathcal{S}$. The extension can be carried out in many ways [9–11,42]. The virtue of the derivation below is that the number of input assumptions is kept to a minimum.

4.1. The Metric on the Embedding E-Phase Space T^*S^+

First, we assign a metric to the embedding bundle $T^*\mathcal{S}^+$; then, we consider the metric it induces on $T^*\mathcal{S}$. The metric of the space $\mathcal{S}^+$ of unnormalized probabilities [13,43] is

$$\delta\ell^2 = g_{ij}\delta\rho^i\delta\rho^j \quad \text{with} \quad g_{ij} = A(|\rho|)n_i n_j + \frac{B(|\rho|)}{2\rho^i}\delta_{ij} \,, \tag{26}$$

where n is a covector with components $n_i = 1$ for all $i = 1 \ldots n$ and $A(|\rho|)$ and $B(|\rho|)$ are smooth scalar functions of $|\rho| = \sum \rho^i$. Since *the only tensor at our disposal* is g_{ij} the length element of $T^*\mathcal{S}^+$ must be of the form

$$\delta\tilde{\ell}^2 = \alpha g_{ij}\delta\rho^i\delta\rho^j + \beta g_i^j \delta\rho^i\delta\pi_j + \gamma g^{ij}\delta\pi_i\delta\pi_j \,, \tag{27}$$

where α, β and γ are constants. Since $\delta\rho^i$ and $\delta\pi_i$ are vectors and covectors, the requirement that $\delta\tilde{\ell}^2$ induce the same magnitudes $g_{ij}\delta\rho^i\delta\rho^j$ on $T\mathcal{S}^+_\rho$ and $g^{ij}\delta\pi_i\delta\pi_j$ on $T^*\mathcal{S}^+_\rho$, as given by information geometry, implies that $\alpha = \gamma = 1$. To fix β, let us consider a curve $\rho = \rho(\tau)$ and $\pi = \pi(\tau)$ on $T^*\mathcal{S}^+$ and its flow-reversed or τ-reversed curve given by $\rho'(\tau) = \rho(-\tau)$ and $\pi'(\tau) = -\pi(-\tau)$. We require that the speed $(d\tilde{\ell}/d\tau)^2$ remains invariant under flow-reversal. Since, under flow-reversal, the mixed $\rho\pi$ terms in (27) change sign, it follows that invariance implies that $\beta = 0$. We emphasize that imposing that the e-phase space be symmetric under flow-reversal does not amount to imposing *time*-reversal invariance; time-reversal violations might still be caused by interaction terms in the Hamiltonian. The resulting line element, *which has been designed to be fully determined by information geometry*, takes the form

$$\delta\tilde{\ell}^2 = G_{\alpha i,\beta j}\delta X^{\alpha i}\delta X^{\beta j} = g_{ij}\delta\rho^i\delta\rho^j + g^{ij}\delta\pi_i\delta\pi_j \,. \tag{28}$$

4.2. A Complex Structure for T^*S^+

The metric tensor G and its inverse G^{-1} can be used to lower and raise indices. In particular, with G^{-1}, we can raise the first index of the symplectic form $\Omega_{\alpha i,\beta j}$ in Equation (9).

$$G^{\alpha i,\gamma k}\Omega_{\gamma k,\beta j} \stackrel{\text{def}}{=} -J^{\alpha i}{}_{\beta j} \,. \tag{29}$$

The tensor J has several important properties. These are most easily derived by writing G and Ω in block matrix form, i.e.,

$$G^{-1} = \begin{bmatrix} g^{-1} & 0 \\ 0 & g \end{bmatrix}, \; \Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; J = \begin{bmatrix} 0 & -g^{-1} \\ g & 0 \end{bmatrix} . \tag{30}$$

We can immediately check that $JJ = -\mathbf{1}$, which shows that J is a square root of the negative identity matrix. Thus, J endows $T^*\mathcal{S}^+$ with a complex structure. *To summarize, in addition to the symplectic Ω and metric G structures, the cotangent bundle T^*S^+ is also endowed with a complex structure J.* Such highly structured spaces are generically known as Kähler manifolds. Here, we deal with a special Kähler manifold where the space of ρs is a statistical manifold and the spaces of πs are flat cotangent spaces. However, ultimately, the geometry of $T^*\mathcal{S}^+$ is only of marginal interest; what matters is the geometry it induces on the e-phase space $T^*\mathcal{S}$ of normalized probabilities, to which we turn next.

4.3. The Metric Induced on the E-Phase Space T^*S

As we saw above the e-phase space $T^*\mathcal{S}$ can be obtained from the space $T^*\mathcal{S}^+$ by the restriction $|\rho| = 1$ and by identifying the gauge equivalent points (ρ^i, π_i) and $(\rho^i, \pi_i + n_i\nu)$. Consider two neighboring points (ρ^i, π_i) and (ρ'^i, π'_i) with $|\rho| = |\rho'| = 1$, the metric induced on $T^*\mathcal{S}$ is defined as the shortest $T^*\mathcal{S}^+$ distance between (ρ^i, π_i) and the points on the ray defined by (ρ'^i, π'_i). Since the $T^*\mathcal{S}^+$ distance between (ρ^i, π_i) and $(\rho^i + \delta\rho^i, \pi_i + \delta\pi_i + n_i\nu)$ is

$$\delta\tilde{\ell}^2(\nu) = g_{ij}\delta\rho^i\delta\rho^j + g^{ij}(\delta\pi_i + n_i\nu)(\delta\pi_j + n_j\nu) \,, \tag{31}$$

the metric on $T^*\mathcal{S}$ is defined by $\delta\tilde{s}^2 = \min_\nu \delta\tilde{\ell}^2$. Imposing $|\delta\rho| = 0$, the value of ν that minimizes (31) is $\nu = -\langle\delta\pi\rangle = -\sum_i \rho^i\delta\pi_i$. Therefore, the metric on $T^*\mathcal{S}$, which measures the distance between neighboring rays, is

$$\delta\tilde{s}^2 = \sum_{i=1}^{n}\left[\frac{B(1)}{2\rho^i}(\delta\rho^i)^2 + \frac{2\rho^i}{B(1)}(\delta\pi_i - \langle\delta\pi\rangle)^2\right] . \tag{32}$$

From now on, we set $B(1) = 1$, which only amounts to a choice of units and has no effect on our results. (In [11], we chose $B(1) = \hbar$.)

Although the metric (32) is expressed in a notation that may be unfamiliar, it turns out to be equivalent to the well-known *Fubini–Study metric*. Thus, the recognition that the

e-phase space is the cotangent bundle of a statistical manifold led us to a novel derivation based on information geometry.

An important feature of the $T^*\mathcal{S}$ metric (32) is that, except for the irrelevant constant $B(1)$, it has turned out to be independent of the particular choices of the *functions* $A(|\rho|)$ and $B(|\rho|)$ (see Equation (26)) that define the geometries of the embedding spaces $\mathcal{S}^+$ and $T^*\mathcal{S}^+$. Therefore, without any loss of generality, we can simplify the analysis considerably by choosing $A(|\rho|) = 0$ and $B(|\rho|) = 1$, which gives the embedding spaces the simplest possible geometries, namely, they are flat. With this choice the $T^*\mathcal{S}^+$ metric, Equation (28) becomes

$$\delta\tilde{\ell}^2 = \sum_{i=1}^{n}\left[\frac{1}{2\rho^i}\delta\rho_i^2 + 2\rho^i\delta\pi_i^2\right] = G_{\alpha i,\beta j}\delta X^{\alpha i}\delta X^{\beta j} \quad \text{with} \quad G_{\alpha i,\beta j} = \begin{bmatrix}\delta_{ij}/2\rho_i & 0\\ 0 & 2\rho_i\delta_{ij}\end{bmatrix} \tag{33}$$

and the tensor J, Equation (30), which defines the complex structure, becomes

$$J^{\alpha i}{}_{\beta j} = -G^{\alpha i,\gamma k}\Omega_{\gamma k,\beta j} = \begin{bmatrix}0 & -2\rho_i\delta_{ij}\\ \delta_{ij}/2\rho_i & 0\end{bmatrix}. \tag{34}$$

4.4. Refining the Choice of Cotangent Space: Complex Coordinates

Having endowed the e-phase spaces $T^*\mathcal{S}^+$ and $T^*\mathcal{S}$ with both metric and complex structures, we can now revisit and refine our choice of cotangent spaces. So far, we assumed the cotangent space $T^*\mathcal{S}_\rho^+$ at ρ to be the flat Euclidean n-dimensional space R^n. It turns out that the cotangent space that is relevant to quantum mechanics requires a further restriction. To see what this is, we use the fact that $T^*\mathcal{S}^+$ is endowed with a complex structure, which suggests a coordinate transformation from (ρ, π) to complex coordinates $(\psi, i\psi^*)$,

$$\psi_i = \rho_i^{1/2}e^{i\pi_i} \quad \text{and} \quad i\psi_i^* = i\rho_i^{1/2}e^{-i\pi_i}, \tag{35}$$

Thus, a point $\psi \in T^*\mathcal{S}^+$ has coordinates

$$\psi^{\mu i} = \begin{pmatrix}\psi^{1i}\\ \psi^{2i}\end{pmatrix} = \begin{pmatrix}\psi_i\\ i\psi_i^*\end{pmatrix}, \tag{36}$$

where the index $\mu = 1, 2$ takes two values (with $\mu, \nu, \ldots$ chosen from the middle of the Greek alphabet).

Since changing the phase $\pi_i \to \pi_i + 2\pi$ yields the same point ψ, we see that the new $T^*\mathcal{S}_\rho^+$ is a flat n-dimensional "hypercube" (its edges have a coordinate length of 2π) with the opposite faces identified (periodic boundary conditions). Thus, the new $T^*\mathcal{S}_\rho^+$ is locally isomorphic to the old R^n, which makes it a legitimate choice of cotangent space. (Strictly, $T^*\mathcal{S}_\rho^+$ is a parallelepiped; from (28), we see that the lengths of its edges are $\ell_i = 2\pi(2\rho_i)^{1/2}$ which vanish at the boundaries of the simplex.)

We can check that the transformation from real (ρ, π) to complex coordinates $(\psi, i\psi^*)$ is canonical, so that

$$\Omega_{\mu i,\nu j} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\delta_{ij}, \tag{37}$$

retains the same form as (9).

Expressed in ψ coordinates, the Hamiltonian flow generated by the normalization constraint (24) is the familiar phase shift $\psi_i(\nu) = \psi_i(0)e^{i\nu}$. Thus, the gauge symmetry induced by the constraint $\tilde{N} = 0$ is the familiar multiplication by a constant phase factor.

In ψ coordinates, the metric G on $T^*\mathcal{S}^+$ Equation (33) becomes

$$\delta\tilde{\ell}^2 = -2i\sum_{i=1}^{n}\delta\psi_i\delta i\psi_i^* = G_{\mu i,\nu j}\,\delta\psi^{\mu i}\delta\psi^{\nu j} \quad \text{where} \quad G_{\mu i,\nu j} = -i\delta_{ij}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{38}$$

Finally, using the inverse $G^{\mu i,\lambda k}$ to raise the first index of $\Omega_{\lambda k,\nu j}$ gives the ψ components of the tensor J,

$$J^{\mu i}{}_{\nu j} = -G^{\mu i,\lambda k}\Omega_{\lambda k,\nu j} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\delta_{ij} \,. \tag{39}$$

5. Hamilton–Killing Flows

In the previous sections we studied those Hamiltonian flows $\bar{K}$ that, in addition to preserving the symplectic form, are generated by a gauge invariant $\tilde{K}$ so they also preserve the normalization constraint $\tilde{N}$. Our next goal is to find those flows that also happen to preserve the metric G of $T^*\mathcal{S}^+$, that is, we want $\bar{K}$ to be a Killing vector. The vector field $\bar{K}$ is determined by the Killing equation [16], $\pounds_K G = 0$, or

$$(\pounds_K G)_{\mu i,\nu j} = K^{\lambda k}\partial_{\lambda k}G_{\mu i,\nu j} + G_{\lambda k,\nu j}\partial_{\mu i}K^{\lambda k} + G_{\mu i,\lambda k}\partial_{\nu j}K^{\lambda k} = 0 \,. \tag{40}$$

Since Equation (38) gives $\partial_{\lambda k}G_{\mu i,\nu j} = 0$, the Killing equation simplifies to

$$(\pounds_K G)_{\mu i,\nu j} = -i\begin{bmatrix} \frac{\partial K^{2j}}{\partial\psi_i} + \frac{\partial K^{2i}}{\partial\psi_j} \,; & \frac{\partial K^{1j}}{\partial\psi_i} + \frac{\partial K^{2i}}{\partial i\psi_j^*} \\ \frac{\partial K^{2j}}{\partial i\psi_i^*} + \frac{\partial K^{1i}}{\partial\psi_j} \,; & \frac{\partial K^{1j}}{\partial i\psi_i^*} + \frac{\partial K^{1i}}{\partial i\psi_j^*} \end{bmatrix} = 0 \,, \tag{41}$$

where $\partial/\partial i\psi_i^* \stackrel{\text{def}}{=} -i\partial/\partial\psi_i^*$. If we further require that $\bar{K}$ is a Hamiltonian flow, $\pounds_K\Omega = 0$, then $K^{\mu i}$ satisfies Hamilton's equations,

$$K^{1i} = \frac{\partial\tilde{K}}{\partial i\psi_i^*} \quad \text{and} \quad K^{2i} = -\frac{\partial\tilde{K}}{\partial\psi_i} \,. \tag{42}$$

Substituting into (41), we find

$$\frac{\partial^2\tilde{K}}{\partial\psi_i\partial\psi_j} = 0 \quad \text{and} \quad \frac{\partial^2\tilde{K}}{\partial\psi_i^*\partial\psi_j^*} = 0 \,. \tag{43}$$

Therefore, in order to generate a flow that preserves both G and Ω, the function $\tilde{K}(\psi,\psi^*)$ must be *linear* in both ψ and ψ^*.

$$\tilde{K}(\psi,\psi^*) = \sum_{i,j=1}^{n}\psi_i^*\hat{K}_{ij}\psi_j + \sum_{i=1}^{n}(\psi_i^*\hat{L}_i + \hat{M}_i\psi_i) + const \,. \tag{44}$$

The kernels $\hat{K}_{ij}$, $\hat{L}_i$ and $\hat{M}_i$ are independent of ψ and ψ^*. Imposing that the flow preserves the normalization constraint $\tilde{N} = const$, Equation (21), implies that $\tilde{K}$ must be invariant under the phase shift $\psi \to \psi e^{i\nu}$. Therefore, $\hat{L}_i = \hat{M}_i = 0$ and we conclude that

$$\tilde{K}(\psi,\psi^*) = \sum_{i,j=1}^{n}\psi_i^*\hat{K}_{ij}\psi_j + const \,. \tag{45}$$

The corresponding HK flow is given by Hamilton's equations

$$\frac{d\psi_i}{d\lambda} = K^{1i} = \frac{\partial\tilde{K}}{\partial i\psi_i^*} = \frac{1}{i}\sum_{j=1}^{n}\hat{K}_{ij}\psi_j \,, \tag{46}$$

$$\frac{di\psi_i^*}{d\lambda} = K^{2i} = -\frac{\partial\tilde{K}}{\partial\psi_i} = -\sum_{j=1}^{n}\psi_j^*\hat{K}_{ji} \,. \tag{47}$$

The constant in (45) can be dropped, because it has no effect on the flow. Taking the complex conjugate of (46) and comparing with (47) show that the kernel $\hat{K}_{ij}$ is Hermitian and that the corresponding Hamiltonian functionals $\tilde{K}$ are real.

$$\hat{K}^*_{ij} = \hat{K}_{ji} \quad \text{and} \quad \tilde{K}(\psi, \psi^*)^* = \tilde{K}(\psi, \psi^*) \,. \tag{48}$$

To summarize, *the preservation of the symplectic structure, the metric structure and the normalization constraint leads to Hamiltonian functions $\tilde{K}$ that are bilinear in ψ and ψ^**, Equation (45). This is the main result of this paper. To appreciate its significance, once again, we adopt a more suggestive notation, i.e., the flow generated by the Hamiltonian function

$$\tilde{H}(\psi, \psi^*) = \sum_{i,j=1}^{n} \psi^*_i \hat{H}_{ij} \psi_j \quad \text{is} \quad \frac{d\psi_i}{d\tau} = \{\psi_i, \tilde{H}\} \quad \text{or} \quad i\frac{d\psi_i}{d\tau} = \sum_{j=1}^{n} \hat{H}_{ij}\psi_j \,, \tag{49}$$

which is recognized as the Schrödinger equation. Beyond being Hermitian, the actual form of the kernel $\hat{H}_{ij}$ remains undetermined.

The central feature of Hamilton's Equations (46) or of the Schrödinger Equation (49) is that they are linear. Given two solutions $\psi^{(1)}$ and $\psi^{(2)}$ and arbitrary constants c_1 and c_2, the linear combination $\psi^{(3)} = c_1\psi^{(1)} + c_2\psi^{(2)}$ is also a solution and this is extremely useful in calculations. Unfortunately, this is an HK flow on the embedding space $T^*\mathcal{S}^+$ and, when the flow is projected onto the e-phase space $T^*\mathcal{S}$, the linearity is severely restricted by normalization. If $\psi^{(1)}$ and $\psi^{(2)}$ are normalized points on $T^*\mathcal{S}$, the superposition $\psi^{(3)}$ is not in general a normalized point on $T^*\mathcal{S}$, unless the constants c_1 and c_2 are chosen appropriately. Furthermore, the states $\psi'^{(1)} = \psi^{(1)}e^{i\nu_1}$ and $\psi'^{(2)} = \psi^{(2)}e^{i\nu_2}$ are supposed to be "physically" equivalent to the original $\psi^{(1)}$ and $\psi^{(2)}$, but, in general, the superposition $\psi'^{(3)} = c_1\psi'^{(1)} + c_2\psi'^{(2)}$ is not equivalent to $\psi^{(3)}$. In other words, *the mathematical linearity of (46) or (49) does not extend to a full-blown superposition principle for physically equivalent states*. On the other hand, any point ψ deserves to be called a "state" in the limited sense that it may serve as the initial condition for a curve in $T^*\mathcal{S}^+$. Since, given two states $\psi^{(1)}$ and $\psi^{(2)}$, their superposition $\psi^{(3)}$ is also a state, we see that the set of states $\{\psi\}$ forms a linear vector space. This is a structure that turns out to be very useful.

6. Hilbert Space

Above we saw that the possible initial conditions for an HK flow, the *points* ψ of $T^*\mathcal{S}^+$, form a linear vector space. To take full advantage of linearity we would like to endow this vector space with the additional structure of an inner product and turn it into a Hilbert space—a term which we loosely use to describe any complex vector space with a Hermitian inner product. The metric tensor G (Equation (38)) and the symplectic form Ω (Equation (37)) are supposed to act on *vectors* $d/d\lambda$; their action on the *points* ψ or (ρ, π) is not defined. However, the choice of inner product for the points ψ is natural, in the sense that the necessary ingredients, G and Ω, are already available.

We adopt the familiar Dirac notation to represent the states ψ as vectors $|\psi\rangle$. In order that the inner product $\langle\psi|\phi\rangle$ be preserved it is defined in terms of the preserved tensors G and Ω,

$$\langle\psi|\phi\rangle \stackrel{\text{def}}{=} \frac{1}{2}\big(G_{\mu i,\nu j} + \alpha\Omega_{\mu i,\nu j}\big)\psi^{\mu i}\phi^{\nu j} \,, \tag{50}$$

where α is a constant and, to follow convention, the overall constant is set to $1/2$. Using Equation (37) and (38), we obtain

$$\langle\psi|\phi\rangle = \frac{1}{2}(\psi_i, i\psi^*_i)(G + \alpha\Omega)\begin{pmatrix}\phi_j \\ i\phi^*_j\end{pmatrix} = \frac{1}{2}\sum_{i=1}^{n}((1 - i\alpha)\psi^*_i\phi_i + (1 + i\alpha)\phi^*_i\psi_i) \,. \tag{51}$$

To fix α, we impose that $\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$, which implies that $\alpha = \pm i$. In order to comply with the standard convention that the inner product $\langle\psi|\phi\rangle$ is anti-linear in the first factor

and linear in the second factor, we select $\alpha = +i$. The result is the familiar expression for the positive definite inner product,

$$\langle\psi|\phi\rangle \stackrel{\text{def}}{=} \frac{1}{2}\big(G_{\mu i,\nu j} + i\Omega_{\mu i,\nu j}\big)\psi^{\mu i}\phi^{\nu j} = \sum_{i=1}^{n} \psi_i^* \phi_i \,. \tag{52}$$

Here we see that the choice of $1/2$ as the overall constant leads to the standard relation $\langle\psi|\psi\rangle = |\rho|$. The map between points and vectors, $\psi \leftrightarrow |\psi\rangle$, is defined by $|\psi\rangle = \sum_i |i\rangle\psi_i$, where $\psi_i = \langle i|\psi\rangle$ and the vectors $\{|i\rangle\}$ form a basis that is orthogonal and complete.

The bilinear Hamilton function $\tilde{K}(\psi, \psi^*)$ with kernel $\hat{K}_{ij}$ can now be written as the expected value, $\tilde{K}(\psi, \psi^*) = \langle\psi|\hat{K}|\psi\rangle$, of the Hamiltonian operator $\hat{K}$ with matrix elements $\hat{K}_{ij} = \langle i|\hat{K}|j\rangle$. The corresponding HK flows are given by

$$i\frac{d}{d\lambda}\langle i|\psi\rangle = \langle i|\hat{K}|\psi\rangle \quad \text{or} \quad i\frac{d}{d\lambda}|\psi\rangle = \hat{K}|\psi\rangle \,, \tag{53}$$

which are described by unitary transformations $|\psi(\lambda)\rangle = \hat{U}_K(\lambda)|\psi(0)\rangle$ where $\hat{U}_K(\lambda) = \exp(-i\hat{K}\lambda)$. Finally, the Poisson bracket of two Hamiltonian functions $\tilde{U}[\psi, \psi^*]$ and $\tilde{V}[\psi, \psi^*]$ can be written in terms of the commutator of the associated operators, $\{\tilde{U}, \tilde{V}\} = -i\langle\psi|[\hat{U}, \hat{V}]|\psi\rangle$. Thus, the Poisson bracket is the expectation of the commutator. This *identity* is much sharper than Dirac's pioneering discovery that the quantum commutator of two quantum variables is *analogous* to the Poisson bracket of the corresponding classical variables.

7. Conclusions

There have been numerous attempts to derive or construct the mathematical formalism of quantum mechanics by adapting the symplectic geometry of classical mechanics. Such phase space methods invariably start from a classical phase space of positions and momenta (q^i, p_i) and, through some series of "quantization rules," posit a correspondence to self-adjoint operators $(\hat{Q}^i, \hat{P}_i)$ which no longer constitute a phase space. The connection to classical mechanics is lost. The interpretation of $\hat{Q}^i$ and $\hat{P}_i$ and even the answer to the question of what is ontic and what is epistemic become highly controversial. Probabilities play a secondary role in such formulations.

In this paper, we take a different starting point that places probabilities at the forefront. We discuss special families of curves—the Hamilton–Killing flows—that promise to be useful for the study of quantum mechanics. We show that the HK flows that preserve the symplectic and the metric structures of the e-phase space reproduce much of the mathematical formalism of quantum theory. It clarifies how the linearity of the Schrödinger equation, complex numbers and the Born rule $\rho_i = |\psi_i|^2$ (the Born rule for generic observables is discussed in [31,32]) follow from the symplectic and metric structures, while the normalization constraint leads to the equivalence of states along rays in a Hilbert vector space.

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References

1. Bell, J. Against 'Measurement'. *Phys. World* **1990**, 33. [CrossRef]
2. Stapp, H.P. The Copenhagen Interpretation. *Am. J. Phys.* **1972**, *40*, 1098. [CrossRef]
3. Schlösshauer, M. Decoherence, the measurement problem, and interpretations of quantum mechanics. *Rev. Mod. Phys.* **2004**, *76*, 1267. [CrossRef]
4. Jaeger, G. *Entanglement, Information, and the Interpretation of Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 2009.
5. Leifer, M.S. Is the Quantum State Real? An Extended Review of Ψ-ontology Theorems. *Quanta* **2014**, *3*, 67. [CrossRef]

6. Nelson, E. *Quantum Fluctuations*; Princeton UP: Princeton, NJ, USA, 1985.
7. Caticha, A. Entropic Dynamics, Time, and Quantum Theory. *J. Phys. A Math. Theor.* **2011**, *44*, 225303. [CrossRef]
8. Goyal, P.; Knuth, K.; Skilling, J. Origin of complex quantum amplitudes and Feynman's rules. *Phys. Rev. A* **2010**, *81*, 022109. [CrossRef]
9. Reginatto, M.; Hall, M.J.W. Quantum theory from the geometry of evolving probabilities. *AIP Conf. Proc.* **2012**, *1443*, 96.
10. Reginatto, M.; Hall, M.J.W. Information geometry, dynamics and discrete quantum mechanics. *AIP Conf. Proc.* **2013**, *1553*, 246.
11. Caticha, A. The Entropic Dynamics approach to Quantum Mechanics. *Entropy* **2019**, *21*, 943. [CrossRef]
12. Amari, S.; Nagaoka, H. *Methods of Information Geometry*; American Mathematical Society: Providence, RI, USA, 2000.
13. Caticha, A. Entropic Physics: Probability, Entropy, and the Foundations of Physics. Available online: Https://www.albany.edu/physics/faculty/ariel-caticha (accessed on 20 June 2021).
14. Arnold, V.I. *Mathematical Methods of Classical Mechanics*; Springer: Berlin/Heidelberg, Germany, 1997; Volume 60.
15. Souriau, J.-M. *Structure of Dynamical Systems—A Symplectic View of Physics*; Translation by Cushman-deVries, C.H.; Birkhäuser: Boston, MA, USA, 1997.
16. Schutz, B. *Geometrical Methods of Mathematical Physics*; Cambridge U.P.: Cambridge, UK, 1980.
17. Hermann, R. Remarks on the Geometric Nature of Quantum Phase Space. *J. Math. Phys.* **1965**, *6*, 1768. [CrossRef]
18. Kibble, T.W.B. Geometrization of Quantum Mechanics. *Commun. Math. Phys.* **1979**, *65*, 189–201. [CrossRef]
19. Heslot, A. Quantum mechanics as a classical theory. *Phys. Rev. D* **1985**, *31*, 1341. [CrossRef]
20. Anandan, J.; Aharonov, Y. Geometry of Quantum Evolution. *Phys. Rev. Lett.* **1990**, *65*, 1697. [CrossRef] [PubMed]
21. Cirelli, R.; Manià, A.; Pizzochero, L. Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Parts I and II. *J. Math. Phys.* **1990**, *31*, 2891, 2898. [CrossRef]
22. Abe, S. Quantum-state space metric and correlations. *Phys. Rev. A* **1992**, *46*, 1667. [CrossRef] [PubMed]
23. Hughston, L.P. Geometric aspects of quantum mechanics. In *Twistor Theory*; Huggett, S.A., Ed.; Marcel Dekker: New York, NY, USA, 1995.
24. Ashtekar, A.; Schilling, T.A. Geometrical Formulation of Quantum Mechanics. In *On Einstein's Path*; Harvey, A., Ed.; Springer: New York, NY, USA, 1998.
25. de Gosson, M.A.; Hiley, B.J. Imprints of the Quantum World in Classical Mechanics. *Found. Phys.* **2011**, *41*, 1415. [CrossRef]
26. Elze, H.-T. Linear dynamics of quantum-classical hybrids. *Phys. Rev. A* **2012**, *85*, 052109. [CrossRef]
27. Wootters, W.K. Statistical distance and Hilbert space. *Phys. Rev. D* **1981**, *23*, 357. [CrossRef]
28. Brodie, D.J.; Hughston, L.P. Statistical Geometry in Quantum Mechanics. *Phil. Trans. R. Soc. Lond. A* **1998**, *454*, 2445. [CrossRef]
29. Goyal, P. From Information Geometry to Quantum Theory. *New J. Phys.* **2010**, *12*, 023012. [CrossRef]
30. Molitor, M. On the relation between geometrical quantum mechanics and information geometry. *J. Geom. Mech.* **2015**, *7*, 169. [CrossRef]
31. Johnson, D.T.; Caticha, A. Entropic dynamics and the quantum measurement problem. *AIP Conf. Proc.* **2012**, *1443*, 104.
32. Vanslette, K.; Caticha, A. Quantum measurement and weak values in entropic quantum dynamics. *AIP Conf. Proc.* **2017**, *1853*, 090003.
33. Nawaz, S.; Caticha, A. Momentum and uncertainty relations in the entropic approach to quantum theory. *AIP Conf. Proc.* **2012**, *1443*, 112.
34. Bartolomeo, D.; Caticha, A. Trading drift and fluctuations in entropic dynamics: Quantum dynamics as an emergent universality class. *J. Phys. Conf. Ser.* **2016**, *701*, 012009. [CrossRef]
35. Bartolomeo, D.; Caticha, A. Entropic Dynamics: The Schrödinger equation and its Bohmian limit. *AIP Conf. Proc.* **2016**, *1757*, 030002.
36. Demme, A.; Caticha, A. The Classical Limit of Entropic Quantum Dynamics. *AIP Conf. Proc.* **2017**, *1853*, 090001.
37. Nawaz, S.; Abedi, M.; Caticha, A. Entropic Dynamics on Curved Spaces. *AIP Conf. Proc.* **2016**, *1757*, 030004.
38. Ipek, S.; Caticha, A. Entropic quantization of scalar fields. *AIP Conf. Proc.* **2015**, *1641*, 345.
39. Ipek, S.; Abedi, M.; Caticha, A. Entropic Dynamics: Reconstructing Quantum Field Theory in Curved Spacetime. *Class. Quantum Grav.* **2019**, *36*, 205013. [CrossRef]
40. Ipek, S.; Caticha, A. The Entropic Dynamics of Quantum Scalar fields coupled to Gravity. *Symmetry* **2020**, *12*, 1324. [CrossRef]
41. Caticha, A.; Carrara, N. The Entropic Dynamics of Spin. *arXiv* **2020**, arXiv:2007.15719.
42. Caticha, A. Entropic Dynamics: Quantum Mechanics from Entropy and Information Geometry. *Ann. Physik* **2019**, *531*, 1700408. [CrossRef]
43. Campbell, L.L. An extended Čencov characterization of the information metric. *Proc. Am. Math. Soc.* **1986**, *98*, 135.